

On the Number of Solutions for the P4P Problem ¹⁾

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Abstract. The perspective n-point (PnP) problem is to find the position and orientation of a camera with respect to a scene object from n correspondence points. In this paper, we prove that with probability one, the P4P problem has a unique solution and this solution could be represented by a set of rational functions in the parameters. For a class of special P4P problems, we give the conditions for the problem to have one, two, three four, and five solutions. We also show that to solve the P4P problem with the rational functions is quite stable and accurate.

Key Words: Camera calibration, Elimination method, Perspective-four-Point Problem (*P4P*), Pose determination, Subresultant, Wu-Ritt's zero decomposition method.

1. Introduction

One of the fundamental goals of computer vision is to discover properties that are intrinsic to a scene by one or several images of this scene. Within this paradigm, an essential process is the determination of the position and orientation of the sensing device (the camera) with respect to objects in the scene. This problem is known as the exterior camera calibration problem and has many interesting applications in robotics and cartography and concerns many important fields, such as computer vision, automation, image analysis, automated cartography, photogrammetry, robotics and model based machine vision system, etc. Fischer and Bolles [2] summarize the problem as follows:

“ Given the relative spatial locations of n control points, and given the angle to every pair of control points from an additional point called the Center of Perspective (C_P), find the lengths of the line segments joining C_P to each of the control points.”

This problem is referred to as the perspective n-point (PnP) problem and there are many results. The P3P problem is the smallest subset of control points that yields a finite number of solutions. The P3P problem could have one, two, three, and four solutions and the probabilities for these four cases are all positive [8]. In order to obtain a unique solution, we need to add more conditions. One of the natural way is to add one control point to consider a P4P problem.

For the P4P problem, Rivers et al. [12] give a set of six quadratic equations with four unknown. Fischer and Bolles attack the problem by finding solutions associated with subsets

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of three points and selecting the solutions that they have in common. Horaud et al. [4] convert the P4P into a special 3-line problem and obtain a fourth degree polynomial equation and show that this equation system has at most four solution. In [11], Quan and Lan present a linear algebra algorithm to solve the P4P problem.

When the control points are coplanar, the P4P problem has a unique solution [1]. If control points are not coplanar, the P4P problem could have five solutions [15]. It is expected that in the general case the P4P problem has a unique solution. In this paper, we prove that with probability one, the P4P problem has a unique solution and this solution could be represented by a set of rational functions in the parameters. For a class of special P4P problems, we give the conditions for the problem to have one, two, three four, and five solutions. We also show that to solve the P4P problem with the rational functions is quite stable and accurate.

In [15], the authors give a theorem which shows that the upper bound of the positive solution of the non-coplanar p4p problem is five and also obtain some sufficient conditions for multiple positive solutions of the non-coplanar P4P problem. In the following Theorem 3.3, we give a geometric proof for the fact that the upper bound of the positive solution of the non-coplanar p4p problem is five. The proof has the advantage that we may tell clearly how the multiple solutions occurs.

The rest of the paper is organized as follows. In section 2, we give a simplified form of the P4P equation system. In Section 3, we present the main result. In Section 4, we give a complete analysis for a special class of P4P problems. In Section 5, we give the experimental results. The proof for the main theorem is given in the appendix.

2. The P4P equation system

Let P be the center of perspective, and A, B, C, D the control points. Let $p = 2 \cos \angle(BPC), q = 2 \cos \angle(APC), r = 2 \cos \angle(APB), s = 2 \cos \angle(CPD), t = 2 \cos \angle(APD), u = 2 \cos \angle(BPD), l_1 = |AB|^2, l_2 = |AC|^2, l_3 = |BC|^2, l_4 = |AD|^2, l_5 = |CD|^2, l_6 = |BD|^2$ (Figure 1).

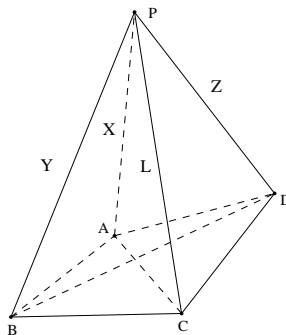


Fig. 1. The P4P problem

From triangles PAB, PAC, PBC, PAD, PBD and PCD , we obtain the P_4P equation

system:

$$\begin{cases} p_1 = X^2 + Y^2 - XYr - l_1 = 0 \\ p_2 = X^2 + L^2 - XLq - l_2 = 0 \\ p_3 = Y^2 + L^2 - YLp - l_3 = 0 \\ p_4 = X^2 + Z^2 - XZs - l_4 = 0 \\ p_5 = Z^2 + L^2 - ZLt - l_5 = 0 \\ p_6 = Y^2 + Z^2 - YZu - l_6 = 0 \end{cases} \quad (1)$$

We need to find the *positive solutions* for X, Y, Z, L . Since $L = |PC|$ is positive, we may make the following variable changes. Let $l_1 = awL^2$, $l_2 = bwL^2$, $l_3 = wL^2$, $l_4 = cwL^2$, $l_5 = dwL^2$, $l_6 = ewL^2$, $X = xL$, $Y = yL$, $Z = zL$. Equation system (1) becomes the following equivalent equation system:

$$\begin{cases} q_1 = x^2 + y^2 - xyr - aw = 0 \\ q_2 = x^2 + 1 - xq - w = 0 \\ q_3 = y^2 + 1 - yp - bw = 0 \\ q_4 = x^2 + z^2 - xzs - cw = 0 \\ q_5 = z^2 + 1 - zt - dw = 0 \\ q_6 = y^2 + z^2 - yzu - ew = 0 \end{cases} \quad (2)$$

From $q_2 = x^2 + 1 - xq - w = 0$ and $|q| < 2$ (see (4)), we have $w = x^2 + 1 - xq = (x - q/2)^2 + 1 - q^2/4 > 0$. L can be uniquely determined by $L = |AC|/\sqrt{w}$. Substituting w into (2), we have the following equivalent equation system:

$$\begin{cases} h_1 = x^2 + y^2 - xyr - a(x^2 + 1 - xq) = 0 \\ h_2 = y^2 + 1 - yp - b(x^2 + 1 - xq) = 0 \\ h_3 = x^2 + z^2 - xzs - c(x^2 + 1 - xq) = 0 \\ h_4 = z^2 + 1 - zt - d(x^2 + 1 - xq) = 0 \\ h_5 = y^2 + z^2 - yzu - e(x^2 + 1 - xq) = 0 \end{cases} \quad (3)$$

The above equation system is simpler than the original *P4P equation system*. This is a parametric equation system with eleven parameters $\mathcal{U} = \{a, b, c, d, e, p, q, r, s, t, u\}$ and three variables x, y, z .

It is clear that we need to add the following “reality conditions”, which are assumed through out the paper.

$$\begin{cases} x > 0, y > 0, z > 0, a > 0, b > 0, c > 0, d > 0, e > 0 \\ -2 < p < 2, -2 < q < 2, -2 < r < 2, -2 < s < 2, -2 < t < 2, -2 < u < 2 \\ \text{Triangles } ABC, ACE, ABE, BCE \text{ do not degenerate to lines.} \end{cases} \quad (4)$$

We also assume that P4P problem has a finite number of solutions. Consider a P3P problem with center of perspective P and control points A, B, C . Assume that A, B, C do not on the same line. It is known that the P3P problem $P-ABC$ has an infinite number of solutions if and only if P is on the circumscribed circle of triangle ABC [8]. Since we assumed that the triangles ABC, ACE, ABE, BCE in Figure 1 do not degenerated to lines, the P4P problem has an infinite number of solutions if and only if point P is on the circumscribed circles of these four triangles, which could happen if and only if points A, B, C, E and P are on the same circle.

3. Number of Solutions for P4P Problem

A set of parametric values is called *feasible* if they are from the real observation in the *P4P problem*. In other words, the P4P equation system has at least one positive solution for this set of parametric values.

Let \mathbf{F} be the set of feasible parametric values for the P4P problem. For $\mathcal{U}_0 \in \mathbf{F}$, let $\mathbf{Z}_{\mathcal{U}_0}$ be the set of solutions of equation (3) for variables x, y, z .

Theorem 3.1 *Use the notations introduced above.*

1. *The probability for $|\mathbf{Z}_{\mathcal{U}_0}| = 1, \mathcal{U}_0 \in \mathbf{F}$ is one. In other words, for almost all feasible parameters, the P4P problem has unique solution.*
2. *We may find rational functions P, Q, R in the parameters such that for almost all feasible parameters, the solutions for the P4P problem can be obtained as follows $x = P, y = Q, z = R$.*

To give the precise geometric meaning for the above theorem, we need the concepts of an irreducible variety and its dimension, which may be found in [3, 6]. Here we use an example to illustrate the concept. For a set of polynomial equations $PS = 0$ and a polynomial D , we use $\text{Zero}(PS)$ to denote the set of solutions for $PS = 0$ and $\text{Zero}(PS/D) = \text{Zero}(PS) - \text{Zero}(D)$. $\text{Zero}(PS)$ and $\text{Zero}(PS/D)$ are called a *variety* and a *quasi-variety* respectively. For instance, the unit sphere $S = \text{Zero}(\{x^2 + y^2 + z^2 - 1\})$ is an irreducible variety of dimension two and the unit circle in the z -plane $C = \text{Zero}(\{x^2 + y^2 - 1, z\})$ is an irreducible variety of dimension one. The dimension represents the number of variables that can take arbitrary values. $\text{Zero}(x^2 - y^2)$ is not irreducible, because it can be written as the union of two varieties $\text{Zero}(x^2 - y^2) = \text{Zero}(x - y) \cup \text{Zero}(x + y)$. Since the dimension for C is one, its area is zero. While the area for S is 4π . In this sense, we may say that the probability for a point $P \in S$ belongs to C is zero and the probability for a point $P \in S$ belongs to $S - C$ is one.

With these concepts, Theorem 3.1 is a consequence of the following result.

Theorem 3.2 *The set of feasible parameters \mathbf{F} is a quasi variety of dimension nine. We may find a subset \mathbf{T} of \mathbf{F} with dimension less than nine and rational functions P, Q, R in the parameters such that for $\mathcal{U}_0 \in \mathbf{F} - \mathbf{T}$, the solutions for equation system (3) can be given by $x = P, y = Q, z = R$.*

The proof can be found in the appendix. In what below, we will show how to compute the three rational functions P, Q, R and give a sketch of the proof.

For polynomials f and g , let $r = \text{Res}(f, g, v)$ be the resultant of f and g with respect to variable v . Then there exist polynomials u and v such that $r = uf + vg$ [10].

Let $f_1 = h_1 - h_2, f_2 = \text{Res}(f_1, h_1, y), f_3 = h_3 - h_4, f_4 = \text{Res}(f_3, h_3, z)$. Then

$$\begin{cases} f_2 = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \\ f_4 = b_0x^4 + b_1x^3 + b_2x^2 + b_3x + b_4 = 0 \end{cases} \quad (5)$$

Table 1. Parametric values for which the P4P problem could have 1,2,3,4,5 solutions

For $n > 4$, similarly, we can get the same result: in the general case, the camera pose is uniquely determined by n control points. The unique solution can be estimated by the equation system (9).

Two solutions P and Q for the P3P problem with control points A, B, C are said to be *symmetric* if P is the symmetric point of Q with respect to plane ABC .

Theorem 3.3 *Let A, B, C, D be four non-planar points. Then the P4P problem for $ABCD$ has at most five solutions. Furthermore, if the P4P problem has five solutions, then these solutions can be divided into two groups $\{P, Q\}$ and $\{U, V, W\}$ such that (1) P and Q are symmetric solutions for the P3P problem with control points A, B, C and PD is perpendicular to plane ABC ; (2) no two of U, V, W are symmetric solutions of the P3P with control points A, B, C .*

Proof. From [2, 8], we know that there is at most eight solution for the three points A, B and C (Figure 2). There is four solutions above the plane ABC and four solution under the plane ABC symmetrically. After added the fourth point D into the three points, we will prove that the following result: if P and Q are the symmetrical solutions for the three points A, B and C , point D must be either on the plane ABC or on the line PQ .

Firstly, since the intersecting angles of the line PD with the line PA, PB and PC are given, the PD is a intersecting line of three tapers. Similar to PD, QD also is a intersecting line of three tapers. We obtain the fact that PD and QD are unique determined by three tapers.

Secondly, suppose that D is not on the plane ABC and on the line PQ and let \bar{D} be the intersecting point of the line PD and the plane ABC . There are two different lines QD and $Q\bar{D}$ which both are the intersecting line of three tapers. It is a contradiction to the above fact proved and point D is either on the plane ABC or the line PQ .

If point D is on the plane ABC , it is known that there is only one solution for the p4p problem. If point D is on the line PQ , it is obvious that there is at most five solutions for the p4p problem.

From the above proof, we know that there exists five solutions if and only if point D is on the line PQ . Suppose that the point D is on the PQ . From the triangles PAO and DAO , we have the following expression: $|PO|^2 = (|AD|^2 - |DO|^2) \frac{s^2}{4-s^2}$. For the triangles PBO, DBO and triangles PCO and DCO , there are similar results. Let $|DO| = h$. Using the notations introduced above, we have the following necessary conditions for the exist of five solutions:

$$\begin{cases} (l_4^2 - h^2)s^2(4 - t^2) = (l_6^2 - h^2)t^2(4 - s^2) \\ (l_4^2 - h^2)s^2(4 - u^2) = (l_5^2 - h^2)u^2(4 - s^2). \end{cases} \quad (10)$$

4. One Special Case

In practice, most man made objects, such as buildings contain symmetries which allow the model to be expressed with far fewer parameters. In this section, we consider one special case in the P4P problem in which we will take these symmetries into account.

In this case, let $a = b = 1$, $c = d = e$, $p = q = r$ and $s = t = u$. There are only two parameters p and s . Then the equation system becomes:

$$\begin{cases} (-py + p)x + y^2 - 1 = 0 \\ y^2 - py - x^2 + px = 0 \\ -x^2 + (2p - sz)x + z^2 - 2 = 0 \\ z^2 - 1 - sz - 2x^2 + 2px = 0 \\ y^2 + z^2 - syz - 2x^2 - 2 + 2px = 0 \end{cases} \quad (11)$$

Using Wu-Ritt's zero decomposition method[6], we have the following result:

$$\text{Zero}((11)) = \bigcup_{i=1}^4 \text{Zero}(C_i) \quad (12)$$

where $C_i (i = 1, \dots, 4)$ are as follows ($s \neq 0, (p - 1) \neq 0$):

$$\begin{cases} p_{11} = x - 1 \\ p_{12} = y - 1 \\ p_{13} = z^2 - sz + 2p - 3 \end{cases} \quad \begin{cases} p_0 = (p - 3)s^2 + p^2 \\ p_{22} = x - p + 1 \\ p_{23} = y - 1 \\ p_{24} = sz - p \end{cases} \quad (13)$$

$$\begin{cases} p_0 = (p - 3)s^2 + p^2 \\ p_{31} = x - 1 \\ p_{32} = y - p + 1 \\ p_{33} = sz - p \end{cases} \quad \begin{cases} p_0 = (p - 3)s^2 + p^2 \\ p_{41} = (p - 1)x - 1 \\ p_{42} = (p - 1)y - 1 \\ p_{43} = (p - 1)sz - p \end{cases} \quad (14)$$

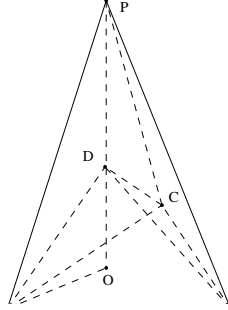


Fig. 2. Case one

Considering the equation p_{13} , we know that there are two solutions for z : $\frac{s}{2} + \frac{\sqrt{\Delta}}{2}$ and $\frac{s}{2} - \frac{\sqrt{\Delta}}{2}$ ($\Delta = s^2 - 4(2p - 3)$). It is easy to check that:

- The equation p_{13} has two positive solutions iff,

$$\begin{aligned} & s^2 - 4(2p - 3) > 0 \text{ and} \\ & s > 0 \text{ and } p > \frac{3}{2}. \end{aligned}$$

- The equation p_{13} has one positive solutions iff,

$$\begin{aligned} & s^2 - 4(2p - 3) > 0 \text{ and } p < \frac{3}{2}. \text{ or} \\ & s^2 - 4(2p - 3) = 0 \text{ and } s > 0. \text{ or} \\ & p = \frac{3}{2} \text{ and } s > 0. \end{aligned}$$

- The equation p_{13} does not have positive solutions iff,

$$\begin{aligned} & s^2 - 4(2p - 3) < 0. \text{ or} \\ & s^2 - 4(2p - 3) = 0 \text{ and } s < 0. \text{ or} \\ & s^2 - 4(2p - 3) > 0 \text{ and } s < 0 \text{ and } p \geq \frac{3}{2}. \end{aligned}$$

Combining the above conditions with other components in the equation systems [13,14], we have the following classification for the P4P problem.

- Point P has one positive solutions iff,

$$\begin{aligned} & s^2 - 4(2p - 3) > 0 \text{ and } p < \frac{3}{2} \text{ and } (p - 3)s^2 + p^2 \neq 0. \text{ or} \\ & s^2 - 4(2p - 3) = 0 \text{ and } s > 0 \text{ and } (p - 3)s^2 + p^2 \neq 0. \text{ or} \\ & p = \frac{3}{2} > 0 \text{ and } s > 0 \text{ and } (p - 3)s^2 + p^2 \neq 0. \end{aligned}$$

- Point P has two positive solutions iff,

$$\begin{aligned} & s^2 - 4(2p - 3) > 0 \text{ and} \\ & s > 0 \text{ and } p > \frac{3}{2} \text{ and } (p - 3)s^2 + p^2 \neq 0. \end{aligned}$$

- Point P has three positive solutions iff,

$$\begin{aligned} & s^2 - 4(2p - 3) < 0 \text{ and } (p - 3)s^2 + p^2 = 0. \text{ or} \\ & s^2 - 4(2p - 3) > 0 \text{ and } s < 0 \text{ and } p \geq \frac{3}{2} \text{ and } (p - 3)s^2 + p^2 = 0. \end{aligned}$$

- Point P has four positive solutions iff,

$$\begin{aligned} & s^2 - 4(2p - 3) > 0 \text{ and } p < \frac{3}{2} \text{ and } (p - 3)s^2 + p^2 = 0. \text{ or} \\ & p = \frac{3}{2} \text{ and } s = \frac{\sqrt{(6)}}{2} \text{ or } -\frac{\sqrt{(6)}}{2}. \end{aligned}$$

- Point P has five positive solutions iff,

$$\begin{aligned} & s^2 - 4(2p - 3) > 0 \text{ and} \\ & s > 0 \text{ and } p > \frac{3}{2} \text{ and } (p - 3)s^2 + p^2 = 0. \end{aligned}$$

Figure 3 is the solution distribution diagram for this special case. L_1 is $p = \frac{12+s^2}{8}$, L_2 is $(p - 3)s^2 + p^2 = 0$, and L_3 is $p = \frac{3}{2}$.

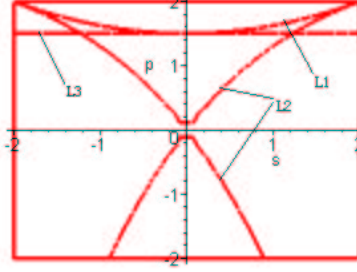


Fig. 3. Solution distribution for case $a = b = 1, c = d = e = 2, q = r = p, u = t = s$

5. Experimental Result

Based on Theorem 3.2, we may give the following algorithm for the P4P problem.

- Compute the a, b, c, d, e and p, q, r, s, t, u from the control points, the image points and the camera calibration matrix K .
- Obtain the solution from the equation system.
- Compute the camera rotation and translation using the absolute orientation[13].

We discuss the accuracy and the stability of the algorithm. For the accuracy of the algorithm, a lot of simulated experiments have been done and therefor show the Main Theorem. The following experiment is done with Maple. The experiment is to show the stability of the solution in Section 2. The detailed simulation results are shown in the following Table 3.

The optical center is located at the origin and the matrix of camera's intrinsic parameters is assumed to be the identity matrix. At each trial, four non-coplanar controls points are generated at random within a cube centered at $(0, 0, 50)$ and of dimension $60 \times 60 \times 60$. Then, the parameters $a, b, c, d, e, p, q, r, s, t,$ and u are computed. Finally, these computed parameters are substituted into equations system (10) in Section 2 to determine (x, y, z) . One hundred trials are carried out and 100 sets of parameters are computed for each trial. For each set of parametric values, two set of solutions are computed: one with the original control points denoted by (x, y, z) ; the other with the control points perturbed by random noises in certain level denoted by $(\hat{x}, \hat{y}, \hat{z})$. If the condition $Max(|x - \hat{x}|, |y - \hat{y}|, |z - \hat{z}|) < 1.5$, then it is considered as a same true solution of the non-coplanar P4P problem. In trial i , let n_i be the number of the parametric values such that the two results are the same and let $\frac{\|n_i - n\|}{n}$ (here $n = 100$) be the relative sets error.

| | | | | | | | |
|---------------------|-------|-------|------|------|------|------|------|
| Noise Level | 0.001 | 0.005 | 0.01 | 0.02 | 0.05 | 0.08 | 0.1 |
| Relative Sets error | 0.05 | 0.09 | 0.11 | 0.15 | 0.19 | 0.23 | 0.27 |

Table 3. Relative Error of for the Algorithm

From the Table 3, we can observe that the algorithms yield very graceful degradation with increasing noises and are, therefore very stable.

6. Appendix: Proof for Theorem 3.2

We first prove a Lemma

Lemma 6.1 *The ideal generated by $h_i, i = 1, \dots, 5$ is a prime ideal of dimension nine.*

We first prove that the ideal I_1 generated by $p_i, i = 1, \dots, 6$ in $\mathbb{1}$ is a prime ideal. Under the variable order $X < Y < L < Z < u < p < q < r < u < t < l_1 < l_2 < l_3 < l_4 < l_5 < l_6$, (1) is an irreducible ascending chain [6]. Since their initials are 1, I_1 is a prime ideal. The dimension for I_1 equals to the number of variables minus the number of the polynomials in the chain [6], which is $16 - 6 = 10$. Next, we will prove that the $q_i, i = 1, \dots, 6$ generate a prime ideal I_2 . Suppose $g_1 g_2 = \sum_{i=1}^6 k_i q_i$, where g_1, g_2 and k_i are polynomials in $a, b, c, d, e, p, q, r, s, t, u, w, x, y, z$. Perform the following substitution in p_i and clear denominators.

$$aw = \frac{l_1}{L^2}, bw = \frac{l_2}{L^2}, w = \frac{l_3}{L^2}, cw = \frac{l_4}{L^2}, dw = \frac{l_5}{L^2}, ew = \frac{l_6}{L^2}, x = \frac{X}{L}, y = \frac{Y}{L}, z = \frac{Z}{L},$$

Then replace w by $\frac{l_3}{L^2}$, we have

$$L^m \bar{g}_1 \bar{g}_2 = \sum_{i=1}^6 \bar{k}_i p_i$$

where m is a positive integer and $\bar{g}_1, \bar{g}_2, \bar{k}_i$ are polynomials in $l_i, i = 1, \dots, 6, p, q, r, s, t, u, X, Y, Z, L$. Since I_1 is a prime ideal and $L \notin I_1$, we have either \bar{g}_1 or \bar{g}_2 is in I_1 . Suppose that $\bar{g}_1 \in I_1$. Then \bar{g}_1 is a linear combination of p_i :

$$\bar{g}_1 = \sum_{i=1}^6 t_i p_i.$$

Perform the following substitution

$$l_1 = awL^2, l_2 = bwL^2, l_3 = wL^2, l_4 = cwL^2, l_5 = dwL^2, l_6 = ewL^2, X = xL, Y = yL, Z = zL$$

we have

$$L^d g_1 = \sum_{i=1}^6 L^2 \bar{t}_i q_i$$

where d is a positive integer. Compare the coefficients for L^d , we have g_1 is a linear combination for q_i and hence belongs to I_2 . Therefore, I_2 is a prime ideal. It is clear that $q_2, q_1, q_3, q_4, q_5, q_6$ consist of a characteristic set for ideal I_2 under the variable order $x < y < z < u < p < q < r < u < t < w < a < b < c < d < e$. Therefore, the dimension for I_2 equals to the number of variables minus the number of the polynomials in the characteristic set [6], which is $15 - 6 = 9$. Finally, we will prove that the ideal I_3 generated by $h_i, i = 1, \dots, 5$ is a prime ideal. It is clear that we have

$$I_3 = I_2 \cap Q[a, b, c, d, e, p, q, r, s, t, u, x, y, z].$$

Then I_3 is clearly a prime ideal. Since $I_2 = (I_3, q_2)$ by introducing a new variable w , I_3 has the same dimension as A_2 , which is nine. ▮

Proof for Theorem 3.2.

Continue with the computation after Theorem 3.2. Substituting x, y, z by $\frac{u_1}{i_1}, \frac{u_2}{i_2}, \frac{u_3}{i_3}$, P, Q, R in h_5 and clear the denominators, we have

$$f_7 = i_1^2 i_3^2 u_2^2 + i_1^2 i_2^2 u_3^2 - u i_1^2 i_2 i_3 u_2 u_3 - e i_2^2 i_3^2 u_1^2 - e i_1^2 i_2^2 i_3^2 + e q i_1 i_2^2 i_3^2 u_1 = i_4 e - u_4$$

where $u_4 = -i_1^2 i_3^2 u_2^2 - i_1^2 i_2^2 u_3^2 + u i_1^2 i_2 i_3 u_2 u_3$ and $i_4 = -i_2^2 i_3^2 u_1^2 - i_1^2 i_2^2 i_3^2 + q i_1 i_2^2 i_3^2 u_1$.

It is clear that the following is a characteristic set for ideal I_3 under the variable order $a < b < c < d < p < q < r < s < t < u < e < x < y < z$.

$$\begin{cases} f_6 = i_5 u^* + u_5 \\ f_7 = i_4 e - u_4 \\ f_5 = i_1 x - u_1 \\ f_8 = i_2 y - u_2 \\ f_9 = i_3 z - u_3 \end{cases} \quad (15)$$

As a consequence

$$\text{Zero}((3)) = \text{Zero}((15)/J) \cup \text{Zero}((3) \cup \{J\})$$

where $J = i_1 i_2 i_3 i_4 i_5$.

Let $S_0 = \text{Zero}(\{f_6, f_7\}/J)$. Then S_0 is a quasi-variety for the parameters with dimension nine. It is clear that for parametric values in S_0 , we may solve x, y, z as follows: $x = u_1/i_1, y = u_2/i_2, z = u_3/i_3$.

By Lemma 6.1, $I_1 = (3)$ is a prime ideal and $J \notin I_1$ (a polynomial $P \in I_1$ if and only if the pseudo-remainder of P with (3) is zero [6]). By the Affine dimension theorem [3], $\text{Zero}((3) \cup \{J\})$ is of lower dimension than I_1 . Applying Wu-Ritt Zero decomposition, we have

$$\text{Zero}((3) \cup \{J\}) = \cup_{j=1}^d \text{Zero}(C_j/J_j)$$

where C_j are ascending chains satisfying $|C_j| > 5$ and J_j are the products of initials of the polynomials in C_j . Let $A_j = C_j \cap Q[a, b, c, d, e, p, q, r, s, t, u]$. Since, we assumed that the P4P problem has a finite number of solutions, each C_j contains three polynomials with leading variables x, y, z . As a consequence $|A_j| > 2$. We may further assume that J_j are polynomials free of x, y, z [6]. Let $S_j = \text{Zero}(A_j/J_j)$. Then for parametric values taken from S_j the variables x, y, z have complex solutions. Let

$$\mathbf{S} = \cup_{j=0}^d S_j.$$

Then \mathbf{S} are the parametric values for which the x, y, z have complex solutions. S_0 is of dimension nine and $S_j, j > 0$ are of dimension lower than nine.

The feasible parametric set

$$\mathbf{F} = \cup_{j=0}^d S'_j$$

where S'_j is obtained by adding some inequalities to S_j so that x, y, z have positive solutions. These inequalities can be obtained similar to [8]. Since $S'_j \subset S_j$, the dimension of \mathbf{F} is less than or equal to nine. To prove it is nine, we need to show that S'_0 is of dimension nine.

Since $|q| < 2, x^2 + 1 - xq > 0$. From (3), we have

$$\begin{cases} a = \frac{x^2+y^2-xyr}{x^2+1-xq} \\ b = \frac{y^2+1-yp}{x^2+1-xq} \\ c = \frac{x^2+z^2-xzs}{x^2+1-xq} \\ d = \frac{z^2+1-zt}{x^2+1-xq} \\ e = \frac{y^2+z^2-yzu}{x^2+1-xq} \end{cases} \quad (16)$$

From the above equation, for any value of $x > 0, y > 0, z > 0, |p| < 2, |q| < 2, |r| < 2, |s| < 2, |t| < 2, |u| < 2$ we may solve a, b, c, d, e . In other words, for the above parametric values, the solution of (2) is a connected real manifold of dimension nine. Then the set of parametric values for which $x > 0, y > 0, z > 0$ have solutions must also be dimension nine. Therefore, \mathbf{F} is a quasi-variety of dimension nine.

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