An improved algebra method and its applications in nonlinear wave equations

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Abstract. In this paper, other types of exact solution of a first-order nonlinear ordinary differential equation, which is including the algebraic method, is further investigated. By using the solutions of this equation, we give some types of solutions of the coupled KdV equation, nonlinear dispersion mK(m,n,k) equation, the variant shallow water wave equation, nonlinear dissipative equation and the higher-order nonlinear Schrodinger equation. These solutions includes compacton solutions, solitary pattern solutions, solitary wave solutions, Weierstrass elliptic function solutions, which may be useful to explain some phenomena.

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1. Introduction

Nonlinear wave phenomena appear in many fields, such as fluid mechanics, plasma physics, biology, hydrodynamics, solid state physics and optical fibres, etc. These nonlinear phenomena are often related to nonlinear wave equations. In order to better understand these nonlinear phenomena as well as further apply them in the practical life, it is important to seek their more exact solutions, if available. Exact solutions not only certify whether or not the obtained numerical solutions is better, but also are used to watch the sport rule of the wave by making the graphs of the exact solutions. Moreover, new exact solutions may help people find new phenomena. Because of the complexity of nonlinear wave equations, there does not exist an uniform method to find all solutions of all nonlinear differential equations. Many powerful methods had been developed such as Backlund transformation[1,2], Darboux transformation[1,3], the inverse scattering transformation[4], the bilinear method[5], the tanh method[6], the sine-cosine method[7-10], the homogeneous balance method[11], the Riccati method[12,13], the Jacobian elliptic function expansion method[14] and its generalization[15,16], the transformation methods in terms of the Weierstrass elliptic function solutions[17-20], etc.

Recently, Fan[21] presented an unified algebraic method to obtain many types of travelling wave solutions based on a nonlinear ordinary differential equation with constant coefficients. In this paper we will improve the algebraic method such that it can be used to obtain more types of solutions, such as compacton solutions, solitary pattern solutions, solitary wave solutions and doubly periodic solutions in terms of the Weierstrass elliptic function.
The rest of this paper is organized as follows: In Sec. 2, we give the mathematical framework of the improved technique. In Sec. 3, we apply it to many types of nonlinear wave equations. Finally some conclusions are given.

2. The simple introduction of the algebraic method and its improved form

For a given nonlinear differential equation, say, in two variables $x, t$

$$F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0.$$ (1a)

where $F$ is a polynomial function with respect to the indicated variables or some function which can be reduced to a polynomial function by using some transformations.

Under the travelling wave transformation $u = u(\xi), \quad \xi = x - \lambda t$, (1a) reduces to an ordinary differential equation with constant coefficients

$$G(u, u', u'', u''', ...) = 0.$$ (1b)

where $G$ is a polynomial of $u$ and its derivatives. If $G$ is not a polynomial of $u$ and its derivatives, then we may use new variable $v(\xi)$ which makes $G$ become a polynomial of $v$ and its derivatives. Otherwise the following transformation will fail to seek solutions of (1a).

A transformation was presented by Fan[21] in the form

$$u(\xi) = A_0 + \sum_{i=1}^{m} A_i \omega^i(\xi).$$ (2a)

with the new variable $\omega$ satisfying

$$\omega'(\xi) = \sqrt{d_0 + d_1 \omega + d_2 \omega^2 + d_3 \omega^3 + d_4 \omega^4}.$$ (2b)

where $\xi = x - \lambda t$ and $A_0, A_i, d_j$ are constants;

Substituting (2a) into (1b) along with (2b), we can determine the parameter $m$ in (1b).

And then by substituting (2a) with the concrete $m$ into (1b) and equating the coefficients of these terms $\omega^i \omega^j (i = 0, 1, 2, \ldots; j = 0, 1)$ we obtain a system of algebraic equations with respect to other parameters $A_0, A_i, d_j, \lambda$. By solving the system, if available, we may determine these parameters. Therefore we establish a transformation (2a) between (1b) and (2b). If we know the solutions of (2b), then we can obtain the solutions of (1b) (or (1a)) by using (2a).

The crucial step of the technique is that we need to give the solutions of (2b). It is difficult to give the general solution of (2b). Fan[21] gave some solutions for some special cases of (2b) which led to some solutions of some nonlinear evolution equations. Motivated by Refs.[17-20], in this paper we will investigate other types of solutions of other some special cases of (2b) such that they are used to obtain new exact solutions of some nonlinear evolution equations such as the coupled KdV equation, nonlinear dispersion mK(m,n,k) equation, the variant shallow water wave equation, nonlinear dissipative equation and the higher-order nonlinear Schrodinger equation.

In what follows we only consider some special cases of (2b), which is important to solve other nonlinear wave equations:
Case 1: Taking (2b) in the form
\[ \omega'(\xi) = \sqrt{d_0 + d_1\omega(\xi) + d_2\omega^2(\xi)}, \]
which has the solution
\[ \omega(\xi) = \sqrt{\frac{d_1^2 - 4d_0d_2}{4d_2^2}} \sin(\sqrt{-d_2}\xi + \xi_0) - \frac{d_1}{2d_2}, \quad d_2 < 0. \]
or the hyperbolic function solutions
\[ \omega(\xi) = \sqrt{\frac{4d_0d_2 - d_1^2}{4d_2^2}} \sinh(\sqrt{d_2}\xi) - \frac{d_1}{2d_2}, \quad d_2 > 0. \]
\[ \omega(\xi) = \sqrt{\frac{d_1^2 - 4d_0d_2}{4d_2^2}} \cosh(\sqrt{d_2}\xi) - \frac{d_1}{2d_2}, \quad d_2 > 0. \]

Remark 1: The case was not considered by Fan[21]. But it is useful to obtain compacton solutions and solitary pattern solutions[23-29], which are of important physical significance.

Case 2: Taking (2b) in the form
\[ \omega'(\xi) = \sqrt{d_0 + d_2\omega^2(\xi) + d_4\omega^4(\xi)}, \]
which has the Weierstrass elliptic function solution
\[ \omega_1(\xi) = \sqrt{\frac{1}{d_4}(\wp(\xi; g_2, g_3) - \frac{1}{3}d_2)}, \]
where
\[ g_2 = \frac{4d_2^2 - 12d_0d_4}{3}, \quad g_3 = \frac{4d_2(-2d_2^2 + 9d_0d_4)}{27}, \]
\[ \omega_2(\xi) = \sqrt{\frac{3d_0}{3\wp(\xi; g_2, g_3) - d_2}}, \]
where \( g_2, g_3 \) satisfy (9),
\[ \omega_3(\xi) = \sqrt{\frac{12d_0\wp(\xi; g_2, g_3) + 2d_0(2d_2 + D)}{12\wp(\xi; g_2, g_3) + D}}, \]
where
\[ D = \frac{-5d_2 \pm \sqrt{9d_2^2 - 36d_0d_4}}{2}, \quad g_2 = -\frac{1}{12}(5d_2D + 4d_2^2 + 33d_0d_2d_4), \]
\[ g_3 = -\frac{1}{216}(-21d_2^2D + 63d_0d_4D - 20d_2^3 + 27d_0d_2d_4), \]
\[ \omega_4(\xi) = \frac{6\sqrt{d_0\wp(\xi; g_2, g_3) + d_2\sqrt{d_0}}}{3\wp(\xi; g_2, g_3)}. \]
where \( \varphi'(\xi; g_2, g_3) = \frac{d\varphi(\xi; g_2, g_3)}{d\xi} \), and

\[
\begin{align*}
g_2 &= \frac{1}{12} d_2^2 + d_0 d_4, \quad g_3 = \frac{d_2}{216} (36d_0 d_4 - d_2^2). \\
\omega_5(\xi) &= \frac{3\sqrt{d_4^{-1}} \varphi'(\xi; g_2, g_3)}{6\varphi(\xi; g_2, g_3) + d_2},
\end{align*}
\]

(14)

where \( g_2, g_3 \) satisfy (14).

**Remark 2:** The case was not considered by Fan[21]. But it is useful to obtain more types of Weierstrass elliptic function solutions of nonlinear wave equations.

**Case 3:** Taking (2b) in the form

\[
\omega'(\xi) = \sqrt{d_0 + d_2 \omega^2(\xi) + d_4 \omega^4(\xi)}, \quad d_0 = \frac{5d_2^2}{36d_4},
\]

(16)

which has the new Weierstrass elliptic function solution

\[
\omega = \frac{d_2 \sqrt{-15d_2/(2d_4)} \varphi(\xi; g_2, g_3)}{3\varphi(\xi; g_2, g_3) + d_2},
\]

(17)

where \( g_2 = 2d_2^2/9, \quad g_3 = d_3^2/54 \).

**Case 4:** Taking (2b) in the form

\[
\omega'(\xi) = \sqrt{d_2 \omega^2(\xi) + d_3 \omega^3(\xi) + d_4 \omega^4(\xi)},
\]

(18)

which has solitary wave solution

\[
\omega(\xi) = \frac{-8d_2 d_3 c_0 \exp(\sqrt{d_2} \xi)}{4d_3^2 \exp(2\sqrt{d_2} \xi) + 4d_3^2 c_0 \exp(\sqrt{d_2} \xi) - c_0^2 (4d_2 d_4 - d_3^2)}
\]

\[
= \frac{-8d_2 d_3 c_0 \text{sech}^2(\sqrt{d_2} \xi)}{4d_3^2 c_0 \text{sech}^2(\sqrt{d_2} \xi) + [4d_3^2 + c_0^2 (4d_2 d_4 - d_3^2)] \tanh(\sqrt{d_2} \xi) + 4d_3^2 - c_0^2 (4d_2 d_4 - d_3^2)},
\]

(19)

where \( c_0 \) is a constant.

3. Examples and exact solutions:

In the following we will use the transformation to consider the nonlinear evolution equations.

**Example 1:** The coupled KdV equation[29,30]

\[
\begin{align*}
&u_t + a v v_x + b u^2 u_x + c u u_x + ru_{xxx} = 0, \\
&v_t + d(uv)_x + e v v_x = 0,
\end{align*}
\]

(20)

where \( a, b, c, d, e, r \) are constants. Recently, Feng[30] obtained three solitary wave solutions.
By travelling wave reduction, \( u = u(\xi), v = v(\xi), \xi = k(x - \lambda t) \), we get the relation
\[
v(\xi) = -\frac{2d}{e}u + \frac{2\lambda}{e},
\]
where \( f \) is an integration constant.

In the following we set
\[
A = b/3, \quad B = \frac{c}{2} + \frac{2ad^2}{e^2}, \quad C = -(\lambda + \frac{4ad\lambda}{e^2}), \quad E = \frac{2a\lambda^2}{e^2}.
\]

**Case 1:** We assume that (21b) has the solution
\[
u(\xi) = \omega(\xi) + \beta,
\]
where \( \omega(\xi) \) satisfies (7). Substituting (22) into (21b) along with (7) yields a system of equations
\[
\begin{align*}
B\beta^2 + A\beta^3 + C\beta + (E + f) &= 0, \\
B + 3A\beta &= 0, \\
3A\beta^2 + k^2 rd_2 + C + 2B\beta &= 0, \\
2k^2 rd_4 + A &= 0.
\end{align*}
\]
from which
\[
\beta = \frac{-B}{3A}, \quad d_4 = -\frac{A}{2k^2r}, \quad d_2 = -\frac{-B^2 + 3CA}{3k^2rA}, \\
f = \frac{1}{27A^2}(-2B^3 + 9CBA - 27EA^2),
\]
According to (8), (23) and (25), we obtain the Weierstrass elliptic function solution
\[
u_1 = \sqrt{\frac{9d_0 k^2 r A}{9k^2 r A \wp(\xi; g_2, g_3) - B^2 + 3AC}} - \frac{B}{3A},
\]
\[
v_1 = -\frac{2d}{e} \sqrt{\frac{9d_0 k^2 r A}{9k^2 r A \wp(\xi; g_2, g_3) - B^2 + 3AC} + \frac{2dB}{3eA} + \frac{2\lambda}{e}},
\]
where
\[
g_3 = \frac{1}{729A^3 r^3 k^6}(-B^2 + 3AC)(-4B^4 + 24B^2 AC - 36C^2 A^2 + 81A^2 k^4 r^2 g_2).
\]
According to (10), (23) and (25), we obtain the Weierstrass elliptic function solution
\[
u_2 = \sqrt{\frac{9d_0 k^2 r A}{9k^2 r A \wp(\xi; g_2, g_3) - B^2 + 3AC}} - \frac{B}{3A},
\]
\[
v_1 = -\frac{2d}{e} \sqrt{\frac{9d_0 k^2 r A}{9k^2 r A \wp(\xi; g_2, g_3) - B^2 + 3AC} + \frac{2dB}{3eA} + \frac{2\lambda}{e}},
\]
where
\[ g^2 = \frac{2}{27k^4r^2A^2}(2B^4 - 12B^2AC + 18C^2A^2 + 27d_0A^3k^2r), \]
\[ g^3 = \frac{2}{729k^6r^3A^3}(-B^2 + 3AC)(4B^4 - 24B^2AC + 36C^2A^2 + 81d_0A^3k^2r). \]

According to (11), (23) and (25), we obtain the Weierstrass elliptic function solution
\[ u_3 = \frac{\sqrt{12d_0}\wp(\xi; g_2, g_3) + 2d_0(2d_2 + D)}{12\wp(\xi; g_2, g_3) + D} - \frac{B}{3A}, \]  
\[ v_3 = -\frac{2d\sqrt{d_0}\wp(\xi; g_2, g_3) + 2d_0(2d_2 + D)}{12\wp(\xi; g_2, g_3) + D} + \frac{2dB}{3eA} + \frac{2\lambda}{e}, \]

where \( d_0 \) is a constant, and \( D, d_2, d_4, g_2, g_3 \) are determined by (12) and (25).

According to (13), (23) and (25), we obtain the Weierstrass elliptic function solution
\[ u_4 = \frac{\sqrt{d_0}(6\wp(\xi; g_2, g_3) - \frac{-B^2 + 3CA}{3k^2rA})}{3\wp(\xi; g_2, g_3)} - \frac{B}{3A}, \]
\[ v_4 = -\frac{2d\sqrt{d_0}(6\wp(\xi; g_2, g_3) - \frac{-B^2 + 3CA}{3k^2rA})}{3e\wp(\xi; g_2, g_3)} + \frac{2dB}{3eA} + \frac{2\lambda}{e}, \]
\[ g_2 = \frac{1}{108k^5r^2A^2}(B^2 - 6B^2CA + 9C^2A^2 - 54d_0A^3k^2r), \]
\[ g_3 = \frac{1}{5832k^6r^3A^3}(-B^2 + 3CA)(162d_0A^3k^2r + B^4 - 6B^2CA + 9C^2A^2). \]

According to (15), (23) and (25), we obtain the Weierstrass elliptic function solution
\[ u_5 = \frac{9k^2r\sqrt{-2k^2rA\wp'(\xi; g_2, g_3)}}{18k^2rA\wp(\xi; g_2, g_3) + B^2 - 3AC} - \frac{B}{3A}, \]
\[ v_5 = -\frac{18dk^2r\sqrt{-2k^2rA\wp'(\xi; g_2, g_3)}}{e(18k^2rA\wp(\xi; g_2, g_3) + B^2 - 3AC)} + \frac{2dB}{3eA} + \frac{2\lambda}{e}, \]

where \( g_2, g_3 \) satisfy (30).

According to (17), (23) and (25), we obtain the Weierstrass elliptic function solution
\[ u_6 = \frac{(B^2 - 3AC)^2\sqrt{5}(B^2 - 3AC)\wp(\xi; g_2, g_3)}{9k^2rA^2\wp(\xi; g_2, g_3) + A(B^2 - 3AC)} - \frac{B}{3A}, \]
\[ v_6 = -\frac{2d(B^2 - 3AC)^2\sqrt{5}(B^2 - 3AC)\wp(\xi; g_2, g_3)}{9eA^2\wp(\xi; g_2, g_3) + A(B^2 - 3AC)} + \frac{2dB}{3eA} + \frac{2\lambda}{e}, \]

where
\[ g_2 = \frac{2(B^2 - 3AC)^2}{81k^4r^2A^2}, \quad g_3 = \frac{(B^2 - 3AC)^3}{1458k^6r^3A^3}. \]

**Case 2:** We assume that (21b) has the solution
\[ u(\xi) = \omega(\xi) + \beta, \]  
(33)
where \( \omega(\xi) \) satisfies (18). Substituting (33) into (21b) along with (18) yields a system of equations. By solving the system we get

\[
d_4 = -\frac{A}{2k^2r}, \quad d_3 = -\frac{2(B + 3A\beta)}{3k^2r}, \quad d_2 = -\frac{3A\beta^2 + 2B\beta + C}{k^2r},
\]

\[
f = -E - B\beta^2 - A\beta^3 - C\beta,
\]

Therefore according to (19), (21a),(33) and (34) we get the solitary wave solution

\[
u_7 = \frac{M_1\text{sech}^2(\sqrt{-\frac{3A\beta^2 + 2B\beta + C}{k^2r}}\xi)}{M_2\text{sech}^2(\sqrt{-\frac{3A\beta^2 + 2B\beta + C}{k^2r}}\xi) + M_3\tanh(\sqrt{-\frac{3A\beta^2 + 2B\beta + C}{k^2r}}\xi) + M_4} + \beta,
\]

\[
u_7 = -\frac{2dM_1\text{sech}^2(\sqrt{-\frac{3A\beta^2 + 2B\beta + C}{k^2r}}\xi) + eM_3\tanh(\sqrt{-\frac{3A\beta^2 + 2B\beta + C}{k^2r}}\xi) + eM_4}{2d + 2\lambda e + 2\lambda},
\]}

where

\[
M_1 = -\frac{16(B + 3A\beta)(3A\beta^2 + 2B\beta + C)}{9k^4r^2}, \quad M_2 = \frac{16(B + 3A\beta)^2c_0}{9k^4r^2},
\]

\[
M_3 = \frac{16(B + 3A\beta)^2}{9k^4r^2} + \frac{18Ac_0(3A\beta^2 + 2B\beta + C) - 4c_0(B + 3A\beta)^2}{9k^4r^2},
\]

\[
M_4 = \frac{16(B + 3A\beta)^2}{9k^4r^2} - \frac{18Ac_0(3A\beta^2 + 2B\beta + C) - 4c_0(B + 3A\beta)^2}{9k^4r^2},
\]

**Example 2:** Nonlinear dispersions \( mk(m,n,k) \)

\[
u^{m-1}u_x + a(u^n)_x + b(u^k)_{xxx} = 0, \quad nk \neq 0,
\]

which is more general form of the \( mk(n,n) \) equation by Wazwaz[24].

Generally speaking, one used the direct method to consider compacton solutions and solitary pattern solutions[22-28]. And an ansatz is only used to obtain one type of solution. But we use the transformation, that is, \( u(\xi) = A\omega^\beta(\xi), \quad \omega'(\xi) = \sqrt{d_0 + d_2\omega^4(\xi)}, \) to obtain many types of compacton solutions and solitary pattern solutions without more anstaz.

(1) Compacton solutions with \( k = n > m \)

\[
u_1 = \left\{ \begin{array}{ll}
\frac{2\lambda k}{\lambda m(k + m)} \sin^2\left(\sqrt{\frac{\lambda}{mb}} - \frac{m}{2k} \xi - \frac{mb}{mb} \xi \right) \right\}^{1/(k-m)} \\
0,
\end{array} \right. \quad 0 \leq \sqrt{\frac{\lambda}{mb} - \frac{m}{2k} \xi} \leq \pi, \quad \text{otherwise}.
\]

(2) Compacton solutions with \( k = m > n \)

\[
u_2 = \left\{ \begin{array}{ll}
\frac{2amk}{\lambda(k + n)} \sin^2\left(\sqrt{\frac{-\lambda}{mb}} - \frac{m}{2k} \xi \right) \right\}^{1/(k-n)} \\
0,
\end{array} \right. \quad 0 \leq \sqrt{-\frac{\lambda}{mb} - \frac{m}{2k} \xi} \leq \pi, \quad \text{otherwise}.
\]
(3) Compacton solutions with $k = m = n > 0$

$$u_3 = \begin{cases} P \left\{ \sin(\sqrt{\frac{ka - \lambda}{kb}} \xi)^{1/k} \right\}, & 0 \leq \sqrt{\frac{ka - \lambda}{kb}} |\xi| \leq \pi, \\ 0, & \text{otherwise.} \end{cases}$$  \hspace{1cm} (40)

(4) Solitary pattern solutions

$$u_4 = \left[ -\frac{2\lambda k}{am(k + m)} \sinh^2\left(\sqrt{-\frac{a}{b}} \frac{k - m}{2k} \xi\right)^{1/(k - m)} \right], \quad k = n > m.$$  \hspace{1cm} (41)

$$u_5 = \left[ -\frac{2amk}{\lambda(k + n)} \sinh^2\left(\sqrt{\frac{\lambda}{mb}} \frac{k - n}{2k} \xi\right)^{1/(k - n)} \right], \quad k = m > n.$$  \hspace{1cm} (42)

$$u_6 = P[\sinh(\sqrt{\frac{\lambda - ka}{kb}} \xi)]^{1/k}, \quad k = n = m.$$  \hspace{1cm} (43)

$$u_7 = \left[ \frac{2\lambda k}{am(k + m)} \cosh^2\left(\sqrt{-\frac{a}{b}} \frac{k - m}{2k} \xi\right)^{1/(k - m)} \right], \quad k = n > m.$$  \hspace{1cm} (44)

$$u_8 = \left[ \frac{2amk}{\lambda(k + n)} \cosh^2\left(\sqrt{\frac{\lambda}{mb}} \frac{k - n}{2k} \xi\right)^{1/(k - n)} \right], \quad k = m > n.$$  \hspace{1cm} (45)

$$u_9 = P[\cosh(\sqrt{\frac{\lambda - ka}{kb}} \xi)]^{1/k}, \quad k = m = n.$$  \hspace{1cm} (46)

where $\xi = x - \lambda t$.

**Example 3:** Nonlinear dissipative equation[17]

$$(v^2 - a)\eta' + b(\eta^2)' + c\eta'' + d(\eta^3)'' + e(\eta')^2 + f(\eta^3)' + q\eta''' = N.$$  \hspace{1cm} (47)

where $'$ denotes $d/d\xi$, $\eta = \eta(\xi)$, $\xi = x - vt$

Porubov[17] had given a Weierstrass elliptic function solution of (47). Here we consider its other types of solutions. According to Case 2, under the conditions $c = \frac{2b}{3f}$, $e = -6d$ and N is taken as a proper constant, we have the Weierstrass elliptic function solutions

$$\eta_1 = \sqrt{-\frac{2q}{f} \psi(\xi; g_2, g_3) - \frac{b}{3f}},$$  \hspace{1cm} (48)

$$\eta_2 = \sqrt{\frac{d_0}{\psi(\xi; g_2, g_3) - \frac{b}{3f}}},$$  \hspace{1cm} (49)

where $\xi = x - \sqrt{\frac{b^2 + 3af}{3f}} t$, $g_2 = \frac{2d_0 f}{q}$, $g_3 = 0$.

$$\eta_3 = \sqrt{\frac{12d_0 \psi(\xi; g_2, g_3) + 2d_0 D}{12\psi(\xi; g_2, g_3) + D}},$$  \hspace{1cm} (50)
where \( D = \pm \frac{3}{2} \sqrt{\frac{2adf}{q}}, \quad g_2 = 0, \quad g_3 = \pm \frac{adf}{32q} \sqrt{\frac{2adf}{q}}. \)

**Example 4:** The higher-order nonlinear Schrödinger equation[31,32]

\[
\psi_z = i \psi_t + i |\psi|^2 \psi + \psi_{ttt} + c_1 (|\psi|^2 \psi)_t + c_2 \psi (|\psi|^2)_t,
\]

which describes the propagation of femtosecond pulses in optical fibers.

If we consider the solution

\[
\psi(z,t) = \psi(\xi) \exp(i\eta), \quad \xi = k(t + \lambda z), \quad \eta = \alpha z + \beta t,
\]

then (51) reduce to

\[
(1 + 3\beta)\psi'' + (1 - c_1 \beta)\psi^3 - (\alpha + \beta^2 + \beta^3)\psi = 0,
\]

under the conditions

\[
\beta = \frac{3 - 3c_1 - 2c_2}{12c_1 + 6c_2}, \quad \alpha = (1 + 3\beta)(\lambda + 2\beta + 3\beta^2) - \beta^2 - \beta^3,
\]

By considering (53), we can obtain the envelope Weierstrass elliptic function solutions of (51)

\[
\psi_1 = \exp(i\eta) \sqrt{\frac{2(1 + 3\beta)k^2}{c_1 \beta - 1}} (\wp(\xi; g_2, g_3) - \frac{(1 + 3\beta)k^2}{\alpha + \beta^2 + \beta^3}),
\]

where

\[
g_2 = \frac{4d_2^2 - 12d_0 d_4}{3}, \quad g_3 = \frac{4d_2 (-2d_2^2 + 9d_0 d_4)}{27}, \quad d_2 = \frac{\alpha + \beta^2 + \beta^3}{(1 + 3\beta)k^2}, \quad d_4 = \frac{c_1 \beta - 1}{(1 + 3\beta)k^2},
\]

\[
\psi_2 = \exp(i\eta) \sqrt{\frac{3d_0 (1 + 3\beta)k^2}{3k^2 (1 + 3\beta)\wp(\xi; g_2, g_3) - (\alpha + \beta^2 + \beta^3)}},
\]

where \( g_2, g_3 \) are determined by (55a).

\[
\psi_3 = \frac{\exp(i\eta) \sqrt{12d_0 \wp(\xi; g_2, g_3) + 2d_0 (2d_2 + D)}}{12\wp(\xi; g_2, g_3) + D},
\]

where \( D, g_2, g_3, d_2, d_4 \) are defined by (12) and (55b).

\[
\psi_4 = \frac{\exp(i\eta) \sqrt{d_0 (6(1 + 3\beta)k^2 \wp(\xi; g_2, g_3) + \alpha + \beta^2 + \beta^3)}}{3(1 + 3\beta)k^2 \wp(\xi; g_2, g_3)},
\]

where

\[
g_2 = \frac{1}{12} d_2^2 + d_0 d_4, \quad g_3 = \frac{d_2}{216} (36d_0 d_4 - d_2^2),
\]

with \( d_2, d_4 \) are determined by (55b).

\[
\psi_5 = \frac{3}{6} \frac{\sqrt{\frac{2(1 + 3\beta)k^2}{c_1 \beta - 1}} \wp'(\xi; g_2, g_3) \exp(i\eta)}{\wp(\xi; g_2, g_3) + d_0^2 (\alpha + \beta^2 + \beta^3)},
\]
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where \( g_2, g_3 \) satisfy (59).

\[ \psi_6 = \exp(i \eta) (\alpha + \beta^2 + \beta^3) \sqrt{-\frac{15(\alpha + \beta^2 + \beta^3)}{c_1 \beta - 1}} \varphi'(\xi; g_2, g_3) \]

\[ \frac{3(1 + 3 \beta) k^2 \varphi'(\xi; g_2, g_3) + \alpha + \beta^2 + \beta^3}{3(1 + 3 \beta) k^2 \varphi'(\xi; g_2, g_3)} \]  

(61)

where \( g_2 = \frac{2(\alpha + \beta^2 + \beta^3)^2}{9(1 + 3 \beta)^2 k^2} \) , \( g_3 = \frac{(\alpha + \beta^2 + \beta^3)^2}{3(1 + 3 \beta)^2 k^2} \).

In summary, by giving more types of solutions of a first order ODE, we obtain many types of solutions of some nonlinear wave equation in mathematical physics. These solutions contain solitary waves solutions, the Weierstrass elliptic function solutions, compacton solution, solitary pattern solution, etc. The transformation is also used to solve many other nonlinear wave equations. In addition, we only consider some special cases of (2b). For other cases, we need to further consider.

References