On Algebrico-Differential Equations-Solving ¹⁾

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Summary. The char-set method of polynomial equations-solving is naturally extended to the differential case which gives rise to an algorithmic method of solving arbitrary systems of algebrico-differential equations. As an illustration of the method, the Devil's Problem of Pommaret is solved in details.

Key Words. Algebrico-Differential Equations, (Differential) Zero-Decomposition Theorem, Riquier-Janet Theory and Method, Integrability d-Polynomial, Compatibility d-Polynomial, Pommaret's Devil Problem.

1. Introduction

Let $y, u_j, j \in J$, be infinitely differentiable functions in independent variables $X = \{x_k, k = 1, \dots, n\}$. A polynomial in various derivatives of y and u_j with respect to x_k with coefficients in the differential field of rational functions of X will be called an *algrbrico-differential polynomial*. Suppose given a finite system of such polynomials $DPS = \{DP_i \mid i \in I\}$. Let us consider the associated system of partial differential equations of y with u_j supposed known:

$$DPS = 0, \text{ or } DP_i = 0, i \in I.$$

Our problem is to determine the integrability conditions for y to be solvable in terms of x_k, u_j and in affirmative case to determine the set of all possible formal solutions of y .

Criteria and even algorithmic methods of solving the above problem were known in quite remote times for which we may cite particularly Riquier ([Rq1,Rq2]), Janet ([J1,J2]), and E.Cartan ([C1,C2]). In recent years J.F.Pommaret had given a systematic *formal intrinsic* way of treatment and had published several voluminous treatises, cf. e.g. [P1,P2,P3]. On the other hand, the present author had given an alternative method in following essentially the steps of Riquier and Janet, cf. [W1]. The present paper is actually a simplified version of the above paper. An example due to Pommaret, the so-called *Devil's Problem* will be treated in details to illustrate the procedure of our method.

For the illustration of our procedure let us first recall our *char-set* method of solving arbitrary purely algebraic polynomial equations. Thus, consider a set of variables $X = \{x_1, \dots, x_n\}$ and polynomials in X with coefficients in the complex field $\mathbf{C}$. We shall introduce some partial ordering among all finite systems of such polynomials. For this purpose let us

¹⁾The present paper is in honor of late Professor R.Thom as a great mathematician, a great scientist, and also a great thinker of modern times.

arrange first the variables x_k in natural ascending order. Any non-constant polynomial P in $\mathbf{C}[X]$ may then be written in a canonical form

$$P = I_d * x_c^d + I_{d-1} * x_c^{d-1} + \cdots + I_0,$$

in which each I_i are themselves polynomials in variables $x_1, \dots, x_{c-1}$ with $I_d \neq 0$. We call then c the *class* of P , d the *degree* of P , x_c the *leading variable* of P , and I_d the *leading coefficient* or *initial* of P . We then introduce a partial ordering of non-zero polynomials first according to class and then to degree, the non-zero constants being considered as polynomials of lowest ordering.

For the partial ordering of polynomial systems let us consider first such polynomial sets well-arranged in the following sense. The polynomials in the set are non-constant ones and may be so arranged with classes c_i steadily increasing:

$$0 < c_1 < \cdots < c_r.$$

The leading coefficient or the initial of the i -th polynomial in the set is either a non-zero constant or has a class less than c_i which, if it is of class c_j , $1 \leq j < i$, should have a degree less than that of j -th polynomial in the set. Such a polynomial set is then called an *ascending set* (abbr. *asc-set*). Some partial ordering is then introduced among the system of all such asc-sets, with the set consisting of a single non-zero constant considered as a *trivial* asc-set to be arranged in the lowest ordering.

Consider now arbitrary finite systems of non-zero polynomials. For such a polynomial system, any asc-set of lowest ordering contained wholly in the given system is called a *basic set* (abbr. *bas-set*) of the system. A partial ordering is then unambiguously introduced among all non-empty polynomial systems according to the partial ordering of their basic sets. Any polynomial system containing a non-zero constant polynomial will be clearly one of lowest ordering.

After the introduction of partial ordering among all finite polynomial systems let us consider now such a given system PS and consider the scheme (S) shown below:

$$\begin{array}{ccccccc} PS = & PS^0 & PS^1 & \cdots & PS^i & \cdots & PS^m \\ & BS^0 & BS^1 & \cdots & BS^i & \cdots & BS^m & = CS \\ & RS^0 & RS^1 & \cdots & RS^i & \cdots & RS^m & = \emptyset. \end{array} \quad (S)$$

In the scheme (S) each BS^i is a basic set of PS^i , each RS^i is the set of non-zero remainders, if any, of polynomials in $PS^i \setminus BS^i$ with respect to BS^i , and $PS^{i+1} = PS \cup BS^i \cup RS^i$ if RS^i is non-empty. It is easily proved that the sequence of BS^i is a steadily decreasing sequence:

$$BS^0 \succ BS^1 \succ \cdots \succ BS^r \succ \cdots$$

Such a sequence cannot be an infinite one and should terminate at certain stage m with $RS^m = \emptyset$. The corresponding basic set $BS^m = CS$ is then called a *characteristic set* (abbr. *char-set*) of the given polynomial system PS . The *zero-set* of PS , $Zero(PS)$, consisting of all possible complex solutions or *zeros* of the system of polynomial equations $PS = 0$, is closely connected with that of CS by the *Well-Ordering Principle* in the form below:

$$\text{Zero}(PS) = \text{Zero}(CS/IP) \cup \text{Zero}(PS \cup \{IP\}),$$

in which IP is the product of all initials of polynomials in CS and $\text{Zero}(CS/IP) = \text{Zero}(CS) \setminus \text{Zero}(IP)$.

Now $PS \cup \{IP\}$ is easily seen to be a polynomial set of lower ordering than PS . If we apply the Well-Ordering Principle to $PS \cup \{IP\}$ and proceed further and further in the same way we should stop in a finite number of steps and arrived at the following

Zero-Decomposition Theorem. For any finite polynomial system PS there is an algorithm which will give in a finite number of steps a finite set of asc-sets CS^s with initial-product IP^s such that

$$\text{Zero}(PS) = \bigcup_s \text{Zero}(CS^s / IP^s). \quad (Z)$$

Now CS^s are all asc-sets. Hence all zero-sets $\text{Zero}(CS^s)$ and all $\text{Zero}(CS^s / IP^s)$ may be considered as well-determined in some natural sense. The formula (Z) gives thus actually an explicit determination of $\text{Zero}(PS)$ for all finite polynomial systems PS which serves for the solving of arbitrary systems of polynomial equations.

Mathematics should be incessantly faced with the solving of various kinds of problems, both theoretical and practical ones. Such problems are abundant in nature, in sciences, in reconstructions, in engineering, in administrative works, etc., besides those in mathematics herself. As the data given and results to be found are usually connected by some form of equations, so equations-solving becomes naturally one of the main concern of mathematics. As the algebraic polynomial equations and differential equations, ordinary or partial, appear as the usual form of equations which arise most often, so the solving of such kinds of equations become naturally our most urgent task to deal with. In this section we have presented an algorithmic method of solving arbitrary systems of polynomial equations. It is conceivable that there will be various kinds of applications of this general method. This is really the case. Among the applications we may cite in particular the mechanical proving on computers of theorems in various kinds of geometries, those of euclidean geometry in particular. See e.g. the book [Ch] of S.C.Chou. For the methods of our theory as well as their applications we refer to the author's book [W2] and the references there. For the extension to differential case we refer to the next sections.

Remark finally that, instead of theorem-proving, polynomial equations-solving occupies a central position throughout the long history of thousands of years of development of Chinese ancient mathematics. In fact, the above general method of polynomial equations-solving had its origin in some of our ancient classic due to the scholar Zhu Shijie in Yuan Dynasty (1271-1368 A.D.), see [Z]. Of course, there are many defects in Zhu's work. However, the main lines of thought and treatment are sound and the above Well-Ordering Principle is just a modified reformulation of Zhu's work in applying the modern techniques even terminologies of the works of J.F.Ritt, see [Rt1,2]. For more details we refer to various writings of the present author, notably the book [W2].

2. Partial Ordering of Algebrico-Differential Polynomials and Algebrico-Differential Polynomial Systems

Let us consider now the case of algebrico-differential polynomials (abbr. *ad-pol* or simply *d-pol*) and such polynomial sets (abbr. *ad-polset* or simply *d-polset*) with notations $X, x_k, y, u_j, DPS, DP_i$, etc. as in the beginning of §1. A d-pol with no y or its partial derivatives actually occurring in it will be called a *trivial d-pol*. For each tuple of n non-negative integers $\mu = (i_1, \dots, i_n)$ let us write $\|\mu\|$ for $i_1 + \dots + i_n$ and ∂_μ for the partial derivative $\frac{\partial^{\|\mu\|}}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}$. We shall arrange all the partial derivatives $\partial_\mu y$ of y in the usual lexicographical order of μ . For any non-trivial d-pol DP the highest derivative occurring in DP is then called the *leading derivative* or simply the *lead* of DP . If the lead is $\partial_\mu y$, then μ is called the *class* of DP . If the highest degree of $\partial_\mu y$ occurring in DP is d (> 0), then d is called the *degree* of DP , with class and degree undefined for trivial d-pols. For non-trivial d-pol DP with class μ and degree d (> 0), we can write DP in the form

$$DP = I * (\partial_\mu y)^d + \text{lower degree terms in } \partial_\mu y.$$

The coefficient I ($\neq 0$) of $(\partial_\mu y)^d$ in DP , which is itself either a trivial d-pol or non-trivial d-pol in partial derivatives of lower ordering than $\partial_\mu y$, is called the *initial* of DP . The formal partial derivative of DP with respect to $\partial_\mu y$ is called the *separant* of DP . Clearly the separant is the same as the initial when degree d of DP is 1, and of the same class but of lower degree than DP if $d > 1$.

We now introduce a partial ordering among all non-trivial d-pols by first according to their class and then to their degree, with trivial d-pols in the lowest ordering. The following proposition is now clear from the very definitions:

Proposition 1. Any sequence of d-pols steadily decreasing in ordering

$$DP_1 \succ DP_2 \succ \dots \succ DP_r \succ \dots$$

is necessarily finite.

For two non-trivial d-pols DP, DQ we say that DQ is *reduced* with respect to DP if no proper derivative of the lead of DP occurs in DQ and the lead itself is either not occurring in DQ , or occurring in DQ with a degree less than the degree of DP .

For the introduction of partial ordering among arbitrary d-polsets let us consider first that of particular d-polsets called *ad-ascending sets* (abbr. *ad-asc-sets* or simply *d-asc-sets*) defined as follows. A d-polset is called a *d-asc-set* if it is either consisting of a single trivial d-pol or a d-polset for which the d-pols are all non-trivial ones and may be arranged in a sequence of d-pols in increasing ordering such that each one in the sequence is reduced with respect to the preceding ones. In the case of a single trivial d-pol the corresponding d-asc-set is then called a *trivial d-asc-set*.

Consider now two non-trivial d-asc-sets DAS, DBS with d-pols arranged in increasing ordering as follows:

$$DAS : DA_1 \prec \dots \prec DA_r$$

$$DBS : DB_1 \prec \dots \prec DB_s$$

We shall say that DAS is of *higher ordering* than DBS or DBS is of *lower ordering* than DAS if either (a) or (b) below holds true:

(a) There is some k ($\leq \min(r, s)$) such that for each $i < k$, DA_i and DB_i are incomparable in ordering while $DA_k \succ DB_k$ as d-pols.

(b) $r < s$ and DA_i, DB_i are incomparable in ordering as d-pols for all $i \leq r$.

It is easy to see that the above definition introduces really a partial ordering among all d-asc-sets with trivial d-asc-sets considered to be in the lowest ordering. As Prop.1 we have also the proposition below for d-asc-sets:

Proposition 2. Any sequence of d-asc-sets steadily decreasing in ordering

$$DAS_1 \succ DAS_2 \succ \cdots \succ DAS_r \succ \cdots$$

is necessarily finite.

Consider now an arbitrary d-polset DP . Any d-asc-set wholly contained in DP will be called an *ad-basic-set* (abbr. *ad-baset* or simply *d-baset*) of DP . We shall introduce now partial ordering among all d-polsets according to the partial ordering of their d-basets. It is easily seen that this is unambiguously well-defined independent of d-basets chosen from the d-polsets.

The above completes the introduction of partial ordering among d-pols and d-polsets. After this preparation we shall show how to solve arbitrary systems of algebrico-differential equations in next section. We remark however that the partial ordering in the present section is only one of many possible ways which will meet our purposes.

3. (Differential) Characteristic-Set Formation and Solving of Arbitrary Algebrico-Differential Polynomial Equations

To extend the notion of char-set and the method of solving polynomial equations in the ordinary case to differential case we need two fundamental procedures of remainder formation and integrability-condition formation to be described in what follows.

For this purpose let us consider a non-trivial d-asc-set DAS as given below:

$$DAS : DA_1 \prec DA_2 \prec \cdots \prec DA_r. \quad (dA)$$

With respect to DAS we have then the following theorem due to J.F.Ritt ([Rt1,2]) which is fundamental for the whole theory:

Ritt's Remainder Theorem. For any non-trivial d-pol DP there are for each $a \in \{1, 2, \dots, r\}$ integers s_a, t_a and certain partial derivatives $\partial_{\tau_{ab}}$ and d-pols C_{ab} for $b \in \{1, 2, \dots, b_a\}$, such that

$$DR = S_1^{s_1} * \cdots * S_r^{s_r} * I_1^{t_1} * \cdots * I_r^{t_r} * DP - \sum_{a,b} C_{ab} * \partial_{\tau_{ab}} DA_a \quad (dR)$$

is reduced with respect to DAS , i.e. reduced with respect to each d-pol in DAS .

The above formula (dR) will be called the *d-Remainder Formula* of DP and the procedure of getting DR from DP the *reduction* of DP with respect to DAS .

Consider now a pair of nontrivial d-pols DP, DQ with classes $\mu = (i_1, \dots, i_n), \nu = (j_1, \dots, j_n)$ respectively. Suppose that neither $i_k \geq j_k$ nor $j_k \geq i_k$ for all $k \in \{1, \dots, n\}$. Such a pair will be called a *legal pair* with respect to DAS . Now for each k let $m_k = \max(i_k, j_k), p_k = m_k - i_k, q_k = m_k - j_k$. Set $\xi = (p_1, \dots, p_n), \eta = (q_1, \dots, q_n)$ and $\zeta = (m_1, \dots, m_n)$. Then $\partial_\xi DP$ and $\partial_\eta DQ$ have the same lead $\partial_\zeta y$ with degree 1. Let the initials of $\partial_\xi DP$ and $\partial_\eta DQ$ be I_ξ and I_η respectively. Then the difference $I_\eta * \partial_\xi DP - I_\xi * \partial_\eta DQ$, after eventually reduction with respect to DAS , will be called the eventually *reduced integrability d-pol* of the legal pair DP and DQ with respect to the d-asc-set DAS .

With the above notions of d-remainder and integrability d-pol, eventually reduced or not, with respect to a non-trivial d-asc-set we can now extend our procedures in ordinary polynomial case as given in scheme (S) of §1 to the differential case as shown in the scheme (dS) below:

$$\begin{array}{llllllll}
 DPS = & DPS^0 & & DPS^1 & & \dots & & DPS^i & & \dots & & DPS^m \\
 & DBS^0 & & DBS^1 & & \dots & & DBS^i & & \dots & & DBS^m & = & DCS \\
 & DRIS^0 & & DRIS^1 & & \dots & & DRIS^i & & \dots & & DRIS^m & = & \emptyset \\
 & DCPS^0 & \cup & DCPS^1 & \cup & \dots & \cup & DCPS^i & \cup & \dots & \cup & DCPS^m & = & DCPS
 \end{array} \quad (dS)$$

In the scheme (dS) DPS is the given d-polset. For each i , DBS^i is a d-baset of DPS^i , and the set $DRIS^i$ is the union of two parts. One is the set of all possible non-zero d-remainders formed from d-pols in $DPS^i \setminus DBS^i$ with respect to DBS^i , while the other is the set of integrability d-pols formed from all possible legal-pairs of d-pols in DPS^i , eventually reduced with respect to DBS^i , so far they contain actually y or its derivatives. On the other hand those containing no y or its derivatives but containing possibly u_j or their derivatives will form a set of *compatibility d-pols* for which the vanishing will form *compatibility conditions* in order that the given set of equations $DPS = 0$ will have solutions. In case $DRIS^i$ is non-empty, then the union $DPS \cup DBS^i \cup DRIS^i$ will form the next d-polset DPS^{i+1} .

As is easily verified, the d-basets DBS^i will form a sequence of steadily decreasing ordering:

$$DBS^0 \succ DBS^1 \succ \dots \succ DBS^s \succ \dots$$

By Prop.2 this sequence can only be a finite one, to be stopped at a certain stage m with $DRIS^m = \emptyset$. The corresponding d-baset $DBS^m = DCS$ is then called an *ad-* or simply *d-characteristic set* (abbr. *d-charset*) of the given d-polset DPS . The union $DCPS$ of all sets $DCPS^i, i = 1, \dots, m$, will form the totality of all possible compatibility d-pols whose vanishing form the compatibility conditions to guarantee the existence of solutions of the partial differential equations $DPS = 0$.

For any d-polset DPS and d-pol DG let $dZero(DPS)$ be the set of all possible solutions of the partial differential equations $DPS = 0$ and $dZero(DPS/DG) = dZero(DPS) \setminus dZero(DG)$. Then we have from scheme (dS) the following (*differential*) *Well-Ordering Principle*:

$$dZero(DPS) = dZero(DCS / DISP) \cup dZero(DPS \cup \{DISP\}), \quad (dW)$$

in which $DISP$ is the product of all initials and separants of d-pols in DCS , so far the compatibility conditions $DCPS = 0$ are assumed to be verified.

As in the case of ordinary polynomial equations-solving, we deduce by successive applications of the above (differential) Well-Ordering Principle the following theorem, which is at the basis of solving arbitrary systems of algebrico-differential polynomial equations:

(Differential) Zero-Decomposition Theorem. For any finite d-polset DPS there is an algorithm which will give in a finite number of steps a finite set of d-asc-sets DCS^s with initial-separant-products $DISP^s$ as well as sets of compatibility d-pols $DCPS^s$ such that

$$dZero(DPS) = \bigcup_s dZero(DCS^s / DISP^s), \quad (dZ)$$

so far some compatibility conditions are supposed to be verified.

In order to give formal explicit solutions of the partial differential equations $DPS = 0$ for d-polset DPS let us consider first the case of a non-trivial d-asc-set DAS as given by (dA). Let the classes of DA_a in (dA) be $\mu_a, a = 1, \dots, r$. Then all partial derivatives of the leads $\partial_{\mu_a} y$, proper or improper, will be called *principal derivatives*, and all the others *parametric* ones.

Consider now any set of constants $c_k \in \mathbf{C}, k = 1, \dots, n$, and also constants $c_\tau \in \mathbf{C}$ for each parametric derivative $\partial_\tau y$. The values $x_k = c_k$ will give definite values to the known functions $u_j, j \in J$, as well as their derivatives. We suppose that the above values will not render zero the initial-separant product ISP of DAS . The set of these constant values c_k, c_τ will then be called an *admissible preliminary constant set* with respect to the d-asc-set DAS .

With values of such admissible preliminary constants substituted in the equations $DA_a = 0, a = 1, \dots, r$, we can solve for them to get values for the proper principal derivative c_{μ_a} for classes μ_a . Let us take any set of such values for each μ_a . By differentiating DA_a and substituting the preliminary values as well as the chosen values of c_{μ_a} , we get also definite constant values c_ν for arbitrary improper principle derivatives $\partial_\nu y$. With these constant values we form now a *Formal Taylor Series FTS* of the form below:

$$FTS = \sum_\tau \frac{c_\tau}{\|\tau\|} * \prod_k (x_k - c_k)^{t_k},$$

in which τ runs over all integer-tuples $\tau = (t_1, \dots, t_n)$ with $\|\tau\| = t_1 + \dots + t_n$.

It is easy to verify the following

Formal Taylor Series Theorem. With a given admissible preliminary value set all possible solutions of the partial differential equations $DAS = 0$ for which the initial-separant product of DAS is non-zero are given by formal Taylor series of above form FTS .

Consider now an arbitrary non-trivial d-polset DPS with a non-trivial d-charset DCS as given in the scheme (dS). Then for any admissible preliminary value set for the d-asc-set DCS , for which the corresponding initial-separant product ISP of DCS is not zero, we will get totality of solutions of the partial differential equations $DPS = 0$, so far all compatibility d-pols in $DCPS$ are rendered zero. We may treat in the same way the d-charsets DCS^s in the (differential) Zero-Decomposition Theorem (dZ) of DPS and get the totality of solutions of $DPS = 0$ in the form of Formal Taylor Series, so far the preliminary value set will not render zero the corresponding initial-separant product ISP^s , while render zero all the d-pols in the corresponding compatibility d-polset.

Remark that the above is only a special case of our general theory and method which we refer to the previously cited paper [W1]. In fact, in the general theory there may be several unknown functions y_1, y_2 , etc. instead of a single one y . Moreover, the coefficient field may be an arbitrary differential field with differential operators obeying the usual differential rules. Furthermore, instead of pairwise determination of integrability d-pols, we may apply the device of multiplicativity of variables originated by Riquier and Janet to make smaller the number of pairs of forming the integrability d-pols. However, as the essence of these methods are actually the same as in the above particular case so we shall not enter into them.

Besides, the relations between the notions of d-charset, d-asc-set, etc. in our procedure and those of passiveness, prolongations, involutiveness, etc. in the procedures of Riquier, Janet, and Cartan require some clarification which we shall do in later occasions. Instead we shall show as an illustration how to solve a particular example by our procedure which will be described in details in the next section.

It is clear that our general method of algebrico-differential equations-solving will have an immense variety of applications as in the case of ordinary polynomial equations-solving. In particular, as in the ordinary case, we have applied our general method of differential equations-solving to the mechanical proving of differential geometry theorems, and to the automated determination of explicit form of relations for which only the existence of the relations is known. We refer these to the author's papers [W3-5] and the paper [C-G] of S.C.Chou and X.S.Gao as well as the references in these papers. On the other hand, we shall leave the studies of various other applications to later occasions.

4. An Example: Pommaret's Devil Problem

For the sake of illustration of his formal intrinsic method of treatment of algebraic partial differential equations, Pommaret had exhibited in details an example, what he called the *Devil's Problem*, in his paper [P1], his treatise [P3], as well as in various courses or lectures taken place in France, in Germany, in Beijing, and elsewhere. We shall treat this Devil's Problem also in details by our own method as exhibited in §3. For this purpose let us reproduce the original statement of the Devil's Problem as well as its final solution from Pommaret's writings as given below:

Devil's Problem. Let u, v, y be 3 functions of the cartesian coordinates x^1, x^2, x^3 on euclidean spaces related by the following system of 2 PDE where $\partial_{33}y = \frac{\partial^2 y}{\partial_{x^3} \partial_{x^3}}, \dots$:

$$\begin{aligned} \partial_{33}y - x^2 \partial_{11}y - u &= 0, \\ \partial_{22}y - v &= 0. \end{aligned} \tag{D}$$

(1) If $u = v = 0$ the space of solutions of the resulting linear system of PDE for y is a vector space over the constants. What is its dimension?

(2) Otherwise, what kind of compatibility conditions must be satisfied by u and v in order to insure the existence of solutions for y ?

(3) Does there exist a "general" way to solve such problems?

The final result of Pommaret on this problem may be described as follows:

The final general solution is furnished by a "Good" Set consisting of 4 "good" algebraic differential polynomials $G_1, \dots, G_4$ below:

$$\begin{aligned} G_1 &= \partial_{1111}y - z, \\ G_2 &= \partial_{112}y - w, \\ G_3 &= \partial_{22}y - v, \\ G_4 &= \partial_{33}y - x^2\partial_{11}y - u. \end{aligned} \tag{G}$$

In (G) w and z are given by

$$\begin{aligned} w &= \frac{1}{2}(\partial_{33}v - x^2\partial_{11}v - \partial_{22}u), \\ z &= \partial_{33}w - \partial_{112}u - x^2\partial_{11}w. \end{aligned}$$

Pommaret solved the above question (2) in deriving two *Compatibility Conditions* $A = 0, B = 0$ with A, B given below:

$$\begin{aligned} \frac{1}{2}A &= \partial_2w - \partial_{11}v, \\ B &= \partial_{3333}w - 2x^2\partial_{1133}w + (x^2)^2\partial_{1111}w - \partial_{11233}u + x^2\partial_{11112}u - \partial_{1111}u. \end{aligned} \tag{CC}$$

Pommaret showed further that the two compatibility conditions $A = 0, B = 0$ are not independent of each other. They are in fact connected by the following differential identity

$$\partial_{3333}A - 2x^2\partial_{1133}A + (x^2)^2\partial_{1111}A - 2\partial_2B = 0.$$

In the case of $u = 0, v = 0$ so that $w = 0, z = 0$ too the "good" sets become simply $\partial_{1111}y = \partial_{112}y = \partial_{22}y = \partial_{33}y - x^2\partial_{11}y = 0$. It follows that there remain only 12 derivatives below which can take arbitrary values, viz.:

$$y, \partial_1y, \partial_{11}y, \partial_{111}y, \partial_2y, \partial_{12}y, \partial_3y, \partial_{13}y, \partial_{113}y, \partial_{1113}y, \partial_{23}y, \partial_{123}y.$$

As a consequence Pommaret solves the question (1) in giving a vector space of dimension 12 spanned by the above vectors.

Let us now apply our own method to the solving of the Devil's Problem. For this purpose let us first change the notations in order to make accordance with those in §3. Thus, instead of x^1, x^2, x^3 we shall write x_1, x_2, x_3 and instead of $\partial_{33}y$ for $\frac{\partial^2 y}{\partial x_3^2 \partial x_3}$ we shall write $\partial_{200}y$, etc., as in §3. In this way the system of 2 PDE (D) of the Devil's Problem will be $DPS = 0$ with $DPS = \{DP_1, DP_2\}$ in which

$$\begin{aligned} DP_1 &= \partial_{200}y - x_2 * \partial_{002}y - u, \\ DP_2 &= \partial_{020}y - v. \end{aligned}$$

As in the scheme (dS) of §3, we have then $DPS^0 = DPS = \{DP_1^0, DP_2^0\}$ with

$$DP_1^0 = DP_1, \quad DP_2^0 = DP_2.$$

The d-baset chosen from DPS^0 is then $DBS^0 = \{DB_1^0, DB_2^0\}$ with

$$DB_1^0 = DP_2^0, \quad DB_2^0 = DP_1^0$$

in increasing ordering. There are clearly no d-remainders but there is one integrability d-pol eventually reduced to be determined from the legal pair (DP_1^0, DP_2^0) . To determine it let us first form

$$\begin{aligned} & \partial_{200}DP_2^0 - \partial_{020}DP_1^0 \\ = & x_2 * \partial_{022}y + 2 * \partial_{012}y - \partial_{200}v + \partial_{020}u \\ = & \Delta, \text{ say.} \end{aligned}$$

We reduce now Δ with respect to the d-baset DBS^0 to get the reduced integrability d-pol

$$\begin{aligned} DI_{12}^0 &= \Delta - x_2 * \partial_{002}DB_1^0 \\ &= 2 * \partial_{012}y - 2 * w, \end{aligned}$$

in which

$$w = \frac{1}{2}(\partial_{200}v - x_2 * \partial_{002}v - \partial_{020}u).$$

We see that $\frac{1}{2}DI_{12}^0$ is actually the same as the "good" set G_2 and w is actually the same as the w given by Pommaret but in different way of notations.

As the integrability d-pol DI_{12}^0 contains actually derivatives of y it will form the set $DRIS^0$ with the corresponding compatibility d-polset $DCPS^0$ empty. Hence, in adjoining DI_{12}^0 or simpler $\frac{1}{2} * DI_{12}^0$ to DPS^0 we get the new d-polset $DPS^1 = \{DP_1^1, DP_2^1, DP_3^1\}$ with

$$\begin{aligned} DP_1^1 &= DP_1^0 &= \partial_{200}y - x_2 * \partial_{002}y - u, \\ DP_2^1 &= DP_2^0 &= \partial_{020}y - v, \\ DP_3^1 &= \frac{1}{2} * DI_{12}^0 &= \partial_{012}y - w. \end{aligned}$$

The d-baset chosen from DPS^1 is then $DBS^1 = \{DB_1^1, DB_2^1, DB_3^1\}$ with

$$DB_1^1 = DP_3^1, \quad DB_2^1 = DP_2^1, \quad DB_3^1 = DP_1^1$$

in increasing ordering. Again there are no d-remainders but there are 3 legal pairs (DP_1^1, DP_2^1) , (DP_1^1, DP_3^1) , (DP_2^1, DP_3^1) . The first pair has already been considered so there remain only two new ones to be considered which give two integrability d-pols eventually reduced below:

$$\begin{aligned} DI_{13}^1 &= \partial_{200}DP_3^1 - \partial_{012}DP_1^1 - x_2 * \partial_{012}DB_3^1 \\ &= \partial_{004}y - z, \\ DCP_{23}^1 &= \partial_{010}DP_3^1 - \partial_{002}DP_2^1 \\ &= -\partial_{010}w + \partial_{002}v, \end{aligned}$$

in which z is given by

$$z = \partial_{200}w - x_2 * \partial_{002}w - \partial_{012}u.$$

It is readily seen that DI_{13}^1 is the same as the "good" set " G_1 " and z is same as that given before, both by Pommaret in different notations. We see that DI_{23}^1 , containig actually

derivatives of y , is to be a d-pol in $DRIS^1$, while DCP_{23}^1 , containing only derivatives in u, v , but not y , is to form the compatibility d-polset $DCPS^1$. Furthermore the compatibility d-pol DCP_{23}^1 is readily seen to be equal to $-\frac{1}{2}A$, where A is the compatibility d-pol also already given by Pommaret in different notations.

Adjoin now DI_{13}^1 to DPS^1 to get $DPS^2 = \{DP_1^2, DP_2^2, DP_3^2, DP_4^2\}$ with

$$DP_1^2 = DP_1^1, DP_2^2 = DP_2^1, DP_3^2 = DP_3^1, DP_4^2 = DI_{13}^1.$$

The d-baset is then $DBS^2 = \{DB_1^2, DB_2^2, DB_3^2, DB_4^2\}$ with

$$\begin{aligned} DB_1^2 &= DP_4^2 = DI_{13}^1, \\ DB_2^2 &= DP_3^2, \\ DB_3^2 &= DP_2^2, \\ DB_4^2 &= DP_1^2 \end{aligned}$$

in increasing ordering. Again there is no d-remainders but there are 3 new legal pairs to be considered: (DP_4^2, DP_1^2) , (DP_4^2, DP_2^2) , (DP_4^2, DP_3^2) which give 3 compatibility d-pols $DCP_{14}^2, DCP_{24}^2, DCP_{34}^2$ as given below:

$$\begin{aligned} DCP_{14}^2 &= \partial_{200}DP_4^2 - \partial_{004}DP_1^2 - x_2 * \partial_{002}DB_1^2 \\ &= -\partial_{200}z + x_2 * \partial_{002}z + \partial_{004}u, \\ DCP_{24}^2 &= \partial_{020}DP_4^2 - \partial_{004}DP_2^2 \\ &= -\partial_{020}z + \partial_{004}v, \\ DCP_{34}^2 &= \partial_{010}DP_4^2 - \partial_{002}DP_3^2 \\ &= -\partial_{010}z + \partial_{002}w. \end{aligned}$$

As $DRIS^2 = \emptyset$, the procedure ends at the stage $m = 2$ so that the scheme (dS) in the present case becomes (dS') below:

$$\begin{array}{llll} DPS = & DPS^0 & DPS^1 & DPS^2 \\ & DBS^0 & DBS^1 & DBS^2 = DCS \\ & DRIS^0 & DRIS^1 & DRIS^2 = \emptyset \\ & DCPS^0 \cup & DCPS^1 \cup & DCPS^2 = DCPS \end{array} \quad (dS')$$

Remark that DCS here is the same as the "good set" $\{G1, G2, G3, G4\}$ given by Pommaret, while the compatibility d-polset here is given by

$$\begin{aligned} DCPS &= DCPS^0 \cup DCPS^1 \cup DCPS^2, \text{ with} \\ DCPS^0 &= \emptyset, \\ DCPS^1 &= \{DCP_{23}^1\}, \\ DCPS^2 &= \{DCP_{14}^2, DCP_{24}^2, DCP_{34}^2\}. \end{aligned}$$

The compatibility d-pols in $DCPS$ are not independent of each other. To determine their interrelations let us first turn these d-pols to be in v and u by means of the expressions of w and z in v and u . Let us consider now these DCP 's as d-pols in v as unknown function while u as known function in $X = \{x_1, x_2, x_3\}$. Then the leads in v of the d-pols $DCP_{23}^1, DCP_{14}^2, DCP_{24}^2, DCP_{34}^2$ are seen to be respectively $\partial_{210}v, \partial_{600}v, \partial_{420}v, \partial_{410}v$. Let us treat now $DCPS$ as a d-polset in v as we have treated DPS as a d-polset in y by means of our

general method. The d-baset is then seen to be consisting of the two d-pols DCP_{23}^1, DCP_{14}^2 with leads $\partial_{210}v, \partial_{600}v$. It is readily found that the d-remainders of DCP_{24}^2, DCP_{34}^2 with respect to the d-baset are both zero with corresponding d-remainder formula given below:

$$\begin{aligned} DCP_{24}^2 &= \partial_{210}DCP_{23}^1 - x_2 * \partial_{012}DCP_{23}^1, \\ DCP_{34}^2 &= \partial_{200}DCP_{23}^1 - x_2 * \partial_{002}DCP_{23}^1. \end{aligned}$$

On the other hand there is only one legal pair (DCP_{23}^1, DCP_{14}^2) for which the integrability d-pol after reduction with respect to the d-baset is found to be zero with the corresponding interrelation given below:

$$\partial_{010}DCP_{14}^2 - \partial_{400}DCP_{23}^1 + 2 * x_2 * \partial_{202}DCP_{23}^1 - x_2^2 * \partial_{004}DCP_{23}^1 = 0.$$

It follows that the procedure ends at the stage $m = 1$ with the d-charset consisting of the two d-pols DCP_{23}^1, DCP_{14}^2 . It is also easily verified that these two d-pols are respectively equal to $-A/2$ and $-B$ of Pommaret, and the interrelations between DCP_{23}^1, DCP_{14}^2 are the same as that between the two compatibility d-pols A, B given by Pommaret, only in different notations.

We see that Pommaret's results are *complete* for the Devil Problem and our results too, though by different ways of treatments. Remark that Pommaret uses a method with quite involved logical reasonings and modern techniques in applying exact sequences, diagram-chasing, etc.. On the other hand our method is highly computational with little mental efforts, and the computations are almost straightforward.

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