Modified nonlinearly dispersive $mK(m, n, k)$ equations: II. Jacobi elliptic function solutions

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Abstract. Recently we have obtained *compacton solutions* and *solitary pattern solutions* of the modified nonlinearly dispersive KdV equations (simply called $mK(m, n, k)$ equations). In this paper the $mK(m, n, k)$ equations are investigated again. By using some transformations we give their some *Jacobi elliptic function solutions*. When the modulus $\mu \rightarrow 1$ or 0 , some of the obtained Jacobi elliptic function solutions degenerate as solitary wave solution.

Keywords: $mK(m, n, k)$ equation; elliptic function; solitary wave solution; periodic wave solution

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1. Introduction

To understand the role of nonlinear dispersion, many nonlinear dispersive equations were presented such as the $K(m, n)$ equation[1-5], the $B(m, n)$ equation[6], the $R(m, n)$ equation[7], the $E(m, n)$ equation[8] and the $ES(m, n)$ equation[9] etc. It was shown that these equations possessed two types of special exact solutions: the *compacton solution*, which is solitary waves with the property that after colliding with other compacton solution, they reemerge with the same coherent shape, and the *solitary pattern solution* having infinite slopes or cusps. Recently Wazwaz[10] presented several generalized forms of the famous $K(m, m)$ equation, that is, $mK(n, n)$ equations in one-dimensional spatial domain

$$u^{n-1}u_t + a(u^n)_x + b(u^n)_{xxx} = 0, \quad (1)$$

where a and b are constants. Some compacton solutions and solitary pattern solutions of (1) were given. More recently, we[11] further extended (1) to be a more general form

$$u^{m-1}u_t + a(u^n)_x + b(u^k)_{xxx} = 0, \quad nk \neq 0, \quad (2)$$

where a and b are constants, which is called the $mK(m, n, k)$ equation. (1) When $m = n = k$, (2) reduces to (1); (2) When $m = 1, k \rightarrow n, n \rightarrow m$, (2) reduces to the $K(m, n)$ equation[1]

$$u_t + a(u^m)_x + b(u^n)_{xxx} = 0, \quad mn \neq 0. \quad (3)$$

We have presented some compacton solutions, solitary pattern solutions, solitary wave solutions and periodic solutions of (2) with differential parameters m, n, k [11]. To my knowledge,

the elliptic function solutions of (2) were not considered before. In this paper we would like to consider Jacobi elliptic function solutions of (2).

It is easy to know that the Jacobi elliptic functions are of the properties[12]: when the modulus $\mu \rightarrow 1$, $\text{sn}\xi$, $\text{cn}\xi$ and $\text{dn}\xi$ degenerate as $\tanh \xi$, $\text{sech}\xi$ and $\text{sech}\xi$, respectively; while when the modulus $\mu \rightarrow 0$, $\text{sn}\xi$, $\text{cn}\xi$ and $\text{dn}\xi$ degenerate as $\sin \xi$, $\cos \xi$ and 1, respectively. Therefore if we can show that one differential equation possesses the Jacobi elliptic function solutions, then it may also possesses solitary wave solutions and/or singly-periodic solutions. Thus it is interesting to consider elliptic function solutions. Recently we presented the extended Jacobi elliptic function expansion method[13-17], which is a powerful method to seek more Jacobi elliptic function solutions of nonlinear evolution equations. In this paper, motivated by our method, we would like to use some transformations to seek Jacobi elliptic function solutions of (2). As a consequence, some Jacobi elliptic functions solutions of (2) are obtained.

2. General formulae of Jacobi elliptic function solutions of (2)

For given equatin (2), we mainly consider its travelling wave solutions in the form

$$u(x, t) = u(\xi), \quad \xi = x - \lambda t. \quad (4)$$

Where λ is a constant. And then (2) reduces to a nonlinear ordinary differential equation

$$-\lambda u^{m-1} \frac{du}{d\xi} + a \frac{du^n}{d\xi} + b \frac{d^3 u^k}{d\xi^3} = 0. \quad (5)$$

In the following we will consider (5). Firstly we introduce the known results which are useful to seek solutions of (5).

By using the extend Jacobi elliptic function expansion method[15-17], we had shown that the reduction equation to mKdV equation

$$\Psi''(R\xi) = \alpha \Psi(R\xi) + \beta \Psi^3(R\xi), \quad (6)$$

where α, β, R are constants, possessed these types of Jacobi elliptic function solutions

$$\Psi_1 = a_0 + a_1 \text{sn}(R\xi) + b_1 \text{cn}(R\xi), \quad \Psi_2 = a_0 + a_1 \text{sn}(R\xi) + b_1 \text{dn}(R\xi), \quad (7)$$

$$\Psi_3 = a_0 + a_1 \text{ns}(R\xi) + b_1 \text{cs}(R\xi), \quad \Psi_4 = a_0 + a_1 \text{ns}(R\xi) + b_1 \text{ds}(R\xi), \quad (8)$$

$$\Psi_5 = a_0 + a_1 \text{sc}(R\xi) + b_1 \text{nc}(R\xi), \quad \Psi_6 = a_0 + a_1 \text{sd}(R\xi) + b_1 \text{nd}(R\xi), \quad (9)$$

$$\Psi_7 = a_0 + a_1 \text{cd}(R\xi) + b_1 \text{nd}(R\xi), \quad \Psi_8 = a_0 + a_1 \text{cn}(R\xi) + b_1 \text{dn}(R\xi), \quad (10)$$

$$\Psi_9 = a_0 + a_1 \text{dc}(R\xi) + b_1 \text{nc}(R\xi), \quad \Psi_{10} = a_0 + a_1 \text{sd}(R\xi) + b_1 \text{cd}(R\xi), \quad (11)$$

$$\Psi_{11} = a_0 + a_1 \text{sc}(R\xi) + b_1 \text{dc}(R\xi), \quad \Psi_{12} = a_0 + a_1 \text{ds}(R\xi) + b_1 \text{cs}(R\xi). \quad (12)$$

It is easy to see that substituting these formal solutions into (6) and equating the coefficients of the corresponding Jacobi elliptic functions can generate a system of algebraic equations of $a_0, a_1, b_1, \alpha, \beta$. Solving the set of equations can determine these parameters $a_0, a_1, b_1, \alpha, \beta$.

Hence we can obtain many Jacobi elliptic function solutions (A.1)-(A.12) of (6), which are listed in the **Appendix A**.

If we can establish a relation between (5) and (6) (for example $u = f(\Psi(R\xi))$, $f(\cdot)$ is an unknown function), then we will obtain the corresponding solutions of (5) by these solutions (A.1)-(A.12) of (6).

To seek Jacobi elliptic function solutions of (5), we assume that there exists the relation

$$u(\xi) = f(\Psi(R\xi)) = A\Psi^B(R\xi), \quad (13)$$

between (5) and (6), where A, B, R are parameters and not equal to zero.

In order to use (6) we need integrate it once to obtain

$$\Psi'^2(\xi) = \alpha\Psi^2(R\xi) + \frac{1}{2}\beta\Psi^4(R\xi) + \gamma, \quad (14)$$

where γ is the constant of integration and $\gamma \neq 0$.

To determine the parameters A, B, R , we need to substitute (13) into (5) along with (6) and (14) to get

$$\begin{aligned} & -\lambda BA^m \Psi^{mB-1}(R\xi) \Psi'(R\xi) + aBnA^n \Psi^{nB-1}(R\xi) \Psi'(R\xi) + bR^2 A^k B^3 k^3 \alpha \Psi^{kB}(R\xi) \Psi'(R\xi) \\ & + \frac{1}{2} bR^2 A^k \beta Bk(Bk+1)(Bk+2) \Psi^{(kB+1)}(R\xi) \Psi'(R\xi) \\ & + \gamma bR^2 A^k Bk(Bk-1)(Bk-2) \Psi^{(kB-3)}(R\xi) \Psi'(R\xi) = 0. \end{aligned} \quad (15a)$$

Because we require that $\Psi(R\xi)$ is not a constant, thus $\Psi'(R\xi) \neq 0$. (15a) further reduces to

$$\begin{aligned} & -\lambda BA^m \Psi^{mB-1}(R\xi) + aBnA^n \Psi^{nB-1}(R\xi) + bR^2 A^k B^3 k^3 \alpha \Psi^{kB}(R\xi) \\ & + \frac{1}{2} bR^2 A^k \beta Bk(Bk+1)(Bk+2) \Psi^{(kB+1)}(R\xi) \\ & + \gamma bR^2 A^k Bk(Bk-1)(Bk-2) \Psi^{(kB-3)}(R\xi) = 0. \end{aligned} \quad (15b)$$

From (15b) we deduce the following four possible systems of algebraic equations:

System 1:

$$\begin{aligned} & kB - 1 = 0, \\ & mB - 1 = kB - 1, \\ & nB - 1 = kB + 1, \\ & -\lambda BA^m + bR^2 A^k B^3 k^3 \alpha = 0, \\ & aBnA^n + \frac{1}{2} bR^2 A^k \beta Bk(Bk+1)(Bk+2) = 0. \end{aligned} \quad (16)$$

System 2:

$$\begin{aligned} & kB - 1 = 0, \\ & nB - 1 = kB - 1, \\ & mB - 1 = kB + 1, \\ & aBnA^n + bR^2 A^k B^3 k^3 \alpha = 0, \\ & -\lambda BA^m + \frac{1}{2} bR^2 A^k \beta Bk(Bk+1)(Bk+2) = 0. \end{aligned} \quad (17)$$

System 3:

$$\begin{aligned}
kB - 2 &= 0, \\
mB - 1 &= kB - 1, \\
nB - 1 &= kB + 1, \\
-\lambda BA^m + bR^2 A^k B^3 k^3 \alpha &= 0, \\
aBnA^n + \frac{1}{2}bR^2 A^k \beta Bk(Bk + 1)(Bk + 2) &= 0.
\end{aligned} \tag{18}$$

System 4:

$$\begin{aligned}
kB - 2 &= 0, \\
nB - 1 &= kB - 1, \\
mB - 1 &= kB + 1, \\
aBnA^n + bR^2 A^k B^3 k^3 \alpha &= 0, \\
-\lambda BA^m + \frac{1}{2}bR^2 A^k \beta Bk(Bk + 1)(Bk + 2) &= 0.
\end{aligned} \tag{19}$$

In the following we use the μ ($0 < \mu < 1$) to denote the modulus of Jacobi elliptic functions[12]. There exist four cases to be discussed respectively:

2.1 Jacobi elliptic function solutions of $mK(m, 3m, m)$ equation

From the system (16) we get

$$m = k, \quad n = 3k = 3m, \quad B = \frac{1}{m}, \quad R^2 = \frac{\lambda}{bm\alpha}, \quad A^{2m} = -\frac{\lambda\beta}{am\alpha}.$$

Therefore we establish the relation between the $mK(m, 3m, m)$ equation (5) and (6)

$$u(\xi) = A\Psi^B(R\xi) = \left\{ \sqrt{-\frac{\lambda\beta}{am\alpha}} \Psi\left(\sqrt{\frac{\lambda}{bm\alpha}}\xi\right) \right\}^{\frac{1}{m}}. \tag{20a}$$

If we set

$$w = u^m = \sqrt{-\frac{\lambda\beta}{am\alpha}} \Psi\left(\sqrt{\frac{\lambda}{bm\alpha}}\xi\right), \tag{20b}$$

then (5) with $n = 3m = 3k$ reduces to

$$-\frac{\lambda}{m} \frac{dw}{d\xi} + a \frac{dw^3}{d\xi} + b \frac{d^3w}{d\xi^3} = 0. \tag{20c}$$

Remark 1: In fact we also establish a relation (20b) between (20c) and (6).

We know that (6) possesses the solutions (A.1)-(A.12)(see Appendix A). Therefore according to (20a) and the known Jacobi elliptic function solutions (A.1)-(A.12) of (6), we get the solutions of $mK(m, 3m, m)$ equations

$$u_1 = \left\{ \sqrt{\frac{2\lambda\mu^2}{am(1+\mu^2)}} \operatorname{sn}\left(\sqrt{-\frac{\lambda}{bm(1+\mu^2)}}\xi\right) \right\}^{\frac{1}{m}}. \tag{21.1}$$

$$u_2 = \left\{ \sqrt{\frac{2\lambda\mu^2}{am(2\mu^2-1)}} \operatorname{cn}\left(\sqrt{\frac{\lambda}{bm(2\mu^2-1)}}\xi\right) \right\}^{\frac{1}{m}}. \quad (21.2)$$

$$u_3 = \left\{ \sqrt{\frac{\lambda\mu^2}{am(2-\mu^2)}} \left[\operatorname{sn}\left(\sqrt{\frac{2\lambda}{bm(\mu^2-2)}}\xi\right) \pm i \operatorname{cn}\left(\sqrt{\frac{2\lambda}{bm(\mu^2-2)}}\xi\right) \right] \right\}^{\frac{1}{m}}, \quad i^2 = -1. \quad (21.3)$$

$$u_4 = \left\{ \sqrt{\frac{2\lambda}{am(2-\mu^2)}} \operatorname{dn}\left(\sqrt{\frac{\lambda}{bm(2-\mu^2)}}\xi\right) \right\}^{\frac{1}{m}}, \quad (21.4)$$

$$u_5 = \left\{ \sqrt{\frac{\lambda}{am(2\mu^2-1)}} \left[\mu \operatorname{sn}\left(\sqrt{\frac{2\lambda}{bm(1-2\mu^2)}}\xi\right) \pm i \operatorname{dn}\left(\sqrt{\frac{2\lambda}{bm(1-2\mu^2)}}\xi\right) \right] \right\}^{\frac{1}{m}}, \quad i^2 = -1. \quad (21.5)$$

$$u_6 = \left\{ \sqrt{\frac{2\lambda}{am(1+\mu^2)}} \operatorname{ns}\left(\sqrt{-\frac{\lambda}{bm(1+\mu^2)}}\xi\right) \right\}^{\frac{1}{m}}. \quad (21.6)$$

$$u_7 = \left\{ \sqrt{\frac{2\lambda}{am(\mu^2-2)}} \operatorname{cs}\left(\sqrt{-\frac{\lambda}{bm(\mu^2-2)}}\xi\right) \right\}^{\frac{1}{m}}. \quad (21.7)$$

$$u_8 = \left\{ \sqrt{\frac{\lambda}{am(2\mu^2-1)}} \left[\operatorname{ns}\left(\sqrt{\frac{2\lambda}{bm(1-2\mu^2)}}\xi\right) \pm \operatorname{cs}\left(\sqrt{\frac{2\lambda}{bm(1-2\mu^2)}}\xi\right) \right] \right\}^{\frac{1}{m}}. \quad (21.8)$$

$$u_9 = \left\{ \sqrt{\frac{2\lambda}{am(1-2\mu^2)}} \operatorname{ds}\left(\sqrt{-\frac{\lambda}{bm(1-2\mu^2)}}\xi\right) \right\}^{\frac{1}{m}}. \quad (21.9)$$

$$u_{10} = \left\{ \sqrt{\frac{\lambda}{am(2-\mu^2)}} \left[\operatorname{ns}\left(\sqrt{\frac{2\lambda}{bm(\mu^2-2)}}\xi\right) \pm \operatorname{ds}\left(\sqrt{\frac{2\lambda}{bm(\mu^2-2)}}\xi\right) \right] \right\}^{\frac{1}{m}}. \quad (21.10)$$

$$u_{11} = \left\{ \sqrt{\frac{2\lambda(1-\mu^2)}{am(\mu^2-2)}} \operatorname{sc}\left(\sqrt{-\frac{\lambda}{bm(\mu^2-2)}}\xi\right) \right\}^{\frac{1}{m}}. \quad (21.11)$$

$$u_{12} = \left\{ \sqrt{\frac{2\lambda(1-\mu^2)}{am(1-2\mu^2)}} \operatorname{nc}\left(\sqrt{\frac{\lambda}{bm(2\mu^2-1)}}\xi\right) \right\}^{\frac{1}{m}}. \quad (21.12)$$

$$u_{13} = \left\{ \sqrt{-\frac{\lambda(1-\mu^2)}{am(1+\mu^2)}} \left[\operatorname{sc}\left(\sqrt{\frac{2\lambda}{bm(1+\mu^2)}}\xi\right) \pm \operatorname{nc}\left(\sqrt{\frac{2\lambda}{bm(1+\mu^2)}}\xi\right) \right] \right\}^{\frac{1}{m}}. \quad (21.13)$$

$$u_{14} = \left\{ \sqrt{\frac{2\lambda\mu^2(\mu^2-1)}{am(1-2\mu^2)}} \operatorname{sd}\left(\sqrt{-\frac{\lambda}{bm(1-2\mu^2)}}\xi\right) \right\}^{\frac{1}{m}}. \quad (21.14)$$

$$u_{15} = \left\{ \sqrt{\frac{2\lambda(\mu^2-1)}{am(\mu^2-2)}} \operatorname{nd}\left(\sqrt{\frac{\lambda}{bm(2-\mu^2)}}\xi\right) \right\}^{\frac{1}{m}}. \quad (21.15)$$

$$u_{16} = \left\{ \sqrt{\frac{\lambda(1-\mu^2)}{am(1+\mu^2)}} [\mu \operatorname{sd}(\sqrt{\frac{2\lambda}{bm(1+\mu^2)}} \xi) \pm \operatorname{nd}(\sqrt{\frac{2\lambda}{bm(1+\mu^2)}} \xi)] \right\}^{\frac{1}{m}}. \quad (21.16)$$

$$u_{17} = \left\{ \sqrt{\frac{2\lambda\mu^2}{am(1+\mu^2)}} \operatorname{cd}(\sqrt{-\frac{\lambda}{bm(1+\mu^2)}} \xi) \right\}^{\frac{1}{m}}. \quad (21.17)$$

$$u_{18} = \left\{ \sqrt{\frac{\lambda}{am(2\mu^2-1)}} [\mu \operatorname{cd}(\sqrt{\frac{2\lambda}{bm(1-2\mu^2)}} \xi) \pm \sqrt{\mu^2-1} \operatorname{nd}(\sqrt{\frac{2\lambda}{bm(1-2\mu^2)}} \xi)] \right\}^{\frac{1}{m}}. \quad (21.18)$$

$$u_{19} = \left\{ \sqrt{\frac{\lambda}{am(1+\mu^2)}} [\mu \operatorname{cn}(\sqrt{\frac{2\lambda}{bm(1+\mu^2)}} \xi) \pm \operatorname{dn}(\sqrt{\frac{2\lambda}{bm(1+\mu^2)}} \xi)] \right\}^{\frac{1}{m}}. \quad (21.19)$$

$$u_{20} = \left\{ \sqrt{\frac{2\lambda}{am(1+\mu^2)}} \operatorname{dc}(\sqrt{-\frac{\lambda}{bm(1+\mu^2)}} \xi) \right\}^{\frac{1}{m}}. \quad (21.20)$$

$$u_{21} = \left\{ \sqrt{\frac{\lambda}{am(2-\mu^2)}} [\operatorname{dc}(\sqrt{-\frac{2\lambda}{bm(2-\mu^2)}} \xi) \pm \sqrt{1-\mu^2} \operatorname{nc}(\sqrt{-\frac{2\lambda}{bm(2-\mu^2)}} \xi)] \right\}^{\frac{1}{m}}. \quad (21.21)$$

$$u_{22} = \left\{ \sqrt{\frac{\lambda\mu^2}{am(2-\mu^2)}} [\sqrt{\mu^2-1} \operatorname{sd}(\sqrt{\frac{2\lambda}{bm(\mu^2-2)}} \xi) \pm \operatorname{cd}(\sqrt{\frac{2\lambda}{bm(\mu^2-2)}} \xi)] \right\}^{\frac{1}{m}}. \quad (21.22)$$

$$u_{23} = \left\{ \sqrt{\frac{\lambda}{am(2\mu^2-1)}} [\sqrt{1-\mu^2} \operatorname{sc}(\sqrt{\frac{2\lambda}{bm(1-2\mu^2)}} \xi) \pm \operatorname{dc}(\sqrt{\frac{2\lambda}{bm(1-2\mu^2)}} \xi)] \right\}^{\frac{1}{m}}. \quad (21.23)$$

$$u_{24} = \left\{ \sqrt{-\frac{\lambda}{am(1+\mu^2)}} [\operatorname{ds}(\sqrt{\frac{2\lambda}{bm(1+\mu^2)}} \xi) \pm \operatorname{cs}(\sqrt{\frac{2\lambda}{bm(1+\mu^2)}} \xi)] \right\}^{\frac{1}{m}}. \quad (21.24)$$

Remark 2: When the modulus $\mu \rightarrow 1$, the solutions $u_i (i = 1, 2, 3)$ degenerate as the solitary wave solutions of $mK(m, 3m, m)$ equations

$$u'_1 = \left\{ \sqrt{\frac{\lambda}{am}} \tanh(\sqrt{-\frac{\lambda}{2bm}} \xi) \right\}^{\frac{1}{m}}. \quad (22.1)$$

$$u'_2 = \left\{ \sqrt{\frac{2\lambda}{am}} \operatorname{sech}(\sqrt{\frac{\lambda}{bm}} \xi) \right\}^{\frac{1}{m}}. \quad (22.2)$$

$$u'_3 = \left\{ \sqrt{\frac{\lambda}{am}} [\tanh(\sqrt{-\frac{2\lambda}{bm}} \xi) \pm i \operatorname{sech}(\sqrt{-\frac{2\lambda}{bm}} \xi)] \right\}^{\frac{1}{m}}, \quad i^2 = -1. \quad (22.3)$$

2.2 Jacobi elliptic function solutions of $mK(3n, n, n)$ equation

From the system (17) we get

$$n = k, \quad m = 3k = 3n, \quad B = \frac{1}{n}, \quad R^2 = -\frac{a}{b\alpha}, \quad A^{2n} = -\frac{ma\beta}{\lambda\alpha}.$$

Therefore we establish the relation between the $mK(3n, n, n)$ equation (5) and (6)

$$u(\xi) = \left\{ \sqrt{-\frac{ma\beta}{\lambda\alpha}} \Psi\left(\sqrt{-\frac{a}{b\alpha}}\xi\right) \right\}^{\frac{1}{n}}. \quad (23a)$$

If we set

$$w = u^n = \sqrt{-\frac{ma\beta}{\lambda\alpha}} \Psi\left(\sqrt{-\frac{a}{b\alpha}}\xi\right), \quad (23b)$$

then (5) with $m = 3k = 3n$ reduces to

$$-\frac{\lambda}{3n} \frac{dw^3}{d\xi} + a \frac{dw}{d\xi} + b \frac{d^3w}{d\xi^3} = 0. \quad (23c)$$

Remark 3: In fact we also establish a relation (23b) between (23c) and (6).

According to (23a) and the known Jacobi elliptic function solutions (A.1)-(A.12) of (6), we get the solutions of $mK(3n, n, n)$ equations

$$u_1 = \left\{ \sqrt{\frac{2ma\mu^2}{\lambda(1+\mu^2)}} \operatorname{sn}\left(\sqrt{\frac{a}{b(1+\mu^2)}}\xi\right) \right\}^{\frac{1}{n}}. \quad (24.1)$$

$$u_2 = \left\{ \sqrt{\frac{2ma\mu^2}{\lambda(2\mu^2-1)}} \operatorname{cn}\left(\sqrt{\frac{a}{b(1-2\mu^2)}}\xi\right) \right\}^{\frac{1}{n}}. \quad (24.2)$$

$$u_3 = \left\{ \sqrt{\frac{ma\mu^2}{\lambda(2-\mu^2)}} \left[\operatorname{sn}\left(\sqrt{\frac{2a}{b(2-\mu^2)}}\xi\right) \pm i \operatorname{cn}\left(\sqrt{\frac{2a}{b(2-\mu^2)}}\xi\right) \right] \right\}^{\frac{1}{n}}, \quad i^2 = -1. \quad (24.3)$$

$$u_4 = \left\{ \sqrt{\frac{2ma}{\lambda(2-\mu^2)}} \operatorname{dn}\left(\sqrt{\frac{a}{b(\mu^2-2)}}\xi\right) \right\}^{\frac{1}{n}}. \quad (24.4)$$

$$u_5 = \left\{ \sqrt{\frac{ma}{\lambda(2\mu^2-1)}} \left[\mu \operatorname{sn}\left(\sqrt{\frac{2a}{b(2\mu^2-1)}}\xi\right) \pm i \operatorname{dn}\left(\sqrt{\frac{2a}{b(2\mu^2-1)}}\xi\right) \right] \right\}^{\frac{1}{n}}, \quad i^2 = -1. \quad (24.5)$$

$$u_6 = \left\{ \sqrt{\frac{2ma}{\lambda(1+\mu^2)}} \operatorname{ns}\left(\sqrt{\frac{a}{b(1+\mu^2)}}\xi\right) \right\}^{\frac{1}{n}}. \quad (24.6)$$

$$u_7 = \left\{ \sqrt{\frac{2ma}{\lambda(\mu^2-2)}} \operatorname{cs}\left(\sqrt{\frac{a}{b(\mu^2-2)}}\xi\right) \right\}^{\frac{1}{n}}. \quad (24.7)$$

$$u_8 = \left\{ \sqrt{\frac{ma}{\lambda(2\mu^2-1)}} \left[\operatorname{ns}\left(\sqrt{\frac{2a}{b(2\mu^2-1)}}\xi\right) \pm \operatorname{cs}\left(\sqrt{\frac{2a}{b(2\mu^2-1)}}\xi\right) \right] \right\}^{\frac{1}{n}}. \quad (24.8)$$

$$u_9 = \left\{ \sqrt{\frac{2ma}{\lambda(1-2\mu^2)}} \operatorname{ds}\left(\sqrt{\frac{a}{b(1-2\mu^2)}}\xi\right) \right\}^{\frac{1}{n}}. \quad (24.9)$$

$$u_{10} = \left\{ \sqrt{\frac{ma}{\lambda(2-\mu^2)}} \left[\operatorname{ns}\left(\sqrt{\frac{2a}{b(2-\mu^2)}}\xi\right) \pm \operatorname{ds}\left(\sqrt{\frac{2a}{b(2-\mu^2)}}\xi\right) \right] \right\}^{\frac{1}{n}}. \quad (24.10)$$

$$u_{11} = \left\{ \sqrt{\frac{2ma(1-\mu^2)}{\lambda(\mu^2-2)}} \operatorname{sc}\left(\sqrt{\frac{a}{b(\mu^2-2)}}\xi\right) \right\}^{\frac{1}{n}}. \quad (24.11)$$

$$u_{12} = \left\{ \sqrt{\frac{2ma(1-\mu^2)}{\lambda(1-2\mu^2)}} \operatorname{nc}\left(\sqrt{\frac{a}{b(1-2\mu^2)}}\xi\right) \right\}^{\frac{1}{n}}. \quad (24.12)$$

$$u_{13} = \left\{ \sqrt{\frac{ma(\mu^2-1)}{\lambda(\mu^2+1)}} \left[\operatorname{sc}\left(\sqrt{-\frac{2a}{b(\mu^2+1)}}\xi\right) \pm \operatorname{nc}\left(\sqrt{-\frac{2a}{b(\mu^2+1)}}\xi\right) \right] \right\}^{\frac{1}{n}}. \quad (24.13)$$

$$u_{14} = \left\{ \sqrt{\frac{2ma\mu^2(1-\mu^2)}{\lambda(2\mu^2-1)}} \operatorname{sd}\left(\sqrt{\frac{a}{b(1-2\mu^2)}}\xi\right) \right\}^{\frac{1}{n}}. \quad (24.14)$$

$$u_{15} = \left\{ \sqrt{\frac{2ma(1-\mu^2)}{\lambda(2-\mu^2)}} \operatorname{nd}\left(\sqrt{\frac{a}{b(\mu^2-2)}}\xi\right) \right\}^{\frac{1}{n}}. \quad (24.15)$$

$$u_{16} = \left\{ \sqrt{\frac{ma(1-\mu^2)}{\lambda(\mu^2+1)}} \left[\mu \operatorname{sd}\left(\sqrt{-\frac{2a}{b(\mu^2+1)}}\xi\right) \pm \operatorname{nd}\left(\sqrt{-\frac{2a}{b(\mu^2+1)}}\xi\right) \right] \right\}^{\frac{1}{n}}. \quad (24.16)$$

$$u_{17} = \left\{ \sqrt{\frac{2ma\mu^2}{\lambda(1+\mu^2)}} \operatorname{cd}\left(\sqrt{\frac{a}{b(1+\mu^2)}}\xi\right) \right\}^{\frac{1}{n}}. \quad (24.17)$$

$$u_{18} = \left\{ \sqrt{\frac{ma}{\lambda(2\mu^2-1)}} \left[\mu \operatorname{cd}\left(\sqrt{\frac{2a}{b(2\mu^2-1)}}\xi\right) \pm \sqrt{\mu^2-1} \operatorname{nd}\left(\sqrt{\frac{2a}{b(2\mu^2-1)}}\xi\right) \right] \right\}^{\frac{1}{n}}. \quad (24.18)$$

$$u_{19} = \left\{ \sqrt{\frac{ma}{\lambda(\mu^2+1)}} \left[\mu \operatorname{cn}\left(\sqrt{-\frac{2a}{b(\mu^2+1)}}\xi\right) \pm \operatorname{dn}\left(\sqrt{-\frac{2a}{b(\mu^2+1)}}\xi\right) \right] \right\}^{\frac{1}{n}}. \quad (24.19)$$

$$u_{20} = \left\{ \sqrt{\frac{2ma}{\lambda(1+\mu^2)}} \operatorname{dc}\left(\sqrt{\frac{a}{b(1+\mu^2)}}\xi\right) \right\}^{\frac{1}{n}}. \quad (24.20)$$

$$u_{21} = \left\{ \sqrt{\frac{ma}{\lambda(2-\mu^2)}} \left[\operatorname{dc}\left(\sqrt{\frac{2a}{b(2-\mu^2)}}\xi\right) \pm \sqrt{1-\mu^2} \operatorname{nc}\left(\sqrt{\frac{2a}{b(2-\mu^2)}}\xi\right) \right] \right\}^{\frac{1}{n}}. \quad (24.21)$$

$$u_{22} = \left\{ \sqrt{\frac{ma\mu^2}{\lambda(2-\mu^2)}} \left[\sqrt{\mu^2-1} \operatorname{sd}\left(\sqrt{\frac{2a}{b(2-\mu^2)}}\xi\right) \pm \operatorname{cd}\left(\sqrt{\frac{2a}{b(2-\mu^2)}}\xi\right) \right] \right\}^{\frac{1}{n}}. \quad (24.22)$$

$$u_{23} = \left\{ \sqrt{\frac{ma}{\lambda(2\mu^2 - 1)}} [\sqrt{1 - \mu^2} \operatorname{sc}(\sqrt{\frac{2a}{b(2\mu^2 - 1)}} \xi) \pm \operatorname{dc}(\sqrt{\frac{2a}{b(2\mu^2 - 1)}} \xi)] \right\}^{\frac{1}{n}}. \quad (24.23)$$

$$u_{24} = \left\{ \sqrt{-\frac{ma}{\lambda(1 + \mu^2)}} [\operatorname{ds}(\sqrt{-\frac{2a}{b(1 + \mu^2)}} \xi) \pm \operatorname{cs}(\sqrt{-\frac{2a}{b(1 + \mu^2)}} \xi)] \right\}^{\frac{1}{n}}. \quad (24.24)$$

Remark 4: When the modulus $\mu \rightarrow 1$, the solutions $u_i (i = 1, 2, 3)$ degenerate as the solitary wave solutions of $mK(3n, n, n)$ equations

$$u'_1 = \left\{ \sqrt{\frac{ma}{\lambda}} \tanh(\sqrt{\frac{a}{2b}} \xi) \right\}^{\frac{1}{n}}. \quad (25.1)$$

$$u'_2 = \left\{ \sqrt{\frac{2ma}{\lambda}} \operatorname{sech}(\sqrt{-\frac{a}{b}} \xi) \right\}^{\frac{1}{n}}. \quad (24.2)$$

$$u'_3 = \left\{ \sqrt{\frac{ma}{\lambda}} [\tanh(\sqrt{\frac{2a}{b}} \xi) \pm i \operatorname{sech}(\sqrt{\frac{2a}{b}} \xi)] \right\}^{\frac{1}{n}}, \quad i^2 = -1. \quad (25.3)$$

2.3 Jacobi elliptic function solutions of $mK(m, 2m, m)$ equation

From the system (18) we get

$$m = k, \quad n = 2k = 2m, \quad B = \frac{2}{m}, \quad R^2 = \frac{\lambda}{4bm\alpha}, \quad A^m = -\frac{3\lambda\beta}{4am\alpha}.$$

Therefore we establish the relation between the $mK(m, 2m, m)$ equation (5) and (6)

$$u(\xi) = \left\{ \sqrt{-\frac{3\lambda\beta}{4am\alpha}} \Psi(\sqrt{\frac{\lambda}{4bm\alpha}} \xi) \right\}^{\frac{1}{m}}. \quad (26a)$$

If we set

$$w = u^m = \sqrt{-\frac{3\lambda\beta}{4am\alpha}} \Psi(\sqrt{\frac{\lambda}{4bm\alpha}} \xi), \quad (26b)$$

then (5) with $n = 2m = 2k$ reduces to

$$-\frac{\lambda}{m} \frac{dw}{d\xi} + a \frac{dw^2}{d\xi} + b \frac{d^3w}{d\xi^3} = 0. \quad (26c)$$

Remark 4: In fact we also establish a relation (26b) between (26c) and (6).

According to (26a) and the known Jacobi elliptic function solutions (A.1)-(A.12) of (6), we get the solutions of $mK(m, 2m, m)$ equations

$$u_1 = \left\{ \sqrt{\frac{3\lambda\mu^2}{2am(1 + \mu^2)}} \operatorname{sn}(\sqrt{-\frac{\lambda}{4bm(1 + \mu^2)}} \xi) \right\}^{\frac{2}{m}}. \quad (27.1)$$

$$u_2 = \left\{ \sqrt{\frac{3\lambda\mu^2}{2am(2\mu^2-1)}} \operatorname{cn}\left(\sqrt{\frac{\lambda}{4bm(2\mu^2-1)}}\xi\right) \right\}^{\frac{2}{m}}. \quad (27.2)$$

$$u_3 = \left\{ \sqrt{\frac{3\lambda\mu^2}{4am(2-\mu^2)}} [\operatorname{sn}\left(\sqrt{\frac{\lambda}{2bm(\mu^2-2)}}\xi\right) \pm i \operatorname{cn}\left(\sqrt{\frac{\lambda}{2bm(\mu^2-2)}}\xi\right)] \right\}^{\frac{2}{m}}, \quad i^2 = -1. \quad (27.3)$$

$$u_4 = \left\{ \sqrt{\frac{3\lambda}{2am(2-\mu^2)}} \operatorname{dn}\left(\sqrt{\frac{\lambda}{4bm(2-\mu^2)}}\xi\right) \right\}^{\frac{2}{m}}. \quad (27.4)$$

$$u_5 = \left\{ \sqrt{\frac{3\lambda}{4am(2\mu^2-1)}} [\mu \operatorname{sn}\left(\sqrt{\frac{\lambda}{2bm(1-2\mu^2)}}\xi\right) \pm i \operatorname{dn}\left(\sqrt{\frac{\lambda}{2bm(1-2\mu^2)}}\xi\right)] \right\}^{\frac{2}{m}}, \quad i^2 = -1. \quad (27.5)$$

$$u_6 = \left\{ \sqrt{\frac{3\lambda}{2am(1+\mu^2)}} \operatorname{ns}\left(\sqrt{-\frac{\lambda}{4bm(1+\mu^2)}}\xi\right) \right\}^{\frac{2}{m}}. \quad (27.6)$$

$$u_7 = \left\{ \sqrt{\frac{3\lambda}{2am(\mu^2-2)}} \operatorname{cs}\left(\sqrt{-\frac{\lambda}{4bm(\mu^2-2)}}\xi\right) \right\}^{\frac{2}{m}}. \quad (27.7)$$

$$u_8 = \left\{ \sqrt{\frac{3\lambda}{4am(2\mu^2-1)}} [\operatorname{ns}\left(\sqrt{\frac{\lambda}{2bm(1-2\mu^2)}}\xi\right) \pm \operatorname{cs}\left(\sqrt{\frac{\lambda}{2bm(1-2\mu^2)}}\xi\right)] \right\}^{\frac{2}{m}}. \quad (28.8)$$

$$u_9 = \left\{ \sqrt{\frac{3\lambda}{2am(1-2\mu^2)}} \operatorname{ds}\left(\sqrt{-\frac{\lambda}{4bm(1-2\mu^2)}}\xi\right) \right\}^{\frac{2}{m}}. \quad (28.9)$$

$$u_{10} = \left\{ \sqrt{\frac{3\lambda}{4am(2-\mu^2)}} [\operatorname{ns}\left(\sqrt{\frac{\lambda}{2bm(\mu^2-2)}}\xi\right) \pm \operatorname{ds}\left(\sqrt{\frac{\lambda}{2bm(\mu^2-2)}}\xi\right)] \right\}^{\frac{2}{m}}. \quad (27.10)$$

$$u_{11} = \left\{ \sqrt{\frac{3\lambda(1-\mu^2)}{2am(\mu^2-2)}} \operatorname{sc}\left(\sqrt{-\frac{\lambda}{4bm(\mu^2-2)}}\xi\right) \right\}^{\frac{2}{m}}. \quad (27.11)$$

$$u_{12} = \left\{ \sqrt{\frac{3\lambda(1-\mu^2)}{2am(1-2\mu^2)}} \operatorname{nc}\left(\sqrt{\frac{\lambda}{4bm(2\mu^2-1)}}\xi\right) \right\}^{\frac{2}{m}}. \quad (27.12)$$

$$u_{13} = \left\{ \sqrt{-\frac{3\lambda(1-\mu^2)}{4am(1+\mu^2)}} [\operatorname{sc}\left(\sqrt{\frac{\lambda}{2bm(1+\mu^2)}}\xi\right) \pm \operatorname{nc}\left(\sqrt{\frac{\lambda}{2bm(1+\mu^2)}}\xi\right)] \right\}^{\frac{2}{m}}. \quad (27.13)$$

$$u_{14} = \left\{ \sqrt{\frac{3\lambda\mu^2(\mu^2-1)}{2am(1-2\mu^2)}} \operatorname{sd}\left(\sqrt{-\frac{\lambda}{4bm(1-2\mu^2)}}\xi\right) \right\}^{\frac{2}{m}}. \quad (27.14)$$

$$u_{15} = \left\{ \sqrt{\frac{3\lambda(\mu^2-1)}{2am(\mu^2-2)}} \operatorname{nd}\left(\sqrt{\frac{\lambda}{4bm(2-\mu^2)}}\xi\right) \right\}^{\frac{2}{m}}. \quad (27.15)$$

$$u_{16} = \left\{ \sqrt{\frac{3\lambda(1-\mu^2)}{4am(1+\mu^2)}} [\mu \operatorname{sd}(\sqrt{\frac{\lambda}{2bm(1+\mu^2)}} \xi) \pm \operatorname{nd}(\sqrt{\frac{\lambda}{2bm(1+\mu^2)}} \xi)] \right\}^{\frac{2}{m}}. \quad (27.16)$$

$$u_{17} = \left\{ \sqrt{\frac{3\lambda\mu^2}{2am(1+\mu^2)}} \operatorname{cd}(\sqrt{-\frac{\lambda}{4bm(1+\mu^2)}} \xi) \right\}^{\frac{2}{m}}. \quad (27.17)$$

$$u_{18} = \left\{ \sqrt{\frac{3\lambda}{4am(2\mu^2-1)}} [\mu \operatorname{cd}(\sqrt{\frac{\lambda}{2bm(1-2\mu^2)}} \xi) \pm \sqrt{\mu^2-1} \operatorname{nd}(\sqrt{\frac{\lambda}{2bm(1-2\mu^2)}} \xi)] \right\}^{\frac{2}{m}}. \quad (27.18)$$

$$u_{19} = \left\{ \sqrt{\frac{3\lambda}{4am(1+\mu^2)}} [\mu \operatorname{cn}(\sqrt{\frac{\lambda}{2bm(1+\mu^2)}} \xi) \pm \operatorname{dn}(\sqrt{\frac{\lambda}{2bm(1+\mu^2)}} \xi)] \right\}^{\frac{2}{m}}. \quad (27.19)$$

$$u_{20} = \left\{ \sqrt{\frac{3\lambda}{2am(1+\mu^2)}} \operatorname{dc}(\sqrt{-\frac{\lambda}{4bm(1+\mu^2)}} \xi) \right\}^{\frac{2}{m}}. \quad (27.20)$$

$$u_{21} = \left\{ \sqrt{\frac{3\lambda}{4am(2-\mu^2)}} [\operatorname{dc}(\sqrt{-\frac{\lambda}{2bm(2-\mu^2)}} \xi) \pm \sqrt{1-\mu^2} \operatorname{nc}(\sqrt{-\frac{\lambda}{2bm(2-\mu^2)}} \xi)] \right\}^{\frac{2}{m}}. \quad (27.21)$$

$$u_{22} = \left\{ \sqrt{\frac{3\lambda\mu^2}{4am(2-\mu^2)}} [\sqrt{\mu^2-1} \operatorname{sd}(\sqrt{\frac{\lambda}{2bm(\mu^2-2)}} \xi) \pm \operatorname{cd}(\sqrt{\frac{\lambda}{2bm(\mu^2-2)}} \xi)] \right\}^{\frac{2}{m}}. \quad (27.22)$$

$$u_{23} = \left\{ \sqrt{\frac{3\lambda}{4am(2\mu^2-1)}} [\sqrt{1-\mu^2} \operatorname{sc}(\sqrt{\frac{\lambda}{2bm(1-2\mu^2)}} \xi) \pm \operatorname{dc}(\sqrt{\frac{\lambda}{2bm(1-2\mu^2)}} \xi)] \right\}^{\frac{2}{m}}. \quad (27.23)$$

$$u_{24} = \left\{ \sqrt{-\frac{3\lambda}{4am(1+\mu^2)}} [\operatorname{ds}(\sqrt{\frac{\lambda}{2bm(1+\mu^2)}} \xi) \pm \operatorname{cs}(\sqrt{\frac{\lambda}{2bm(1+\mu^2)}} \xi)] \right\}^{\frac{2}{m}}. \quad (27.24)$$

Remark 6: When the modulus $\mu \rightarrow 1$, the solutions $u_i (i = 1, 2, 3)$ degenerate as the solitary wave solutions of $mK(m, 2m, m)$ equations

$$u'_1 = \left\{ \sqrt{\frac{3\lambda}{4am}} \tanh(\sqrt{-\frac{\lambda}{8bm}} \xi) \right\}^{\frac{2}{m}}. \quad (28.1)$$

$$u'_2 = \left\{ \sqrt{\frac{3\lambda}{2am}} \operatorname{sech}(\sqrt{\frac{\lambda}{4bm}} \xi) \right\}^{\frac{2}{m}}. \quad (28.2)$$

$$u'_3 = \left\{ \sqrt{\frac{3\lambda}{4am}} [\tanh(\sqrt{-\frac{\lambda}{2bm}} \xi) \pm i \operatorname{sech}(\sqrt{-\frac{\lambda}{2bm}} \xi)] \right\}^{\frac{2}{m}}, \quad i^2 = -1. \quad (28.3)$$

2.4 Jacobi elliptic function solutions of $mK(2n, n, n)$ equation

From the system (19) we get

$$n = k, \quad m = 2k = 2n, \quad B = \frac{2}{n}, \quad R^2 = -\frac{a}{4b\alpha}, \quad A^n = -\frac{3ma\beta}{4\lambda\alpha}.$$

Therefore we establish the relation between the $mK(2n, n, n)$ equation (5) and (6)

$$u(\xi) = \left\{ \sqrt{-\frac{3ma\beta}{4\lambda\alpha}} \Psi\left(\sqrt{-\frac{a}{4b\alpha}}\xi\right) \right\}^{\frac{1}{n}}. \quad (29a)$$

If we set

$$w = u^n = \sqrt{-\frac{3ma\beta}{4\lambda\alpha}} \Psi\left(\sqrt{-\frac{a}{4b\alpha}}\xi\right), \quad (29b)$$

then (5) with $m = 2k = 2n$ reduces to

$$-\frac{\lambda}{2n} \frac{dw^2}{d\xi} + a \frac{dw}{d\xi} + b \frac{d^3w}{d\xi^3} = 0. \quad (29c)$$

Remark 7: In fact we also establish a relation (29b) between (29c) and (6).

According to (29a) and the known Jacobi elliptic function solutions (A.1)-(A.12) of (6), we get the solutions of $mK(2n, n, n)$ equations

$$u_1 = \left\{ \sqrt{\frac{3ma\mu^2}{2\lambda(1+\mu^2)}} \operatorname{sn}\left(\sqrt{\frac{a}{4b(1+\mu^2)}}\xi\right) \right\}^{\frac{2}{n}}. \quad (30.1)$$

$$u_2 = \left\{ \sqrt{\frac{3ma\mu^2}{2\lambda(2\mu^2-1)}} \operatorname{cn}\left(\sqrt{\frac{a}{4b(1-2\mu^2)}}\xi\right) \right\}^{\frac{2}{n}}. \quad (30.2)$$

$$u_3 = \left\{ \sqrt{\frac{3ma\mu^2}{4\lambda(2-\mu^2)}} \left[\operatorname{sn}\left(\sqrt{\frac{a}{2b(2-\mu^2)}}\xi\right) \pm i \operatorname{cn}\left(\sqrt{\frac{a}{2b(2-\mu^2)}}\xi\right) \right] \right\}^{\frac{2}{n}}, \quad i^2 = -1. \quad (30.3)$$

$$u_4 = \left\{ \sqrt{\frac{3ma}{2\lambda(2-\mu^2)}} \operatorname{dn}\left(\sqrt{\frac{a}{4b(\mu^2-2)}}\xi\right) \right\}^{\frac{2}{n}}. \quad (30.4)$$

$$u_5 = \left\{ \sqrt{\frac{3ma}{4\lambda(2\mu^2-1)}} \left[\mu \operatorname{sn}\left(\sqrt{\frac{a}{2b(2\mu^2-1)}}\xi\right) \pm i \operatorname{dn}\left(\sqrt{\frac{a}{2b(2\mu^2-1)}}\xi\right) \right] \right\}^{\frac{2}{n}}, \quad i^2 = -1. \quad (30.5)$$

$$u_6 = \left\{ \sqrt{\frac{3ma}{2\lambda(1+\mu^2)}} \operatorname{ns}\left(\sqrt{\frac{a}{4b(1+\mu^2)}}\xi\right) \right\}^{\frac{2}{n}}. \quad (30.6)$$

$$u_7 = \left\{ \sqrt{\frac{3ma}{2\lambda(\mu^2-2)}} \operatorname{cs}\left(\sqrt{\frac{a}{4b(\mu^2-2)}}\xi\right) \right\}^{\frac{2}{n}}. \quad (30.7)$$

$$u_8 = \left\{ \sqrt{\frac{3ma}{4\lambda(2\mu^2 - 1)}} [\text{ns}(\sqrt{\frac{a}{2b(2\mu^2 - 1)}}\xi) \pm \text{cs}(\sqrt{\frac{a}{2b(2\mu^2 - 1)}}\xi)] \right\}^{\frac{2}{n}}. \quad (30.8)$$

$$u_9 = \left\{ \sqrt{\frac{3ma}{2\lambda(1 - 2\mu^2)}} \text{ds}(\sqrt{\frac{a}{4b(1 - 2\mu^2)}}\xi) \right\}^{\frac{2}{n}}. \quad (30.9)$$

$$u_{10} = \left\{ \sqrt{\frac{3ma}{4\lambda(2 - \mu^2)}} [\text{ns}(\sqrt{\frac{a}{2b(2 - \mu^2)}}\xi) \pm \text{ds}(\sqrt{\frac{a}{2b(2 - \mu^2)}}\xi)] \right\}^{\frac{2}{n}}. \quad (30.10)$$

$$u_{11} = \left\{ \sqrt{\frac{3ma(1 - \mu^2)}{2\lambda(\mu^2 - 2)}} \text{sc}(\sqrt{\frac{a}{4b(\mu^2 - 2)}}\xi) \right\}^{\frac{2}{n}}. \quad (30.11)$$

$$u_{12} = \left\{ \sqrt{\frac{3ma(1 - \mu^2)}{2\lambda(1 - 2\mu^2)}} \text{nc}(\sqrt{\frac{a}{4b(1 - 2\mu^2)}}\xi) \right\}^{\frac{2}{n}}. \quad (30.12)$$

$$u_{13} = \left\{ \sqrt{\frac{3ma(\mu^2 - 1)}{4\lambda(\mu^2 + 1)}} [\text{sc}(\sqrt{-\frac{a}{2b(\mu^2 + 1)}}\xi) \pm \text{nc}(\sqrt{-\frac{a}{2b(\mu^2 + 1)}}\xi)] \right\}^{\frac{2}{n}}. \quad (30.13)$$

$$u_{14} = \left\{ \sqrt{\frac{3ma\mu^2(1 - \mu^2)}{2\lambda(2\mu^2 - 1)}} \text{sd}(\sqrt{\frac{a}{4b(1 - 2\mu^2)}}\xi) \right\}^{\frac{2}{n}}. \quad (30.14)$$

$$u_{15} = \left\{ \sqrt{\frac{3ma(1 - \mu^2)}{2\lambda(2 - \mu^2)}} \text{nd}(\sqrt{\frac{a}{4b(\mu^2 - 2)}}\xi) \right\}^{\frac{2}{n}}. \quad (30.15)$$

$$u_{16} = \left\{ \sqrt{\frac{3ma(1 - \mu^2)}{4\lambda(\mu^2 + 1)}} [\mu \text{sd}(\sqrt{-\frac{a}{2b(\mu^2 + 1)}}\xi) \pm \text{nd}(\sqrt{-\frac{a}{2b(\mu^2 + 1)}}\xi)] \right\}^{\frac{2}{n}}. \quad (30.16)$$

$$u_{17} = \left\{ \sqrt{\frac{3ma\mu^2}{2\lambda(1 + \mu^2)}} \text{cd}(\sqrt{\frac{a}{4b(1 + \mu^2)}}\xi) \right\}^{\frac{2}{n}}. \quad (30.17)$$

$$u_{18} = \left\{ \sqrt{\frac{3ma}{4\lambda(2\mu^2 - 1)}} [\mu \text{cd}(\sqrt{\frac{a}{2b(2\mu^2 - 1)}}\xi) \pm \sqrt{\mu^2 - 1} \text{nd}(\sqrt{\frac{a}{2b(2\mu^2 - 1)}}\xi)] \right\}^{\frac{2}{n}}. \quad (30.18)$$

$$u_{19} = \left\{ \sqrt{\frac{3ma}{4\lambda(\mu^2 + 1)}} [\mu \text{cn}(\sqrt{-\frac{a}{2b(\mu^2 + 1)}}\xi) \pm \text{dn}(\sqrt{-\frac{a}{2b(\mu^2 + 1)}}\xi)] \right\}^{\frac{2}{n}}. \quad (30.19)$$

$$u_{20} = \left\{ \sqrt{\frac{3ma}{2\lambda(1 + \mu^2)}} \text{dc}(\sqrt{\frac{a}{4b(1 + \mu^2)}}\xi) \right\}^{\frac{2}{n}}. \quad (30.20)$$

$$u_{21} = \left\{ \sqrt{\frac{3ma}{4\lambda(2 - \mu^2)}} [\text{dc}(\sqrt{\frac{a}{2b(2 - \mu^2)}}\xi) \pm \sqrt{1 - \mu^2} \text{nc}(\sqrt{\frac{a}{2b(2 - \mu^2)}}\xi)] \right\}^{\frac{2}{n}}. \quad (30.21)$$

$$u_{22} = \left\{ \sqrt{\frac{3ma\mu^2}{4\lambda(2-\mu^2)}} [\sqrt{\mu^2-1} \operatorname{sd}(\sqrt{\frac{a}{2b(2-\mu^2)}}\xi) \pm \operatorname{cd}(\sqrt{\frac{a}{2b(2-\mu^2)}}\xi)] \right\}^{\frac{2}{n}}. \quad (30.22)$$

$$u_{23} = \left\{ \sqrt{\frac{3ma}{4\lambda(2\mu^2-1)}} [\sqrt{1-\mu^2} \operatorname{sc}(\sqrt{\frac{a}{2b(2\mu^2-1)}}\xi) \pm \operatorname{dc}(\sqrt{\frac{a}{2b(2\mu^2-1)}}\xi)] \right\}^{\frac{2}{n}}. \quad (30.23)$$

$$u_{24} = \left\{ \sqrt{-\frac{3ma}{4\lambda(1+\mu^2)}} [\operatorname{ds}(\sqrt{-\frac{a}{2b(1+\mu^2)}}\xi) \pm \operatorname{cs}(\sqrt{-\frac{a}{2b(1+\mu^2)}}\xi)] \right\}^{\frac{2}{n}}. \quad (30.24)$$

Remark 8: When the modulus $\mu \rightarrow 1$, the solutions $u_i (i = 1, 2, 3)$ degenerate as the solitary wave solutions of $mK(2n, n, n)$ equations

$$u'_1 = \left\{ \sqrt{\frac{3ma}{4\lambda}} \tanh(\sqrt{\frac{a}{8b}}\xi) \right\}^{\frac{2}{n}}. \quad (31.1)$$

$$u'_2 = \left\{ \sqrt{\frac{3ma}{2\lambda}} \operatorname{sech}(\sqrt{-\frac{a}{4b}}\xi) \right\}^{\frac{2}{n}}. \quad (31.2)$$

$$u'_3 = \left\{ \sqrt{\frac{3ma}{4\lambda}} [\tanh(\sqrt{\frac{a}{2b}}\xi) \pm i \operatorname{sech}(\sqrt{\frac{a}{2b}}\xi)] \right\}^{\frac{2}{n}}, \quad i^2 = -1. \quad (31.3)$$

3. Conclusions

In summary, we had obtained some compacton solutions and solitary pattern solutions of (2) in recent paper[11]. In this paper we presented many types of Jacobi elliptic function solutions of the $mK(m, n, k)$ equations with different parameters again. In particular, when the modulus $\mu \rightarrow 1$, some Jacobi elliptic function solutions degenerate as solitary wave solutions. If the parameters m, n, k are taken as other relations, further study is needed to see whether (2) has the Jacobi elliptic function solutions or other types of solutions.

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Appendix A

Substituting these formal solutions (7)-(12) into (6), respectively and equating the coefficients of the corresponding Jacobi elliptic functions can generate a system of algebraic equations of a_0, a_1, b_1, α . Solving the set of equations can determine these parameters a_0, a_1, b_1, α . Hence we can obtain the following Jacobi elliptic function solutions of (6):

(1) Taking $\Psi(R\xi) = a_0 + a_1 \operatorname{sn}(R\xi) + b_1 \operatorname{cn}(R\xi)$, we have the solution of (6)

$$\begin{aligned}\Psi_1(R\xi) &= \mu \sqrt{\frac{2}{\beta}} \operatorname{sn}(R\xi), \quad \alpha = -(1 + \mu^2), \\ \Psi_2(R\xi) &= \mu \sqrt{-\frac{2}{\beta}} \operatorname{cn}(R\xi), \quad \alpha = 2\mu^2 - 1, \\ \Psi_3(R\xi) &= \mu \sqrt{\frac{1}{2\beta}} [\operatorname{sn}(R\xi) \pm i \operatorname{cn}(R\xi)], \quad i = \sqrt{-1}, \quad 2\alpha = \mu^2 - 2.\end{aligned}\tag{A.1}$$

(2) Taking $\Psi(R\xi) = a_0 + a_1 \operatorname{sn}(R\xi) + b_1 \operatorname{dn}(R\xi)$, we get

$$\begin{aligned}\Psi_4(R\xi) &= \sqrt{-\frac{2}{\beta}} \operatorname{dn}(R\xi), \quad \alpha = 2 - \mu^2, \\ \Psi_5(R\xi) &= \sqrt{\frac{1}{2\beta}} [\mu \operatorname{sn}(R\xi) \pm i \operatorname{dn}(R\xi)], \quad 2\alpha = 1 - 2\mu^2,\end{aligned}\tag{A.2}$$

(3) Taking $\Psi(R\xi) = a_0 + a_1 \operatorname{ns}(R\xi) + b_1 \operatorname{cs}(R\xi)$, we have

$$\begin{aligned}\Psi_6(R\xi) &= \sqrt{\frac{2}{\beta}} \operatorname{ns}(R\xi), \quad \alpha = -1 - \mu^2, \\ \Psi_7(R\xi) &= \sqrt{\frac{2}{\beta}} \operatorname{cs}(R\xi), \quad \alpha = 2 - \mu^2. \\ \Psi_8(R\xi) &= \sqrt{\frac{1}{2\beta}} [\operatorname{ns}(R\xi) \pm \operatorname{cs}(R\xi)], \quad 2\alpha = 1 - 2\mu^2,\end{aligned}\tag{A.3}$$

(4) Taking $\Psi(R\xi) = a_0 + a_1 \operatorname{ns}(R\xi) + b_1 \operatorname{ds}(R\xi)$, we get

$$\begin{aligned}\Psi_9(R\xi) &= \sqrt{\frac{2}{\beta}} \operatorname{ds}(R\xi), \quad \alpha = 2\mu^2 - 1, \\ \Psi_{10}(R\xi) &= \sqrt{\frac{1}{2\beta}} [\operatorname{ns}(R\xi) \pm \operatorname{ds}(R\xi)], \quad 2\alpha = \mu^2 - 2,\end{aligned}\tag{A.4}$$

(5) Taking $\Psi(R\xi) = a_0 + a_1 \operatorname{sc}(R\xi) + b_1 \operatorname{nc}(R\xi)$, we derive

$$\begin{aligned}\Psi_{11}(R\xi) &= \sqrt{\frac{2(1 - \mu^2)}{\beta}} \operatorname{sc}(R\xi), \quad \alpha = 2 - \mu^2, \\ \Psi_{12}(R\xi) &= \sqrt{\frac{2(1 - \mu^2)}{\beta}} \operatorname{nc}(R\xi), \quad \alpha = 2\mu^2 - 1, \\ \Psi_{13}(R\xi) &= \sqrt{\frac{1 - \mu^2}{2\beta}} [\operatorname{sc}(R\xi) \pm \operatorname{nc}(R\xi)], \quad 2\alpha = 1 + \mu^2,\end{aligned}\tag{A.5}$$

(6) Setting $\Psi(R\xi) = a_0 + a_1\text{sd}(R\xi) + b_1\text{nd}(R\xi)$, we obtain

$$\begin{aligned}\Psi_{14}(R\xi) &= \mu\sqrt{\frac{2(\mu^2-1)}{\beta}}\text{sd}(R\xi), \quad \alpha = 2\mu^2 - 1, \\ \Psi_{15}(R\xi) &= \sqrt{\frac{2(\mu^2-1)}{\beta}}\text{nd}(R\xi), \quad \alpha = 2 - \mu^2, \\ \Psi_{16}(R\xi) &= \sqrt{\frac{\mu^2-1}{2\beta}}[\mu\text{sd}(R\xi) \pm \text{nd}(R\xi)], \quad 2\alpha = 1 + \mu^2.\end{aligned}\tag{A.6}$$

(7) Setting $\Psi(R\xi) = a_0 + a_1\text{cd}(R\xi) + b_1\text{nd}(R\xi)$, we obtain

$$\begin{aligned}\Psi_{17}(R\xi) &= m\sqrt{\frac{2}{\beta}}\text{cd}(R\xi), \quad \alpha = -1 - \mu^2, \\ \Psi_{18}(R\xi) &= \sqrt{\frac{1}{2\beta}}[\mu\text{cd}(R\xi) \pm \sqrt{\mu^2-1}\text{nd}(R\xi)], \quad 2\alpha = 1 - 2\mu^2.\end{aligned}\tag{A.7}$$

(8) Setting $\Psi(R\xi) = a_0 + a_1\text{cn}(R\xi) + b_1\text{dn}(R\xi)$, we obtain

$$\Psi_{19}(R\xi) = \sqrt{-\frac{1}{2\beta}}[\mu\text{cn}(R\xi) \pm \text{dn}(R\xi)], \quad 2\alpha = 1 + \mu^2.\tag{A.8}$$

(9) Setting $\Psi(R\xi) = a_0 + a_1\text{dc}(R\xi) + b_1\text{nc}(R\xi)$, we obtain

$$\begin{aligned}\Psi_{20}(R\xi) &= \sqrt{\frac{2}{\beta}}\text{dc}(R\xi), \quad \alpha = -1 - \mu^2, \\ \Psi_{21}(R\xi) &= \sqrt{\frac{1}{2\beta}}[\text{dc}(R\xi) \pm \sqrt{1-\mu^2}\text{nc}(R\xi)], \quad 2\alpha = \mu^2 - 2,\end{aligned}\tag{A.9}$$

(10) Setting $\Psi(R\xi) = a_0 + a_1\text{sd}(R\xi) + b_1\text{cd}(R\xi)$, we obtain

$$\Psi_{22}(R\xi) = \mu\sqrt{\frac{1}{2\beta}}[\sqrt{\mu^2-1}\text{sd}(R\xi) \pm \text{cd}(R\xi)], \quad 2\alpha = -2 + \mu^2,\tag{A.10}$$

(11) Setting $\Psi(R\xi) = a_0 + a_1\text{sc}(R\xi) + b_1\text{dc}(R\xi)$, we obtain

$$\Psi_{23}(R\xi) = \sqrt{\frac{1}{2\beta}}[\sqrt{1-\mu^2}\text{sc}(R\xi) \pm \text{dc}(R\xi)], \quad 2\alpha = 1 - 2\mu^2,\tag{A.11}$$

(12) Setting $\Psi(R\xi) = a_0 + a_1\text{ds}(R\xi) + b_1\text{cs}(R\xi)$, we obtain

$$\Psi_{24}(R\xi) = \sqrt{\frac{1}{2\beta}}[\text{ds}(R\xi) \pm \text{cs}(R\xi)], \quad 2\alpha = 1 + \mu^2.\tag{A.12}$$

where μ is the modulus of Jacobi elliptic functions, and R is an arbitrary constant.

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