Jacobi elliptic function solutions of nonlinear wave equations via the new sinh-Gordon equation expansion method

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Abstract. In this paper, based on the well-known sinh-Gordon equation, a new sinh-Gordon equation expansion method is developed. This method transforms the problem of solving nonlinear partial differential equations into the problem of solving the corresponding systems of algebraic equations. With the aid of symbolic computation, the procedure can be carried out in computer. Many nonlinear wave equations in mathematical physics are chosen to illustrate the method such as the KdV-nKdV equation, (2+1)-dimensional coupled Davey-Stewartson equation, the new integrable Davey-Stewartson-type equation, the modified Boussinesq equation, (2+1)-dimensional mKP equation and (2+1)-dimensional generalized KdV equation. As a consequence, many new doubly-periodic (Jacobi elliptic function) solutions are obtained. When the modulus $m \to 1$ or 0, the corresponding solitary waves solutions and singly-periodic solutions are also found. This approach can be also applied to solve other nonlinear differential equations.

Keywords: nonlinear wave equation; sinh-Gordon equation; Jacobi elliptic function; soliton solution

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1. Introduction

Up to now more and more nonlinear evolution equations were presented, which described the motion of the isolated waves, localized in a small part of space, in many fields such as hydrodynamic, plasma physics, nonlinear optic, etc. The investigation of exact solutions of these nonlinear evolution equations is interesting and important. In the past several decades, many authors mainly had paid attention to study soliton solutions of nonlinear wave equations by using various methods, such as Backlund transformation[1,5], Darboux transformation[2], Inverse scattering method[3], Hirota’s bilinear method[4], the tanh method[6], the sine-cosine method[7,10], the homogeneous balance method[8,9], the Riccati expansion method with constant coefficients[11,12] or variable coefficients[13], etc. But there were also a few papers to consider the doubly-periodic solutions which were expressed by using the Jacobi elliptic functions, Weierstrass elliptic function, the Theta functions, etc(see Refs.[14-19] and therein). Some methods were presented to seek the doubly-periodic solutions such as the Jacobi elliptic function expansion method[14], the extended Jacobi elliptic function expansion method[15-17] and the algebraic method[20], etc. When the modulus $m \to 1$ or 0,
the Jacobi elliptic functions degenerate as soliton solutions and trigonometric function solutions. Therefore seeking the Jacobi elliptic function solutions of nonlinear wave equations is of important significance.

In order to seek more types of the Jacobi elliptic function solutions, in this paper, we would like to develop a transformation from the sinh-Gordon equation[1] which reveals a relationship between differential nonlinear wave equations. The transformation and sinh-Gordon equation are used to construct Jacobi elliptic function solutions of nonlinear wave equations. Under the transformation $u(x,t) = u(\xi)$, $\xi = k(x - \lambda t)$, the famous sinh-Gordon equation

$$\frac{\partial^2 \phi}{\partial x \partial t} = \alpha \sinh \phi,$$  

(1)

which appears in many branches of nonlinear science[1], where $\alpha$ is a constant, reduces to an ordinary differential equation

$$\frac{d^2 \phi}{d\xi^2} = -\frac{\alpha}{k\lambda} \sinh \phi.$$  

(2)

where $k$ and $\lambda$ are the wave number and wave speed respectively. Integrating (2) once yields

$$\left(\frac{d}{d\xi} \frac{1}{2} \phi \right)^2 = -\frac{\alpha}{k\lambda} \sinh^2 \left(\frac{1}{2} \phi \right) + c.$$  

(3)

with integration constant $c$. If we set $c = 0$, $-\frac{\alpha}{k\lambda} = 1$, $\frac{1}{2} \phi = w$, then (3) becomes

$$\frac{dw(\xi)}{d\xi} = \sinh w(\xi).$$  

(4)

By using the solution of the equation (4), one can seek soliton solutions of nonlinear equations.

In this paper we would like to consider the case $c \neq 0$. In order to use (3) conveniently, we set $\phi = 2w$, $-\frac{\alpha}{k\lambda} = 1$, thus (3) reduces to

$$\left(\frac{dw}{d\xi} \right)^2 = \sinh^2 w + c, \quad \text{or} \quad \frac{dw}{d\xi} = \sqrt{\sinh^2 w + c},$$  

(5)

which is useful in the following method, where $c$ is a constant of integration.

If we take $c = 1 - m^2$, where $m(0 < m < 1)$ is the modulus of the Jacobi elliptic functions[18], then we know that (5) with $c = 1 - m^2$ has the general solution

$$\sinh[w(\xi)] = \cs(\xi; m),$$  

(6a)

or

$$\cosh[w(\xi)] = \ns(\xi; m),$$  

(6b)

which are Jacobi elliptic functions and have the properties

$$\frac{d\cs(\xi; m)}{d\xi} = -\ns(\xi; m)\ds(\xi; m), \quad \frac{d\ns(\xi; m)}{d\xi} = -\cs(\xi; m)\ds(\xi; m),$$  

where $\cs, \ns, \ds$ are the Jacobi elliptic functions.
Jacobi elliptic function solutions

\[ \text{ns}^2(\xi; m) = 1 + \text{cs}^2(\xi; m). \]  

(7)

In what follows we would like to use the solution (6) of (5) to construct Jacobi elliptic function solutions of nonlinear wave equations.

The rest of this paper is arranged as follows. In Sec. 2 we give the computational steps of the Method I, which is used for nonlinear ODEs with constant coefficients and nonlinear PDEs that can reduce to nonlinear ODEs with constant coefficients, and the Method II, which is used for nonlinear ODEs and nonlinear PDEs. In Sec. 3 we would like to use the Method I to construct doubly-periodic solutions of some complex nonlinear wave equations in (1+1)-dimensional and (2+1)-dimensional space, such the combined KdV-mKdV equation, (2+1)-dimensional coupled Davey-Stewartson equation, the new integrable Davey-Stewartson-type equation, the modified Boussinesq equation, (2+1)-dimensional mKP equation and the (2+1)-dimensional generalized KdV equation. Finally we given some conclusions in Sec. 4.

2. The sinh-Gordon equation expansion method and its algorithm

2.1 Method I- Seeking the travelling wave type of Jacobi elliptic function solutions

For a given nonlinear partial differential equation, say, in two variables \( x, t \)

\[ F(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \]  

(8)

we seek its travelling wave solution, if available, in the form \( u(x, t) = u(\xi), \xi = k(x - \lambda t). \) By using the new variable \( w = w(\xi), \) we assume that (8) has the solution in the form

\[ u(\xi) = u(w(\xi)) = A_0 + \sum_{i=1}^{n} \cosh^{i-1} w[A_i \sinh w + B_i \cosh w]. \]  

(9)

where \( A_i (i = 0, 1, \ldots, n), B_j (j = 1, 2, \ldots, n) \) are constants to be determined later and \( w = w(\xi) \) satisfies (5).

According to (7) and (9), we define a polynomial degree function as \( D(u(w)) = n, \) thus we have

\[ D\left( u^p(w) \left( \frac{d^q u(w)}{d \xi^s} \right)^q \right) = np + q(n + s). \]  

(10)

Therefore we can determine the parameter \( n \) by balancing the highest order derivative term with nonlinear terms in (8).

The method is summed up as the following steps:

**Step 1:** Reduce the given nonlinear equation to an ODE by using the travelling wave transformation \( u(x, t) = u(\xi), \xi = k(x - \lambda t). \)

**Step 2:** Determine the parameter \( n \) in (9) by balancing the highest order derivative terms and nonlinear terms and thus give the formal solution (9) (Remark: If \( n \) is not a positive integers, then we firstly make the transformation \( u = v^n, \) and then we perform the second step again.)

**Step 3:** Substitute (9) with the known \( n \) along with (5) into the obtained ODE and obtain a hyperbolic polynomial for \( u^p \sinh^q w \cosh^p w (i = 0, 1; s = 0, 1; j = 0, 1, 2, \ldots). \)
Step 4: Set to zero the coefficients of $w'\sinh w \cosh w(i = 0, 1; s = 0, 1; j = 0, 1, 2, ...)$ to get a set of algebraic equations with respect to the unknowns $k, \lambda, A_j(j = 0, 1, ..., n)$ and $B_j(j = 1, 2, ..., n)$.

Step 5: Solve the set of algebraic equations, which may not be consistent, finally derive the doubly periodic solutions of the given nonlinear equations by using the $u(x, t) = u(\xi), \xi = k(x - \lambda t)$ and the known solution (6).

Remark 1: The method is an indirect method which is used to find Jacobi elliptic function solutions of equations by using a transformation (9) and the target equation (5) with $c = 1 - m^2$ whose solutions are known. Bases on the symbolic computation, the procedure can be carried out in computer.

Remark 2: Because when $m \to 1$, $cs(\xi; m) \to \operatorname{csch} \xi$ and $ns(\xi; m) \to \coth \xi$; while $m \to 0$, $cs(\xi; m) \to \cot \xi$ and $ns(\xi; m) \to \csc \xi$, thus it is easy to see that the method is used to obtain both soliton solutions and Jacobi elliptic function solutions.

2.2 Method II-Seeking the non-travelling wave type of Jacobi elliptic function solutions

We know that Method I is only used for these nonlinear ODEs with constant coefficients or nonlinear partial differential equations that can be reduced to be the corresponding ODEs with constant coefficients by using some transformations, otherwise the Method I will not work. In order to overcome the disadvantage of the Method I, we change the Method I into a general form as follows:

If we set $\xi \to \psi(x, t)$, then (5)

$$\left( \frac{dw(\psi)}{d\psi} \right)^2 = \sinh^2 w(\psi) + c, \quad \text{or} \quad \frac{dw(\psi)}{d\psi} = \sqrt{\sinh^2 w(\psi) + c}. \quad (11)$$

Where $c$ is a constant of integration and $\psi(x, t)$ is an unknown function of $x, t$.

If we set $c = 1 - m^2 (0 < m < 1)$, then by solving (11), we know it has the solution

$$\sinh[w(\psi)] = \text{cs}[\psi(x, t); m], \quad (12a)$$
$$\cosh[w(\psi)] = \text{ns}[\psi(x, t); m]. \quad (12b)$$

For the given nonlinear partial differential equation (8), we do not need to firstly make the travelling wave transformation to reduce (8) to an ODE with constant coefficients. We can directly assume that (8) has the generalized formal solution

$$u(x, t) = A_0(x, t) + \sum_{i=1}^{n} \cosh^{i-1} w(\psi)[A_i(x, t) \sinh w(\psi) + B_i(x, t) \cosh w(\psi)], \quad (13)$$

where the $w = w(\psi)$ satisfies (11) with $c = 1 - m^2$ and $A_i(x, t)'s, B_i(x, t)'s$ and $\psi(x, t)$ are functions to be determined later. Similar to the steps mentioned in Method I. Substituting (13) with (11) into (8) yields a set of differential equations w.r.t $A_i's, B_i's$ and $\psi$. By solving the set of differential equations, if available, and using (12), we may can obtain more Jacobi elliptic function solutions. When the modulus $m \to 1$ or 0, we may obtain soliton-like solutions and more types of singly-periodic solutions.
Remark 3: If we take \( \psi(x, t) \) to be the form \( \psi(x, t) = \xi = k(x - \lambda t)(k, \lambda \text{ constants}) \), then the Method II reduces to the Method I. But if we can obtain the case that the function \( \psi(x, t) \) is not of the linearly combined form of \( x \) and \( t \), then we will have new Jacobi elliptic function solutions of (8).

Remark 4: We know that the Method I transforms (8) into a system of nonlinear algebraic equations (SNAEs) with respect to unknown variables, but the Method II transforms (8) into a system of nonlinear partial differential equations (SNPDEs) with respect to unknown variables. Generally speaking, solving the SNPDEs is more difficult than the SNAEs. Thus the Method II is more complicated to be used than the Method I.

Recently, we have found new Jacobi elliptic function solutions of some simple nonlinear equations[21]. In what follows we would like to apply the Method I to some more complicated nonlinear equations, such as the combined KdV-mKdV equation, \((2+1)\)-dimensional coupled Davey-Stewartson equation and \((2+1)\)-dimensional generalized KdV equation, etc.

As a consequence, some new Jacobi elliptic function solutions are obtained.

3. Some examples to illustrate the Method I and their solutions

In this section we illustrate our Method I to some nonlinear wave equations.

Example 3.1 the KdV-mKdV equation[1]

\[
\begin{align*}
  u_t + (\alpha + \beta u)u_x + u_{xxx} &= 0 \\
  &\text{(14)}
\end{align*}
\]

According to Step 1, under the travelling wave transformation \( u(x, t) = u(\xi), \xi = k(x - \lambda t), \) (7) reduces to be

\[
C - \lambda u + \frac{1}{2}\alpha u^2 + \frac{1}{3}\beta u^3 + k^2 \frac{d^2 u}{d\xi^2} = 0. \\
&\text{(15)}
\]

where \( C \) is the integration constant. Fan[20] gave some exact solutions. In what follows we will give other types of Jacobi elliptic function solutions.

According to Step 2, we assume that it has the solution

\[
\begin{align*}
  u(\xi) &= A_0 + A_1 \sinh w(\xi) + B_1 \cosh w(\xi), \\
  &\text{(16)}
\end{align*}
\]

and \( w \) satisfying (1.8) or (1.12), \( \xi = k(x - \lambda t) \).

By the aid of Maple, substituting (16) into (15) along with (5), we have the polynomial of \( w^5 \sinh^3 w \cos \omega w \). Setting their coefficients to zero yields a set of algebraic equations

\[
\begin{align*}
  1/3\beta B_1^2 + 2B_1k^2 + \beta A_1^2B_1k &= 0, \\
  \beta B_1^2A_1 + 1/3\beta A_1^3 + 2A_1k^2 &= 0, \\
  \beta A_0B_1^2 + 1/2\alpha A_1^2 + 1/2\alpha B_1^2 + \beta A_0A_1 &= 0, \\
  A_1k^2c - A_1\lambda + \alpha A_0A_1 + \beta A_0A_1 - 1/3\beta A_1^3 - A_1k^2 &= 0, \\
  -2B_1k^2 - B_1\lambda + \alpha A_0B_1 - \beta A_1^2B_1 + B_1k^2c + \beta A_0^2B_1 &= 0, \\
  \alpha A_1B_1 + 2\beta A_0A_1B_1 &= 0, \\
  -1/2\alpha A_1^2 - \lambda A_0 + C - \beta A_0A_1^2 + 1/2\alpha A_0^2 + 1/3\beta A_0^3. \\
  &\text{(17)}
\end{align*}
\]
From which we have

\[ A_0 = -\frac{\alpha}{2\beta}, \quad A_1 = 0, \quad B_1 = \pm \sqrt{\frac{6(4\lambda\beta + \alpha^2)}{4\beta^2(1 + m^2)}}, \quad k = \pm \sqrt{\frac{4\lambda\beta + \alpha^2}{4\beta(1 + m^2)}}, \quad (18) \]

\[ A_0 = -\frac{\alpha}{2\beta}, \quad A_1 = 0, \quad A_2 = \pm 6\frac{(4\lambda\beta + \alpha^2)}{4\beta^2(2 - m^2)}, \quad k = \pm \sqrt{\frac{4\lambda\beta + \alpha^2}{4\beta(2 - m^2)}}, \quad (19) \]

\[ A_0 = -\frac{\alpha}{2\beta}, \quad A_2^2 = B_1^2 A_1 = \pm \sqrt{\frac{3(4\lambda\beta + \alpha^2)}{4\beta^2(2m^2 - 1)}}, \quad k = \pm \sqrt{-\frac{4\lambda\beta + \alpha^2}{2\beta(1 - 2m^2)}}, \quad (20) \]

Therefore we have three new Jacobi elliptic function solutions

\[ u_1 = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{6(4\lambda\beta + \alpha^2)}{4\beta^2(1 + m^2)}} \csch \left( \sqrt{-\frac{4\lambda\beta + \alpha^2}{4\beta(1 + m^2)}} \xi \right), \quad (21) \]

\[ u_2 = -\frac{\alpha}{2\beta} \pm \sqrt{-\frac{6(4\lambda\beta + \alpha^2)}{4\beta^2(2 - m^2)}} \text{ns} \left( \sqrt{\frac{4\lambda\beta + \alpha^2}{4\beta(2 - m^2)}} \xi \right), \quad (22) \]

\[ u_3 = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{3(4\lambda\beta + \alpha^2)}{4\beta^2(2m^2 - 1)}} \left[ \text{cs} (k\xi) \pm \text{ns} (k\xi) \right], \quad k = \sqrt{-\frac{4\lambda\beta + \alpha^2}{2\beta(1 - 2m^2)}}, \quad (23) \]

In particular (1) when the modulus \( m \to 1 \), we have the soliton solutions from (21)-(23)

\[ u_4 = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{3(4\lambda\beta + \alpha^2)}{4\beta^2}} \text{csch} \left( \sqrt{-\frac{4\lambda\beta + \alpha^2}{8\beta}} \xi \right), \]

\[ u_5 = -\frac{\alpha}{2\beta} \pm \sqrt{-\frac{6(4\lambda\beta + \alpha^2)}{4\beta^2}} \text{coth} \left( \sqrt{\frac{4\lambda\beta + \alpha^2}{4\beta}} \xi \right), \]

\[ u_6 = \frac{\alpha}{2\beta} \pm \sqrt{\frac{3(4\lambda\beta + \alpha^2)}{4\beta^2}} \left[ \text{csch} (k\xi) \pm \text{coth} (k\xi) \right], \quad k = \sqrt{\frac{4\lambda\beta + \alpha^2}{2\beta}}, \]

which is singular soliton solutions that imply that for the certain time \( t = t_0 \), these solutions blow up at the point \( x = x_0 \).

(2) when the modulus \( m \to 0 \), we have the singly-periodic solutions from (21)-(23)

\[ u_7 = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{6(4\lambda\beta + \alpha^2)}{4\beta^2}} \cot \left( \sqrt{-\frac{4\lambda\beta + \alpha^2}{4\beta}} \xi \right), \]

\[ u_8 = -\frac{\alpha}{2\beta} \pm \sqrt{-\frac{6(4\lambda\beta + \alpha^2)}{8\beta^2}} \csc \left( \sqrt{\frac{4\lambda\beta + \alpha^2}{8\beta}} \xi \right), \]

\[ u_9 = -\frac{\alpha}{2\beta} \pm \sqrt{-\frac{3(4\lambda\beta + \alpha^2)}{4\beta^2}} \left[ \text{csc} (k\xi) \pm \cot (k\xi) \right], \quad k = \sqrt{\frac{4\lambda\beta + \alpha^2}{2\beta}}. \]
Example 3.2: \((2+1)\)-dimensional coupled Davey-Stewartson equation\cite{1,22}

\[iu_t + u_{xx} - u_{yy} - 2|u|^2u - 2uv = 0,\]
\[v_{xx} + v_{yy} + 2(|u|^2)_{xx} = 0.\]  \((24)\)

Fan\cite{20} gave three Jacobi elliptic function solutions. In what follows we will obtain other types of Jacobi elliptic function solutions. We first introduce the transformations

\[u(x,t) = \exp(i\eta)u(\xi), \quad v(x,t) = v(\xi), \quad \eta = \alpha x + \beta y + \gamma t, \quad \xi = k(x + py - \lambda t),\]  \((25)\)

where \(\alpha, \beta, \gamma, k, p, \lambda\) are constants to be determined later.

Substituting \((25)\) into \((24)\) we have

\[k^2(1 - p^2)u'' + (-\gamma - \alpha^2 + \beta^2)u - 2u^3 - 2uv = 0,\]
\[(1 + p^2)v'' + 2(u^2)'' = 0,\]
\[\lambda = 2(\alpha - \beta p).\]  \((26)\)

Assume that \((26)\) has the solutions by using the Method I

\[u(\xi) = A_0 + A_1 \sinh w + B_1 \cosh w,\]
\[v(\xi) = a_0 + a_1 \sinh w + b_1 \cosh w + a_2 \sinh w \cosh w + b_2 \cosh^2 w.\]  \((27)\)

where \(A_0, A_1, B_1, a_0, a_1, a_2, b_1, b_2\) are constants to be determined later. According to the Step 3-5 in the Method I, we have the three new Jacobi elliptic function solutions

\[u_1 = k\sqrt{1 + p^2} \exp(i\eta) \text{cs}\xi, \quad v_1 = -2k^2 \text{cs}^2 \xi + \frac{C}{1 + p^2},\]  \((28)\)

where \(\eta = \alpha x + \beta y + \gamma t, \quad \xi = k(x + py - 2(\alpha - \beta) t), \quad k = \sqrt{\frac{2C/(1+p^2) + \gamma + \alpha^2 - \beta^2}{(2 - m^2)(1 - p^2)}}\) and \(C\) is an arbitrary constant.

\[u_2 = k\sqrt{1 + p^2} \exp(i\eta) \text{ns}\xi, \quad v_2 = -2k^2 \text{ns}^2 \xi + \frac{C}{1 + p^2},\]  \((29)\)

where \(\eta = \alpha x + \beta y + \gamma t, \quad \xi = k(x + py - 2(\alpha - \beta) t), \quad k = \sqrt{\frac{2C/(1+p^2) + \gamma + \alpha^2 - \beta^2}{(1 + m^2)(p^2 - 1)}}\) and \(C\) is an arbitrary constant.

\[u_3 = \frac{1}{2} k\sqrt{1 + p^2} \exp(i\eta)[\text{cs} \pm \text{ns}]\xi, \quad v_3 = -k^2[\text{cs}^2 \xi \pm \text{cs} \text{ns} \xi] - \frac{1}{2} k^2 + \frac{C}{1 + p^2},\]  \((30)\)

where \(\eta = \alpha x + \beta y + \gamma t, \quad \xi = k(x + py - 2(\alpha - \beta) t), \quad k = \sqrt{\frac{2(2C/(q+p^2) + \gamma + \alpha^2 - \beta^2)}{(1-2m^2)(1-p^2)}}\) and \(C\) is an arbitrary constant.

Similar to Example 1, when the modulus \(m \to 1\) or 0, we can also get the soliton solutions and singly-periodic solutions. We omit them here.
Example 3.3: The new integrable Davey-Stewartson-type equation\cite{23}

\[
\begin{align*}
\imath \Psi_\tau + L_1 \Psi + \Psi \Phi + \Psi \chi &= 0, \\
L_2 \chi &= L_3 |\Psi|^2, \\
\Phi_\xi &= \chi_\eta + \mu |\Psi|^2, \\
\mu &= \mp 1. 
\end{align*}
\]

where the linear differential operators are given by

\[
\begin{align*}
L_1 &= \frac{b^2 - a^2}{4} \frac{\partial^2}{\partial \xi^2} - a \frac{\partial^2}{\partial \xi \partial \eta} - \frac{\partial^2}{\partial \eta^2}, \\
L_2 &= \frac{b^2 + a^2}{4} \frac{\partial^2}{\partial \xi^2} + a \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}, \\
L_3 &= \pm \frac{1}{4} \left( b^2 + a^2 + \frac{8b^2(a - 1)}{(a - 2)^2 - b^2} \right) \frac{\partial^2}{\partial \xi^2} \pm \left( a + \frac{2b^2}{(a - 2)^2 - b^2} \right) \frac{\partial^2}{\partial \xi \partial \eta} \pm \frac{\partial^2}{\partial \eta^2},
\end{align*}
\]

\(a, b\) are real parameters, and \(\Psi = \Psi(\xi, \eta, \tau)\) is complex while \(\Phi = \Phi(\xi, \eta, \tau), \chi = \chi(\xi, \eta, \tau)\) are real. This equation was presented firstly by Maccari\cite{23} from the Konopelchenko-Dubrovsky (KD) equation\cite{24} by using the reduction method.

We first introduce the transformations

\[
\begin{align*}
\Psi(\xi, \eta, \tau) &= \Psi(X) \exp(iY), \\
\Phi(\xi, \eta, \tau) &= \Phi(X), \\
\chi(\xi, \eta, \tau) &= \chi(X), \\
X &= kZ = k(\xi + \eta + \lambda \tau), \\
Y &= \alpha \xi + \beta \eta + \gamma \tau. 
\end{align*}
\]

where \(k, l, \lambda, \alpha, \beta, \gamma\) are constants to be determined later.

Substituting (33) into (31), we have

\[
\begin{align*}
&k^2 M_1 \frac{d^2 \Psi(X)}{dX^2} + M_0 \Psi(X) + \Psi(X) \Phi(X) + \Psi(X) \chi(X) = 0, \\
&k^2 M_2 \frac{d^2 \chi(X)}{dX^2} = k^2 M_3 \frac{d^2 \Psi^2(X)}{dX^2}, \\
&k \frac{d\Phi(X)}{dX} = kl \frac{d\chi(X)}{dX} + \mu kl \frac{d\Psi^2(X)}{dX},
\end{align*}
\]

under the condition

\[
\lambda = -\frac{\alpha}{4} (b^2 - a^2) + a(\beta + \alpha l) + 2l \beta,
\]

where

\[
\begin{align*}
M_0 &= -\gamma - \frac{1}{4} \alpha^2 (b^2 - a^2) + a \alpha \beta + \beta^2, \\
M_1 &= \frac{b^2 - a^2}{4} - al - l^2, \\
M_2 &= \frac{b^2 + a^2}{4} + al + l^2, \\
M_3 &= \pm \frac{1}{4} (b^2 + a^2 + \frac{8b^2(a - 1)}{(a - 2)^2 - b^2}) \pm \left( a + \frac{2b^2}{(a - 2)^2 - b^2} \right) l \pm l^2.
\end{align*}
\]
According to the Method I, we assume that (34a) has the solution in the form

\[
\Psi(X) = A_0 + A_1 \sinh w + B_1 \cosh w, \\
\Phi(\xi) = a_0 + a_1 \sinh w + b_1 \cosh w + a_2 \sinh w \cosh w + b_2 \cosh^2 w, \\
\chi(\xi) = e_0 + e_1 \sinh w + f_1 \cosh w + e_2 \sinh w \cosh w + f_2 \cosh^2 w. 
\]

where \(A_0, A_1, B_1, a_0, a_1, a_2, b_1, b_2, c_0, e_1, e_2, f_1, f_2\) are constants to be determined later. According to the Step 3-5 in the Method I, we have the three new Jacobi elliptic function solutions

\[
\begin{align*}
\Psi_1 &= \sqrt{-\frac{2(M_0 + c_1 + l c_1 + c_2)M_2}{(1 + m^2)(M_3 + l M_3 + \mu M_2)^{\frac{1}{2}}}} \sqrt{\frac{M_0 + c_1 + l c_1 + c_2}{(1 + m^2)M_1}} Z \exp(iY), \\
\chi_1 &= -\frac{2M_0(M_0 + c_1 + l c_1 + c_2)}{(1 + m^2)(M_2 + l M_2 + \mu M_3)^{\frac{1}{2}}^{\frac{1}{2}}} \sqrt{\frac{M_0 + c_1 + l c_1 + c_2}{(1 + m^2)M_1}} Z + c_1, \\
\Phi_1 &= -\frac{2(M_0 + c_1 + l c_1 + c_2)M_2}{(1 + m^2)(M_3 + l M_3 + \mu M_2)^{\frac{1}{2}}} \sqrt{\frac{M_0 + c_1 + l c_1 + c_2}{(1 + m^2)M_1}} Z + c_1 + c_2, \\
\Psi_2 &= \sqrt{-\frac{2(M_0 + c_1 + l c_1 + c_2)M_2}{(m^2 - 2)(M_3 + l M_3 + \mu M_2)^{\frac{1}{2}}}} \sqrt{\frac{M_0 + c_1 + l c_1 + c_2}{(m^2 - 2)M_1}} Z \exp(iY), \\
\chi_2 &= -\frac{2M_0(M_0 + c_1 + l c_1 + c_2)}{(m^2 - 2)(M_2 + l M_2 + \mu M_3)^{\frac{1}{2}}^{\frac{1}{2}}} \sqrt{\frac{M_0 + c_1 + l c_1 + c_2}{(m^2 - 2)M_1}} Z + c_1, \\
\Phi_2 &= -\frac{2(M_0 + c_1 + l c_1 + c_2)M_2}{(m^2 - 2)(M_3 + l M_3 + \mu M_2)^{\frac{1}{2}}} \sqrt{\frac{M_0 + c_1 + l c_1 + c_2}{(m^2 - 2)M_1}} Z + c_1 + c_2, \\
\Psi_3 &= \sqrt{-\frac{(M_0 + c_1 + l c_1 + c_2)M_2}{(2m^2 - 1)(M_3 + l M_3 + \mu M_2)^{\frac{1}{2}}}} \sqrt{\frac{2(M_0 + c_1 + l c_1 + c_2)}{(2m^2 - 1)M_1}} Z \\
&\quad \pm \sqrt{\frac{2(M_0 + c_1 + l c_1 + c_2)}{(2m^2 - 1)M_1}} Z \exp(iY), \\
\chi_3 &= -\frac{(M_0 + c_1 + l c_1 + c_2)M_3}{(2m^2 - 1)(M_3 + l M_3 + \mu M_2)^{\frac{1}{2}}} \sqrt{\frac{2(M_0 + c_1 + l c_1 + c_2)}{(2m^2 - 1)M_1}} Z + 1, \\
\Phi_3 &= -\frac{(M_0 + c_1 + l c_1 + c_2)M_3}{(2m^2 - 1)(M_3 + l M_3 + \mu M_2)^{\frac{1}{2}}} \sqrt{\frac{2(M_0 + c_1 + l c_1 + c_2)}{(2m^2 - 1)M_1}} Z + 1 \\
&\quad \pm \sqrt{\frac{2(M_0 + c_1 + l c_1 + c_2)}{(2m^2 - 1)M_1}} Z \exp(iY) \sqrt{\frac{2(M_0 + c_1 + l c_1 + c_2)}{(2m^2 - 1)M_1}} Z + c_1, \\
\end{align*}
\]
Example 3.4: The modified Boussinesq equation[25]

\[ P_t = (Q - \frac{3}{2} k^2 P_x^2), \]

\[ Q_t = -3k^2(P_{xx} - PQ + k^2 P^3)_x. \]  

(40)

where \( k \) is a constant.

Under the travelling wave transformation

\[ P(x, t) = P(\xi), \quad Q(x, t) = Q(\xi), \quad \xi = \alpha(x + \lambda t), \]

(41)

(40) reduces to

\[ \lambda \frac{dP}{d\xi} = \frac{dQ}{d\xi} - \frac{3}{2} k^2 \frac{dP^2}{d\xi}, \]

\[ \lambda \frac{dQ}{d\xi} = -3k^2(\alpha^2 \frac{d^3P}{d\xi^3} - \frac{d(PQ)}{d\xi} + k^2 \frac{dP^3}{d\xi}). \]

(42)

Assume that (42) has the solutions by using the Method I

\[ P(\xi) = A_0 + A_1 \sinh w + B_1 \cosh w, \]

\[ Q(\xi) = a_0 + a_1 \sinh w + b_1 \cosh w + a_2 \sinh w \cosh w + b_2 \cosh^2 w. \]

(43)

where \( A_0, A_1, B_1, a_0, a_1, a_2, b_1, b_2 \) are constants to be determined later. According to the Step 3-5 in the Method I, we have the three new Jacobi elliptic function solutions

\[ P_1 = \sqrt{\frac{2(\lambda^2 - 2k^2 c_1)}{k^4(1 + m^2)}} \text{ns}\left[ \sqrt{\frac{\lambda^2 - 2k^2 c_1}{2k^2(1 + m^2)}}(x + \lambda t) \right] - \frac{\lambda}{3k^2}, \]  

(44a)

\[ Q_1 = \frac{3(\lambda^2 - 2k^2 c_1)}{k^2(1 + m^2)} \text{ns}^2\left[ \sqrt{\frac{\lambda^2 - 2k^2 c_1}{2k^2(1 + m^2)}}(x + \lambda t) \right] - \frac{\lambda^2}{6k^2} + c_1. \]  

(44b)

\[ P_2 = \sqrt{\frac{2(\lambda^2 - 2k^2 c_1)}{k^4(2m^2 - 2)}} \text{cs}\left[ \sqrt{\frac{\lambda^2 - 2k^2 c_1}{2k^2(2m^2 - 2)}}(x + \lambda t) \right] - \frac{\lambda}{3k^2}, \]  

(45a)

\[ Q_2 = \frac{3(\lambda^2 - 2k^2 c_1)}{k^2(2m^2 - 2)} \text{cs}^2\left[ \sqrt{\frac{\lambda^2 - 2k^2 c_1}{2k^2(2m^2 - 2)}}(x + \lambda t) \right] - \frac{\lambda^2}{6k^2} + c_1. \]  

(45b)

\[ P_3 = \sqrt{\frac{\lambda^2 - 2k^2 c_1}{k^4(2m^2 - 1)}} \text{ns}\left[ \sqrt{\frac{\lambda^2 - 2k^2 c_1}{k^2(2m^2 - 1)}}(x + \lambda t) \right] \pm \text{cs}\left[ \sqrt{\frac{\lambda^2 - 2k^2 c_1}{k^2(2m^2 - 1)}}(x + \lambda t) \right] - \frac{\lambda}{3k^2}, \]  

(46a)

\[ Q_3 = \frac{3(\lambda^2 - 2k^2 c_1)}{2k^2(2m^2 - 1)} \text{cs}^2\left[ \sqrt{\frac{\lambda^2 - 2k^2 c_1}{k^2(2m^2 - 1)}}(x + \lambda t) \right] \]  

\[ \pm 2\text{ns}\left[ \sqrt{\frac{\lambda^2 - 2k^2 c_1}{k^2(2m^2 - 1)}}(x + \lambda t) \right] \text{cs}\left[ \sqrt{\frac{\lambda^2 - 2k^2 c_1}{k^2(2m^2 - 1)}}(x + \lambda t) + 1 \right] - \frac{\lambda^2}{6k^2} + c_1, \]  

(46b)

where \( c_1 \) is a constant.
Example 3.5: (2+1)-dimensional mKP equation[24, 26]

\[ q_t + \frac{1}{8}(q_{xxx} - 6q_x^2q_x + 6q_xq_y^{-1}q_y + 3q_x^{-1}q_{yy}) = 0. \]  

(47)

Dai[26] used a direct method to decompose (47) into two (1+1)-dimensional soliton equations. By using the Darboux transformation of the two (1+1)-dimensional soliton equations, some soliton solutions of (47) were obtained. In what follows we consider its Jacobi elliptic function solutions.

According to the Method I, we can obtain three new Jacobi elliptic function solutions of (47)

\[ q_1 = \sqrt{\frac{9/2l^2 + 8\lambda}{1 + m^2}} \text{ns}\left[ \sqrt{\frac{9/2l^2 + 8\lambda}{1 + m^2}}(x + ly + \lambda t) \right] + \frac{1}{2} l. \]  

(48)

\[ q_2 = \sqrt{\frac{9/2l^2 + 8\lambda}{m^2 - 2}} \text{cs}\left[ \sqrt{\frac{9/2l^2 + 8\lambda}{m^2 - 2}}(x + ly + \lambda t) \right] + \frac{1}{2} l. \]  

(49)

\[ q_3 = \sqrt{\frac{9/2l^2 + 8\lambda}{2(2m^2 - 1)}} \text{ns}\left[ \sqrt{\frac{9/2l^2 + 16\lambda}{2m^2 - 1}}(x + ly + \lambda t) \right] \pm \text{cs}\left[ \sqrt{\frac{9/2l^2 + 16\lambda}{2m^2 - 1}}(x + ly + \lambda t) \right] + \frac{1}{2} l. \]  

(50)

Example 3.6: The (2+1)-dimensional generalized KdV equation[27]

\[ u_t + a_1 u_{xxx} + a_2 u_{yyy} + a_3 u_x + a_4 u_y - 3a_1(u_x^{-1}u_x)_x - 3a_2(u_x^{-1}u_y)_y = 0. \]  

(51)

which is obtained by Boiti[27] form the general equation, where \(a_i(i = 1, 2, 3, 4)\) are constants.

When \(a_3 = a_4 = 0\), (22) reduces to be the Nizhnhik-Novikov-Veselov equation[28].

According to the above-mentioned method I, we can obtain two new types of Jacobi elliptic function solutions

\[ u_1 = 2k^2p \cos^2 \xi - \frac{p(a_4p - \lambda + (4a_2k^2p^3 + 4a_1k^2)(2 - m^2) + a_3)}{6(a_1 + a_2p^3)}. \]  

(52)

\[ u_2 = k^2p(\cos^2 \xi \pm \cos \xi \sin \xi) + \frac{1}{2} k^2p - \frac{p(a_4p - \lambda + (2a_2k^2p^3 + 2a_1k^2)(1 - 2m^2) + a_3)}{6(a_1 + a_2p^3)}. \]  

(53)

Similar to Example 1, when the modulus \(m \to 1\) or 0, we can also get the soliton solutions and singly-periodic solutions.

\[ u_3 = 2k^2p \cosh^2 \xi - \frac{p(a_4p - \lambda + 4a_2k^2p^3 + 4a_1k^2 + a_3)}{6(a_1 + a_2p^3)}, \]  

(54)

\[ u_4 = 2k^2p \cot^2 \xi - \frac{p(a_4p - \lambda + 8a_2k^2p^3 + 8a_1k^2 + a_3)}{6(a_1 + a_2p^3)}, \]  

(55)

\[ u_5 = k^2p(\cosh^2 \xi \pm \cosh \xi \sinh \xi) + \frac{1}{2} k^2p - \frac{p(a_4p - \lambda - (2a_2k^2p^3 + 2a_1k^2) + a_3)}{6(a_1 + a_2p^3)}, \]  

(56)

\[ u_6 = k^2p(\cot^2 \xi \pm \csc \xi \cot \xi) + \frac{1}{2} k^2p - \frac{p(a_4p - \lambda + (2a_2k^2p^3 + 2a_1k^2) + a_3)}{6(a_1 + a_2p^3)}. \]  

(57)
where $\xi = k(x + py - \lambda t)$.

4. Conclusions

In summary, based on the second-order sinh-Gordon equation, we have developed a new method (called sinh-Gordon equation expansion method). The Method I is used to the combined KdV-mKdV equation, (2+1)-dimensional coupled Davey-Stewartson equation, the new integrable Davey-Stewartson-type equation, a modified Boussinesq equation, (2+1)-dimensional mKdV equation and the (2+1)-dimensional generalized KdV equation such that some new solutions are obtained. When the module $m \to 1$ or 0, the corresponding solitary waves and singly periodic solutions are also found. Therefore it is easily seen that the method is simple and powerful as well as can be carried out in computer with the aid of symbolic computation. But we know that the method I is only used to these nonlinear ODEs with constant coefficients and nonlinear PDEs that can reduce to nonlinear ODEs with constant coefficients. If the Method I does not work for some nonlinear differential equations, then we may use the Method II to solve them. When one applies the Method II to nonlinear differential equations, it is complex to solve. Sometime we may not obtain the more results by using Method II than Method I. About the applications of Method II, we will give some examples in future.

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References