Entropy of Partitions on Quantum Logic

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Abstract Partition and entropy of partitions in quantum logic are introduced and their properties are investigated. The results are generalized to the general case of T-norm and T-conorm.

1. Introduction

In the framework of axiomatic approach known as the quantum logic approach, it is usually assumed that a “quantum logic”, that is, a mathematical representation of the set of all experimentally verifiable propositions about a physical system (or equivalently, the set of all random events of a physical experiment), is a σ-orthomodular lattice, with a full set of states. And the entropy, as a tool to measure the amount of uncertainty in random events, has been applied in many fields. Using the notion of the state of quantum logic, we can introduce the entropy of partitions in quantum logic, which is a useful tool in studying the isomorphism of dynamical systems and has been applied in many other structures [3], [4], [9], [10].

In this paper we introduce the corresponding definitions of the partition and the entropy of partitions in quantum logic and investigate the entropy of partition on general T-norm and T-conorm (For the concept of T-norm and T-conorm, please refer to [5]).

Definition 1 A quantum logic (QL) is a σ-orthomodular lattice, i.e., a lattice L (L, ≤, ∨, ∧, 0, 1) which contains the smallest element 0 and the greatest element 1, with the following properties:

(i) L carries a bijective map a → a′ such that ∀a, b ∈ L, a′′ = a, a ≤ b ⇒ b′ ≤ a′, a ∨ a′ = 1, a ∧ a′ = 0;

(ii) Given any countable sequence (aᵢ)ᵢ∈I ⊆ L, aᵢ ≤ aᵢ′, ∀i ≠ j, the join ∨ᵢ∈Iaᵢ exists in L;

(iii) L is orthomodular: a ≤ b ⇒ b = a ∨ (b ∧ a′).

Two elements a, b in a QL are called orthogonal (denoted as a ⊥ b) if a ≤ b′. A family (aᵢ)ᵢ∈I is orthogonal if aᵢ ⊥ aⱼ whenever i ≠ j.

Definition 2 A (σ-additive) state on a QL L is a map s : L → [0, 1] such that s(1) = 1, and for any orthogonal sequence (aᵢ)ᵢ∈I, s(∨ᵢ∈Iaᵢ) = ∑ᵢ∈Is(aᵢ).

A finite system P = (a₁, a₂, · · · , aₙ) of elements of the QL L is said to be a ∨-orthogonal system if

(∨ᵢ∈Iaᵢ) ⊥ a_k₊₁, for k = 1, 2, · · · , n − 1.
Lemma 1 For any $\lor$-orthogonal system $P = (a_1, a_2, \cdots, a_n)$ and any state $s$ of QL $L$, it holds that

$$s(\lor_{i=1}^na_i) = \sum_{i=1}^ns(a_i).$$

Proof: Since $(\lor_{i=1}^ka_i \perp a_{k+1},$ for $k = 1, 2, \cdots, n - 1,$ we have $\lor_{i=1}^ka_i \leq a'_k$. Therefore $a_i \leq a'_k$ for $i \leq k,$ i.e., $a_i \perp a'_k$ for $i \leq k, k = 1, 2, \cdots, n - 1.$ We have $a_i \perp a_j$ for $i < j,$ that is, $\forall i \neq j, a_i \perp a_j.$ From the definition of a state, we have $s(\lor_{i \in I}a_i) = \sum_{i \in I}s(a_i).$ □

2. The Entropy of Partition

Now we can introduce a partition on a quantum logic. Let $L$ be a quantum logic and let $s$ be a state on $L$.

Definition 3 A system $P = (a_1, a_2, \cdots, a_n) \subset L$ is said to be the partition of $L$ corresponding to the state $s$ iff

(1) $P$ is a $\lor$-orthogonal system;
(2) $s(\lor_{i=1}^na_i) = 1$.

Let the system $(b_1, b_2, \cdots, b_m)$ be any partition corresponding to a state $s$ and $a \in L$. We say the state $s$ has Bayes’ Property if

$$s(\lor_{i=1}^m(a \land b_i)) = s(a).$$

Lemma 2 Let $Q = (b_1, b_2, \cdots, b_m)$ be a partition on $L$, $a \in L$, and the state $s$ has Bayes’ Property. Then

$$\sum_{i=1}^ms(a \land b_i) = s(a).$$

Proof: We first show that $(a \land b_1, a \land b_2, \cdots, a \land b_m)$ is a $\lor$-orthogonal system. Set $c_i = a \land b_i, i = 1, 2, \cdots, m.$ We need to prove that $c_1 \perp c_2, (c_1 \lor c_2) \perp c_3, \cdots, (c_1 \lor c_2 \lor \cdots \lor c_{m-1}) \perp c_m.$ From the $\lor$-orthogonality of the system $Q$, we have $b_1 \leq b_3$. Using the monotonicity of operations $\lor$ and $\land$, we obtain $a \land b_1 \leq b_1 \leq b_3 \leq a' \lor b'_2 = a \land b_2$. Hence $c_1 \perp c_2$. Similarly, we can prove that $(c_1 \lor c_2) \perp c_3$, which is equivalent to $(a \land b_1) \lor (a \land b_2) \leq (a \land b_3)'$. Accounting to that $(a \land b_3)' = a' \lor b'_3$ and $b_1 \lor b_2 \leq b'_3$ ($Q$ is $\lor$-orthogonal), we can write $(a \land b_1) \lor (a \land b_2) \leq b_1 \lor b_2 \leq b'_3 \leq a' \lor b'_3$ = $(a \land b_3)'$. So $(c_1 \lor c_2) \perp c_3$.

Second, for the $\lor$-orthogonal system $Q$, from Lemma 1 and Bayes’ Property of a state $s$, we have that

$$\sum_{i=1}^ms(a \land b_i) = s(\lor_{i=1}^m(a \land b_i)) = s(a).$$

Definition 4 Let the system $P = (a_1, a_2, \cdots, a_n)$ be a partition of a QL $L$ corresponding to a state $s$. Then the entropy $H_s$ of the partition $P$ with respect to state $s$ is defined by

$$H_s(P) = -\sum_{i=1}^ns(a_i)\log s(a_i)$$

with $0 \log 0 = 0.$
Definition 5  Let $P = (a_1, a_2, \ldots, a_n)$ and $Q = (b_1, b_2, \ldots, b_m)$ be two partitions of QL $L$ corresponding to a state $s$. Then the common refinement of these partitions is defined as the system

$$P \cup Q = (a_i \land b_j; a_i \in P, b_j \in Q, \; i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, m).$$

Lemma 3  If the state $s$ has Bayes’ Property, $P$ and $Q$ are partitions defined above, then the system $P \cup Q$ is also a partition of $L$.

Proof: Let $P = (a_1, a_2, \ldots, a_n)$ and $Q = (b_1, b_2, \ldots, b_m)$ be partitions of QL corresponding to a state $s$. Set $c_{ij} = a_i \land b_j$, $i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, m$. Similar to the proof in the first part of lemma 2, we can prove that the system $P \cup Q = (c_{ij}; i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ is $\lor$-orthogonal. And from the Bayes’ Property of the state $s$ and the $\lor$-orthogonal of the system $P \cup Q$, we have $s(\lor_{i=1}^n a_i \land b_j) = s(\lor_{j=1}^m (a_i \land b_j)) = \sum_{i=1}^n s(\lor_{j=1}^m (a_i \land b_j)) = \sum_{i=1}^n s(a_i) = s(\lor_{i=1}^n a_i) = 1$. \(\Box\)

Theorem 1  Let $P$ be partitions of a QL $L$ corresponding to a state $s$ having Bayes’ Property. Then

$$H_s(P \cup Q) \leq H_s(P) + H_s(Q).$$

Proof: Assume $P = (a_1, a_2, \ldots, a_n)$, $Q = (b_1, b_2, \ldots, b_m)$ be partitions of QL corresponding to a state $s$. First we define the condition state. Let $a, b \in L$ and $s$ be a state on $L$. Then the condition state is given by

$$s(a/b) = \begin{cases} 0, & s(b) = 0 \\ s(a \land b)/s(b), & s(b) > 0 \end{cases}$$

Set $f(x) = x \log x, \; x \geq 0 (f(0) = 0)$. From the convexity of the function $f(x)$ and Jensen’s inequality we have

$$f(\sum_{j=1}^m \alpha_j x_j) \leq \sum_{j=1}^m \alpha_j f(x_j), \quad (1)$$

where $\sum_{j=1}^m \alpha_j = 1$ and $\alpha_j, x_j \in [0, 1]$. Let $\alpha_j = s(b_j)$ and $x_j = s(a_i/b_j)$, $j = 1, 2, \ldots, m$. We have $\alpha_j, x_j \in [0, 1], \; \sum_{j=1}^m \alpha_j = \sum_{j=1}^m s(b_j) = 1$. Using the definition of a conditional state and lemma 2, we have the relations

$$\sum_{j=1}^m \alpha_j x_j = \sum_{j=1}^m s(b_j)s(a_i/b_j) = \sum_{j=1}^m s(b_j) s(a_i \land b_j) = \sum_{j=1}^m s(a_i \land b_j) = s(a_i)$$

and

$$f(\sum_{j=1}^m \alpha_j x_j) = f(s(a_i)), \; \forall i = 1, 2, \ldots, n.$$
Similarly, we have
\[
\sum_{j=1}^{m} \alpha_j f(x_j) = \sum_{j=1}^{m} s(b_j)f(s(a_i/b_j)) = \sum_{j=1}^{m} s(b_j)s(a_i/b_j) \log(s(a_i/b_j)) = \sum_{j=1}^{m} s(b_j) \frac{s(a_i \land b_j)}{s(b_j)} \log \frac{s(a_i \land b_j)}{s(b_j)} = \sum_{j=1}^{m} s(a_i \land b_j)(\log s(a_i \land b_j) - \log s(b_j)) = \sum_{j=1}^{m} s(a_i \land b_j) \log s(a_i \land b_j) - \sum_{j=1}^{m} s(a_i \land b_j) \log s(b_j) = \sum_{j=1}^{m} f(s(a_i \land b_j)) - \sum_{j=1}^{m} s(a_i \land b_j) \log s(b_j).
\]

Then the inequality (1) has the form
\[
f(s(a_i)) \leq \sum_{j=1}^{m} f(s(a_i \land b_j)) - \sum_{j=1}^{m} s(a_i \land b_j) \log s(b_j), \text{ for } i = 1, 2, \ldots, n.
\]

Summarizing, we obtain
\[
\sum_{i=1}^{n} f(s(a_i)) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} f(s(a_i \land b_j)) - \sum_{i=1}^{n} \sum_{j=1}^{m} s(a_i \land b_j) \log s(b_j).
\]

From lemma 2, we have \(\sum_{i=1}^{m} s(a_i \land b_j) = s(b_j), \text{ } j = 1, 2, \ldots, m. \) Therefore
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} s(a_i \land b_j) \log s(b_j) = \sum_{j=1}^{m} \left( \sum_{i=1}^{n} s(a_i \land b_j) \right) \log s(b_j) = \sum_{j=1}^{m} f(s(b_j)).
\]

Hence
\[
\sum_{i=1}^{n} f(s(a_i)) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} f(s(a_i \land b_j)) - \sum_{j=1}^{m} f(s(b_j)).
\]

Therefore
\[
-H_s(P) \leq -H_s(P \cup Q) + H_s(Q), \quad H_s(P \cup Q) \leq H_s(P) + H_s(Q).
\]

\(\square\)

**Corollary 1** Let \(P_1, P_2, \ldots, P_k\) be partitions of a QL \(L\) corresponding to a state \(s\) which has Bayes's Property. Then
\[
H_s(P_1 \cup P_2 \cup \cdots \cup P_k) \leq \sum_{i=1}^{k} H_s(P_i),
\]

where \(P_1 \cup P_2 \cup \cdots \cup P_k\) is the common refinement of partitions \(P_1, P_2, \ldots, P_k\) as in definition 5.
3. General Case

The above discussions on the properties of partitions and entropy is in fact not restricted to the case of Quantum Logic. It can be generalized to more general cases.

Assume $L$ is a partial ordered set (poset) $(L, \leq)$ with a smallest element $0$ and a greatest element $1$.

**Definition 6** A $t_L$-norm $T$ is an increasing, associative and commutative $L^2 \rightarrow L$ mapping that satisfies the boundary condition $(\forall a \in L)(T(a, 1) = a)$.

Similarly, a $t_L$-conorm $S$ is an increasing, associative and commutative $L^2 \rightarrow L$ mapping that satisfies the boundary condition $(\forall a \in L)(S(a, 0) = a)$.

If $L = [0, 1]$, then the $t_L$-norm and $t_L$-conorm become $t$-norm and $t$-conorm as usual[5].

**Definition 7** A complementation in $L$ is a mapping $a \mapsto a'$ of $L$ onto itself such that

(i) $(a')' = a$,

(ii) $a \leq b \Rightarrow b' \leq a'$.

Any $t_L$-norm $T$ corresponds to a dual $t_L$-conorm $S$ defined by $s(a, b) = (T(a', b'))'$, $\forall a, b \in L$, i.e., $t_L$-norm $T$ and $t_L$-conorm $S$ are dual.

In the quantum logic, $t_L$-norm $T$ and $t_L$-conorm $S$ correspond to the operators $\wedge$ and $\vee$ respectively. While in MV-algebra, they correspond to the operators $\odot$ and $\oplus$ respectively. Thus, we also can define state map $s$, $S$-orthogonal system, partition and entropy in poset $L$ as in quantum logic, replacing the operators $\vee$ and $\wedge$ with operators $S$ and $T$. Therefore we have

**Theorem 2** Let $P, Q$ be the partitions of a poset $L$ corresponding to a state $s$ with Bayes’ property. Then

$$H_s(P \cup Q) \leq H_s(P) + H_s(Q).$$

We have introduced the partition and the entropy of partitions in quantum logic, similar to the case of MV-algebra. The results are generalized to the general case of $T$-norm and $T$-conorm. The possible applications to the logic in quantum computations will be discussed in the forthcoming paper.

**References**


