Rational Quadratic Approximation to Real Plane Algebraic Curves¹⁾

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Abstract. An algorithm is proposed to give a global approximation to an implicit real plane algebraic curve with rational quadratic B-splines. The algorithm consists of three steps: curve segmentation, segment approximation and curve tracing. The curve is first divided into so-called triangle convex segments. Then each segment is approximated with several rational quadratic Bézier curves. At last, the curve segments are connected into several maximal branches and each branch is represented by a B-spline curve resulting in a C^1 global parameterization for the curve branch. Due to the detailed geometric analysis, high accuracy of approximation may be achieved with a small number of quadratic segments. The final approximation based on quadratic spline curves keeps many important geometry features and gives a refined topological structure of the original curve. Extension of the method to spatial curves is also illustrated.

Keywords: plane algebraic curve, parametrization, approximation, quadratic Bézier curve, quadratic B-spline curve, topology determination.

1. Introduction

An implicit real plane algebraic curve of degree n is defined by $f(x, y) = 0$ where $f(x, y)$ is a polynomial of degree n . The curve is said to be *rational* if it can be additionally represented by a rational parametric equation $x = \frac{x(t)}{d(t)}$ and $y = \frac{y(t)}{d(t)}$, where $x(t), y(t), d(t) \in \mathcal{R}[t]$ are of degrees at most n . Implicit and parametric representations of algebraic curves have important applications in CAGD [18].

It is desirable to generate the parametric representation for an implicitly given algebraic curve, because with the parametric representation we can easily compute points on the curve and plot it. Unfortunately, only a small subset of real algebraic curves are rational. In general, an algebraic curve of arbitrary degree is rational if and only if its genus is zero [21]. Methods to find parametric equations of implicit curves are given, for instance, in [1, 8, 19, 20]. For general implicit curves, approximation methods are therefore put forward to give a rational form. These methods might be categorized mainly into three classes: the linear approximation [13], the points sampling approximation [16, 17, 22] and the approximation based on power series [2, 6]. Ihm surveyed a number of techniques for generating a piecewise

¹⁾Partially supported by a National Key Basic Research Project of China (NO. G1998030600) and by a US NSF grant CCR-0201253.

linear approximation for an algebraic curve [13]. Given a model shape, curve or surface, expressed by a set of sample points on it, Pottmann used an active B-spline curve or surface to approximate it with an optimization method [16]. The active B-spline curve method is further refined in [22]. For a polygonal curve in space, whose vertices can be generated through adaptive sampling, Safonova and Rossignac gave a compressed piecewise-circular approximation for it [17]. Chuang and Hoffmann gave a local approximation to a plane curve with power series [6]. Bajaj and Xu used a combination of algebraic and numerical techniques to construct a C^1 -continuous, piecewise rational approximation of a plane algebraic curve [2]. There the Hensel lifting technique and Padé approximation are used.

As an extension of our previous work on approximate implicitization [10], we consider in this paper the approximate rational parametrization problem for a plane algebraic curve $\mathcal{C}$ within a bounding box B (written as $\mathcal{C}_B$ for short) from a geometrical viewpoint. The resulted approximations are several rational quadratic B-spline curves, each of which is obtained from piecewise rational quadratic Bézier segments. The rational quadratic segment is used since it is the freeform curve with the lowest degree and has many nice properties [7, 14]. The algorithm consists of three steps: curve segmentation, segment approximation and curve tracing. First, the curve is divided into triangle convex segments (definition in section 2.1.). Then each triangle convex segment is approximated with several rational quadratic Bézier curves. At last, the curve segments are connected into several maximal branches and each branch is represented globally by a B-spline curve resulting in a C^1 representation for the curve branch.

As compared with previous approaches, our method is based on a detailed topology analysis of $\mathcal{C}_B$. Previous methods such as [2] also assumed that the curve topology was known. In this paper, after a topology is determined with method in [11, 12], we further try to find the inflection points and tangent directions of the curve so that triangle convex segments are obtained. The triangle convexity of the segment make it possible to give a nice quadratic approximation with a method called shoulder point approximation. Due to these facts, high accuracy of approximation may be achieved with a small number of quadratic segments. The final approximation based on quadratic spline curves keeps many important geometry features of the original curve such as the convexity and sharp points. The branches obtained in the tracing step provide a global parameterization and a refined topological structure of the curve.

The rest of the paper is organized as follows. The curve segmentation, the segment approximation and the curve tracing are illustrated in sections 2., 3. and 4. respectively. After the main algorithm and some examples are given in section 5., we conclude this paper in section 7..

2. Curve segmentation

Throughout this paper, we assume that $f(x, y) \in \mathbb{Z}[x, y]$ is an irreducible polynomial of degree greater than two. A plane algebraic curve $\mathcal{C}$ is defined by $f(x, y) = 0$ and $\mathcal{C}_B$ is taken for the part of $\mathcal{C}$ inside a bounding box $B = \{(x, y) : x_l \leq x \leq x_r, y_b \leq y \leq y_u\}$. And the approximation error bound is assumed to be a small positive number δ .

2.1. Preliminaries about plane algebraic curves

A point $P = (x_0, y_0)$ is said to be a *singular* point of $\mathcal{C}$ if $f(x_0, y_0) = f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. P is said to be a *vertical point* of $\mathcal{C}$ if it is a non-singular point and $f(x_0, y_0) = f_y(x_0, y_0) = 0$. The *inflection points*, *flexes* for short, of $\mathcal{C}$ are its non-singular points satisfying the Hessian equation $H(f) = 0$ [21].

A *curve segment* S of $\mathcal{C}$ is an open ended and continuous part of $\mathcal{C}$ with two endpoints P_0 and P_1 . The left (right) endpoint is the one with smaller (larger) x coordinate. If P_0 and P_1 have the same x coordinate, then the left (right) endpoint is the one with smaller (larger) y coordinate.

Let P_0 be an endpoint of a curve segment S . Then a tangent direction $T_0 = (x_t, y_t)$ of S at P_0 always exists [21]. If P_0 is the left endpoint, T_0 is called the *left tangent direction* of S , and is denoted by T_- . The left slope k_- of S at P_0 is defined as follows: if $x_t \neq 0$, $k_- = y_t/x_t$; otherwise $k_- = +\infty$ or $-\infty$ depending on $y_t > 0$ or $y_t < 0$. The right slope k_+ and the right tangent direction T_+ can be defined in a similar way.

We use $S[P_0, P_2]$ to denote a curve segment of curve $\mathcal{C}$ with left endpoint P_0 and right endpoint P_2 and $S[P_0, T_0, P_2, T_2]$ is also used when the left and right tangent directions T_0 and T_2 are also prescribed.

A curve segment $S[P_0, T_0, P_2, T_2]$ is said to be *triangle convex* if either

1. the lines passing through P_0 and P_2 and with directions T_0 and T_2 respectively meet at a point P_1 and the line segment P_0P_2 and S form a convex region inside the *control triangle* $P_0P_1P_2$ of S ; or
2. T_0 and T_2 are parallel and the line segment P_0P_2 and the curve segment S form a convex region.

S_P is said to be a *shoulder point* on a triangle convex segment $S[P_0, P_2]$ if S_P has the maximal distance to the line P_0P_2 .

Lemma 2.1 *The shoulder point for a triangle convex segment $S[P_0, P_2]$ is unique.*

Proof. Suppose that there are two shoulder points S_1 and S_2 . Since $S[P_0, P_2]$ is triangle convex, the line segment S_1S_2 should lie inside the region formed by line P_0P_2 and S . Since S_1 and S_2 have maximal distance to P_0P_2 , the line segment S_1S_2 must be coincident with S . This is impossible because $f(x, y)$ is irreducible and of degree greater than two. $\blacksquare$

From a set of curve segments $\mathcal{S}_S$, we can generate a plane graph $\mathcal{G}_S$ with a map $\mathcal{U} : \mathcal{S}_S \rightarrow \mathcal{G}_S$ such that

1. $\mathcal{G}_S$ takes the set of the endpoints of the segments in $\mathcal{S}_S$ as its vertex set $V(\mathcal{G}_S)$, and
2. there exists an edge between two vertices v_1, v_2 if and only if v_1, v_2 are the endpoints of a segment in $\mathcal{S}_S$. We use $E(\mathcal{G}_S)$ to denote the set of edges in $\mathcal{G}_S$.

It can be seen that $\mathcal{U}$ is a bijection map and $\mathcal{U}^{-1}$ is used to denote the reverse.

The *degree* of a vertex v in $\mathcal{G}_S$ is the number of edges in $\mathcal{G}_S$ incident with v .

2.2. Topology determination

The topology determination of $\mathcal{C}_B$ produces a plane graph $\mathcal{G}_T$ which is topologically equivalent to $\mathcal{C}_B$ [11, 12]. This is the starting point of our algorithm. Roughly speaking, the algorithm works as follows.

Algorithm 2.2 (Topology Determination[11, 12]) *The input is a plane algebraic curve $\mathcal{C}_B$. The output is a plane graph $\mathcal{G}_T$ topologically equivalent to $\mathcal{C}_B$.*

1. Compute the discriminant $D(x)$ of $f(x, y)$ with respect to y and determine its real roots within B : $x_l \leq \alpha_1 < \dots < \alpha_{s-1} \leq x_r$. Suppose $\alpha_1 \neq x_l$, $\alpha_{s-1} \neq x_r$ and let $\alpha_0 = x_l$, $\alpha_s = x_r$.
2. For every α_i , $0 \leq i \leq s$, compute within B the real roots of $f(\alpha_i, y)$, $\beta_{i,0} < \dots < \beta_{i,t_i}$.
3. At each point $P_{i,j} = (\alpha_i, \beta_{i,j})$, count the numbers of branches of $\mathcal{C}_B$ to the right and to the left.
4. For each $0 \leq i < s$, the number of branches to the right of points $P_{i,j}$ must be the same as the number of branches to the left of points $P_{i+1,j}$. Connect the points $P_{i,j}$ to the other endpoint of the segment with edges, obeying the branch counts and get the graph $\mathcal{G}_T$.
5. Label the generated curve segments from left to right (i), top to bottom (j) as

$$\mathcal{S}_T = \{S_{i,j}, 1 \leq i \leq s, 1 \leq j \leq s_i\} \quad (1)$$

where s_i denotes the number of curve segments of $\mathcal{C}_B$ with x in the interval (α_{i-1}, α_i) .

Let $V = \{(\alpha_i, \beta_{i,j}), 0 \leq i \leq s, 0 \leq j \leq t_i\}$ and decompose it into $V = V_V \cup V_S \cup V_O$, where V_V is the set of vertical points, V_S is the set of singular points, and V_O is the other simple points.

Consider the curve defined by the following equation

$$\mathcal{C}_0 : f_0(x, y) = 2x^4 - 3x^2y + y^2 - 2y^3 + y^4 = 0. \quad (2)$$

The figure and the corresponding topology graph of $\mathcal{C}_0$ is shown in Figure 1.

Fig. 1. A curve and its topology graph

With the topology $\mathcal{S}_T$ for $\mathcal{C}_B$, we can do the following basic operations for curve segments.

Algorithm 2.3 *The inputs are a segment $S_{i_0,j_0} = S[P_0, P_2]$ in $\mathcal{S}_T$ for $\mathcal{C}_B$ with $P_i = (x_i, y_i)$ and an $\bar{x} \in (x_0, x_2)$. The output is a point $\bar{P} = (\bar{x}, \bar{y})$ on S .*

1. Let $g(y) = f(\bar{x}, y)$ and find all the solutions $y_1 > \dots > y_s$ of $g(y) = 0$ within B . Note that $g(y) = 0$ has no repeated roots.

2. Since S_{i_0, j_0} is the j_0 -th segment of $\mathcal{C}_B$ in the interval (x_0, x_2) from top to bottom, $\bar{P} = (\bar{x}, y_{j_0})$ should be on S_{i_0, j_0} .

Algorithm 2.4 *The inputs are a point $\bar{P} = (\bar{x}, \bar{y})$ on $\mathcal{C}_B$ and the segment set $\mathcal{S}_T$ of $\mathcal{C}_B$. The output is a pair of footnotes (i_0, j_0) such that $\bar{P}$ is on segment $S_{i_0, j_0} \in \mathcal{S}_T$.*

1. Select a unique α_{i_0} such that $\alpha_{i_0-1} \leq \bar{x} < \alpha_{i_0}$. If there exists only one segment S_{i_0, j_0} in the interval $(\alpha_{i_0-1}, \alpha_{i_0})$, output (i_0, j_0) .
2. If each segment $S_{i_0, j}$ in the interval $(\alpha_{i_0-1}, \alpha_{i_0})$ is triangle convex and there exists just one segment S_{i_0, j_0} with $\bar{P}$ contained in its control triangle, output (i_0, j_0) .
3. Let $g(y) = f(\bar{x}, y)$ and isolate all the real roots of $g(y) = 0$ within B : $y_1 > \dots > y_r$. Let y_{j_0} be the one nearest to $\bar{y}$. Then output (i_0, j_0) .

2.3. Flex computation and generation of triangle convex segments

The cusps and flexes on rational curves are computed in [15]. A method to compute the real inflection points of cubic plane algebraic curves is given in [5]. There seems no work on computing the flexes of general implicit algebraic curves. Since this is not the central topics of this paper, we compute the flexes of $\mathcal{C}_B$ directly from its definition by solving the equation system $f(x, y) = 0$ and $H(f) = 0$ with well known methods based on resultant computation. Let V_F be the set of the flexes in $\mathcal{C}_B$.

Algorithm 2.5 (Division at Flex) *The inputs are $\mathcal{S}_T$ and V_F . The output is a set of triangle convex segments $\mathcal{S}_F = \{S_{i,j,k}, 1 \leq i \leq s, 1 \leq j \leq s_i, 1 \leq k \leq s_{ij}\}$ and its corresponding graph $\mathcal{G}_F$.*

1. For each $S_{i,j} \in \mathcal{S}_T$, find all the points in $V_F \cap S_{i,j}$ with Algorithm 2.4. List these points from left to right according to the x coordinate: $P_{i,j,k}, 1 \leq k \leq s_{ij} - 1$.
2. Divide the segment $S_{i,j}$ at the points $P_{i,j,k}, 1 \leq k \leq s_{ij} - 1$, ending with the curve segments $S_{i,j,k}, 1 \leq k \leq s_{ij}$.
3. If there is no flex on $S_{i,j}$, let $S_{i,j,1} = S_{i,j}$ and $s_{ij} = 1$.
4. Let $\mathcal{S}_F = \{S_{i,j,k}, 1 \leq i \leq s, 1 \leq j \leq s_i, 1 \leq k \leq s_{ij}\}$ and $\mathcal{G}_F = \mathcal{U}(\mathcal{S}_F)$. It is clear that $\mathcal{S}_F$ takes $V = V_V \cup V_S \cup V_O \cup V_F$ as the endpoints of its segments.

Theorem 2.6 *Each segment $\mathcal{S}[P_0, P_2]$ in $\mathcal{S}_F$ is triangle convex.*

Proof. Let $k(P)$ be the slope of the tangent line of $\mathcal{S}$ at point P on $\mathcal{S}$. Since there is no cusp points, flexes and vertical points on $\mathcal{S}$, $k(P)$ must be monotonic and continuous. It could happen that $k(P_0)$ and $k(P_2)$ are equal to $\pm\infty$. Without loss of generality, suppose $k(P_0) > k(P_2)$. Then, there exist four possible cases as shown in Figure 2. In the first three cases, the right tangent line at P_0 and the left tangent line at P_2 have an intersection point, say P_1 . Since the slope $k(P)$ is decreasing starting from P_0 , the segment is below the line P_0P_1 . Similarly, the segment is also below the line P_1P_2 . To show that the segment is above line P_0P_2 , let p' be a point on $\mathcal{S}$ between P_0 and P_2 such that $k(p') = 0$. If such a P'

does not exist, let $P' = P_2$. Then from $P_0(P_2)$ to P' the distance from $\mathcal{S}$ to line P_0P_2 is increasing. This fact implies that the segment is above line P_0P_2 . Therefore, the segment is inside triangle $P_0P_1P_2$. Since there exists no flexes and no cusp points, the convexity of the curve segment does not change and hence the segment forms a convex region with line segment P_0P_2 . The fourth case can be proved similarly. $\blacksquare$

Fig. 2. Triangle convex segments

The C_0 in Figure 1 does not have flexes. Then its topology graph need not to be modified. The curve in Figure 14 has a flex point v_3 .

2.4. Tangent direction computation

The tangent direction of a segment at a simple point can be easily obtained from its definition. We give a method to compute the tangent directions at a singular point.

Let K be an algebraic closed field, and $\mathcal{C}$ a curve defined by $f(x, y) = 0$ over K . Suppose that all derivatives of $f(x, y)$, up to and including $(r - 1)$ -th, vanish at P_0 but that at least one r -th derivative does not vanish. The tangent directions (λ, μ) to $\mathcal{C}$ at P_0 , correspond to the roots of

$$g(\lambda, \mu) = f_{x^r} \lambda^r + \binom{r}{1} f_{x^{r-1}y} \lambda^{r-1} \mu + \dots + \binom{r}{r} f_{y^r} \mu^r = 0 \quad (3)$$

where all the partial derivatives are evaluated at P_0 . But for a real algebraic curve defined by $f(x, y) \in \mathcal{R}[x, y]$, there does not exist such a one-one correspondence between the real roots of the equation $g(\lambda, \mu) = 0$ and the tangent directions of the components containing P_0 . For example, let the plane curve $\mathcal{C}$ defined by

$$f(x, y) = y^3 - 2y^2x + 15yx^4 - x^5.$$

The real roots of $g(\lambda, \mu)$ at $P_0 = (0, 0)$ are $(1, 0)$ and $(1, 2)$. However, the curve has no real component with the tangent direction $(1, 0)$ at P_0 (Figure 3). In fact, the set of the tangent directions (λ, μ) of all real components containing P_0 is just a subset of the real roots set of $g(\lambda, \mu) = 0$.

Fig. 3. Unexpected tangent direction

Algorithm 2.7 *The input is the segment set $\mathcal{S}_F$. The output is a set of segments $\mathcal{S}_{\bar{F}}$ with the tangent directions at the endpoints of each segment and its corresponding graph $\mathcal{G}_{\bar{F}}$.*

1. Let $V = V_V \cup V_S \cup V_O \cup V_F$ be the endpoints of the segments in $\mathcal{S}_F$.
2. For every point $P_0 = (x_0, y_0)$ in V_S compute the left tangent directions for the segments $S_j, j = 0, \dots, r$ with P_0 as the left endpoint. We assume that $S_0, \dots, S_r$ are from top to bottom.
 - Find all the solutions $(\lambda_i, \mu_i), 0 \leq i \leq s$, of the homogeneous algebraic equation $g(\lambda, \mu) = 0$ defined in (3). Let $k_i = \frac{\mu_i}{\lambda_i}$ with $k_i = +\infty$ when $\lambda_i = 0$ and $\mu_i > 0$ and $k_i = -\infty$ when $\lambda_i = 0$ and $\mu_i < 0$.
 - Sort k_i in a descending order and rename them as $k_i, 0 \leq i \leq s$.
 - Select a $\bar{\delta} < \delta$ such that $I_i \cap I_{i+1} = \emptyset, 0 \leq i \leq s-1$, where $I_i = [k_i - \bar{\delta}, k_i + \bar{\delta}], 0 \leq i \leq s$. If $k_s = +\infty$, set $\bar{k}_s = \max(0, 2k_{s-1})$ and $I_s = [\bar{k}_s, +\infty)$; if $k_0 = -\infty$, set $\bar{k}_0 = \min(0, 2k_1)$ and $I_0 = (-\infty, \bar{k}_0]$.
 - Set $\epsilon = \frac{x_1 - x_0}{100}$, where x_1 is the minimal x coordinates of the right endpoints of S_j .
 - Find a point $(x_0 + \epsilon, \bar{y}_j)$ on S_j with Algorithm 2.3. Let $\bar{k}_j = \frac{\bar{y}_j - y_0}{\epsilon}$, which could be considered as an approximation to the slope of S_j . If there exists a $\bar{k}_j$ which is not in $\bigcup I_i$, set $\epsilon := \epsilon/10$ and repeat this step. This step will end because $\bar{k}_j$ is approaching to the slope of some segment at point P_0 .
 - Suppose that $\bar{k}_j$ is in I_{n_j} . Then the left tangent direction of S_j is $(\lambda_{n_j}, \mu_{n_j})$.
3. Compute the right tangent directions T_+ in a similar way by taking $-\epsilon$ instead of ϵ .
4. Compute the left tangent and right tangent directions at the points $V \setminus V_S$ according to the definition.
5. Add the tangent information to each segment in $\mathcal{S}_F$ to obtain $\mathcal{S}_{\bar{F}}$. Set $\mathcal{G}_{\bar{F}} = \mathcal{U}(\mathcal{S}_{\bar{F}})$.

2.5. Segment combination

We will combine some segments under the condition that the triangle convexity of the segments will be kept.

Algorithm 2.8 *The input is $\mathcal{S}_{\bar{F}}$. The output is a set of segments $\mathcal{S}_{\bar{C}}$ for $\mathcal{C}_B$ with less segments than that of $\mathcal{S}_{\bar{F}}$.*

1. Let the endpoints of the segments in $\mathcal{S}_{\bar{F}}$ be $V(\mathcal{S}_{\bar{F}}) = V_V \cup V_S \cup V_O \cup V_F$.
2. For all $P_0 \in V_O$ and segments $S_{i_0, j_0, k_0}[P_1, P_0], S_{i_1, j_1, k_1}[P_0, P_2]$, combine them as one segment S_J where $J = \{\{i_0, j_0, k_0\}, \{i_1, j_1, k_1\}\}$. The information in J is needed, e.g. in Algorithm 2.3.
3. Keeping those segments containing no points in V_O unchanged, we obtain $\mathcal{S}_{\bar{C}}$. Let $\mathcal{G}_{\bar{C}} = \mathcal{U}(\mathcal{S}_{\bar{C}})$. We have $V(\mathcal{G}_{\bar{C}}) = V_V \cup V_S \cup V_F$.

Fig. 4. Segment comb. and the graph-1

Since only two segments sharing a simple point are combined, the topology of the graph does not change. Figure 4 shows the combined segments and its corresponding graph of the curve in Figure 1.

The following algorithm tries to combine segments at singular points which are intersections of two curve branches ensuring that the resulted approximate curves have the same topology with the original curve.

Algorithm 2.9 *The input is the same with Algorithm 2.8. The output is a refined plane graph $\tilde{\mathcal{G}}_{\vec{C}}$ such that each segment in $\mathcal{S}_{\vec{C}} = \mathcal{U}^{-1}(\tilde{\mathcal{G}}_{\vec{C}})$ is triangle convex.*

1. Simplify $\mathcal{G}_{\vec{F}}$ to $\tilde{\mathcal{G}}_{\vec{C}}$ with Algorithm 2.8.
2. Let $V_f \subset V_S$ be the set of singular points with degree (in the graph) less than or equal to four and not all the left or right tangent directions at the point are the same.
3. For a point $P_S \in V_f$ let E_S be the set of edges in $\tilde{\mathcal{G}}_{\vec{C}}$ with P_S as an endpoint. If E_S is empty, go to step 7.
4. Find two segments $E_{i_0, j_0, k_0}[P_\alpha, T_\alpha, P_S, T_S]$ and $E_{i_1, j_1, k_1}[P_S, T_S, P_\beta, T_\beta]$ in E_S such that they are at the same side of the line passing through P_S and with direction T_S . In Figure 5 the left figure satisfies this condition while the right one does not.

Fig. 5. Convexity maintenance

5. Combine the edges E_{i_0, j_0, k_0} and E_{i_1, j_1, k_1} into a new edge and refine $\tilde{\mathcal{G}}_{\vec{C}}$ as step 2 in Algorithm 2.8.
6. Let $E_S = E_S \setminus \{E_{i_0, j_0, k_0}, E_{i_1, j_1, k_1}\}$ and go to step 4 until no such segments exist.
7. Let $V_f = V_f \setminus \{P_S\}$ go to step 3 until V_f is empty.

8. Set $\bar{S}_{\bar{C}} = \mathcal{U}^{-1}(\bar{\mathcal{G}}_{\bar{C}})$.

Figure 6 shows the combined segments and its corresponding graph of the curve in Figure 1 with Algorithm 2.9.

Fig. 6. Segments comb. and the graph-2

3. Segment approximation

An algorithm will be shown to approximate a triangle convex segment in this section. It mainly consists of two parts: the shoulder point computation and the segment approximation.

3.1. Quadratic rational Bézier curve

Any quadratic segment can be expressed by a *rational quadratic Bézier curve* with the following form [7, 14],

$$P(\omega, t) = \frac{P_0\phi_0(t) + \omega P_1\phi_1(t) + P_2\phi_2(t)}{\phi_0(t) + \omega\phi_1(t) + \phi_2(t)}; \quad 0 \leq t \leq 1, \quad (4)$$

where $\omega \in \mathcal{R}$, $P_i \in \mathcal{R}^2$, and ϕ_i , $0 \leq i \leq 2$, are the quadratic Bernstein basis [7, 14]. The quadratic Bézier curve (4) has the following properties [7, 14].

(P1) *Convex Hull*: The segment $P(\omega, t)$, lies in its control triangle $P_0P_1P_2$ for $\omega > 0$, and is triangle convex.

(P2) *Endpoints Interpolation*: $P(t)$ passes through the endpoints P_0, P_2 with the corresponding tangent directions $P'(0)$ and $P'(1)$ parallel to P_0P_1 and P_1P_2 .

(P3) *Control Point at Infinity*: If the tangent lines at the endpoints are parallel, it can be written as below

$$P(\omega, t) = \frac{P_0\phi_0(t) + \omega T\phi_1(t) + P_2\phi_2(t)}{\phi_0(t) + \phi_2(t)}; \quad 0 \leq t \leq 1, \quad (5)$$

where T is the tangent vector at either of the endpoints.

(P4) *Shoulder Point*: The point $S_P = P(\frac{1}{2})$ is called the *shoulder point* of $P(t)$, which can be computed from

$$S_P = \frac{1}{2}(Q_0 + Q_2), \quad (6)$$

where $Q_0 = \frac{P_0 + \omega P_1}{1 + \omega}$, $Q_2 = \frac{\omega P_1 + P_2}{1 + \omega}$, or $Q_0 = P_0 + \omega T$, $Q_2 = \omega T + P_2$, when (5) is used. S_P is the unique point in the curve $P(t)$, $0 \leq t \leq 1$, that has the maximum distance to P_0P_2 .

Let $S[P_0, T_0, P_2, T_2]$ be a triangle convex curve segment, and P_1 the intersection point of the tangent lines at P_0 and P_2 if it exists. The curve family $P(\omega, t)$ with $\omega > 0$ in (4)

Fig. 7. Approximation curve family

interpolates points P_0, P_2 and with tangent directions T_0, T_2 at P_0, P_2 , and thus provides a G^1 approximation to S . Suppose that the solid curve in Figure 7 is the curve segment S and the dotted curves are the quadratic curve family $P(\omega, t)$. In the next section, we give a method called *shoulder point approximation* to select a proper value for ω so that it has an optimal approximation to the segment.

3.2. Shoulder point computation

From the definition of the shoulder point, we can see that the gradient (f_x, f_y) of $\mathcal{C}$ at the shoulder point S_P of $S[P_0, P_2]$ is perpendicular to $(k_0, k_1) = P_2 - P_0$. The following equations system is therefor to be solved to get S_P

$$F(x, y) : \begin{cases} f(x, y) = 0, \\ h(x, y) = k_0 f_x + k_1 f_y = 0. \end{cases} \quad (7)$$

Since the shoulder point is needed in each process of segment approximation, it should be computed in a fast way. The following algorithm provides a fast method to compute the shoulder point.

Fig. 8. Find a shoulder point

Algorithm 3.1 *The input is a triangle convex segment $S[P_0, T_0, P_2, T_2]$. The output is the shoulder point S_P of S if it is found.*

1. Select an initial point I_0 . Let $N = \frac{P_0 + P_2}{2}$. If T_0 is not parallel to T_2 , let $M = \frac{N + P_1}{2} = \frac{P_0 + 2P_1 + P_2}{4}$; otherwise let $M = N$. Let $M = (M_x, M_y)$. Find a point $I_0 = (M_x, I_y)$ on S with Algorithm 2.3.

3.3. Segment approximation

Let $P = (x(t), y(t)), t \in [0, 1]$ be a parametric curve and $S[P_0, P_2]$ a curve segment for $f(x, y) = 0$. The distance from $P = (x(t), y(t))$ to a curve $f(x, y) = 0$ can be approximated by the following *approximation error function* [6, 16],

$$e(t) = \frac{f(x(t), y(t))}{[(f_x(x(t), y(t))^2 + f_y(x(t), y(t))^2)^{\frac{1}{2}}]}. \quad (8)$$

The *approximation error* can then be set as the following optimization problem,

$$e = \max_{0 \leq t \leq 1} (e(t)). \quad (9)$$

In practice, we sample t as $t_i = \frac{i}{n}$, $i = 0, \dots, n$, for a proper value of n , say $n = 20$, and set the approximation error as $\max(|d(t_i)|)$.

Let $S[P_0, T_0, P_2, T_2]$ be a triangle convex segment, and δ a small positive number. By a δ -*approximation* for S , we mean a sequence of points $\bar{P}_i = (\bar{x}_i, \bar{y}_i), 0 \leq i \leq m$, satisfying $\bar{P}_0 = P_0, \bar{P}_m = P_2, \bar{x}_0 < \bar{x}_1 < \dots < \bar{x}_m$ and rational quadratic Bézier curve segments $P_i(t), 1 \leq i \leq m$, such that

- $P_i(t), i = 1, \dots, m$ pass through $\bar{P}_{i-1}, \bar{P}_i$ and have the same tangent directions with S at these points; and
- the approximation error between S and $P_i(t)$ is less than δ with t varying in the interval $(0, 1)$.

Algorithm 3.3 (Segment approximation) *The inputs are a triangle convex curve segment $S[P_0, T_0, P_2, T_2]$ and a small positive number δ . The output is a δ -approximation to S .*

1. Let $P_i = (x_i, y_i), i = 0, 1, 2$ and $T_0 = (t_x, t_y)$.
2. According to the interpolating requirements at the endpoints of $P(\omega, t)$, set $P(\omega, t)$ as (4), or (5) if T_0 and T_2 are parallel.
3. Find the shoulder point $S_P = (P_x, P_y)$ on S with Algorithm 3.2.
4. Let the shoulder point of $P(\omega, t)$ be $S(\omega)$, as expressed in equation (6). A specific value ω_0 will be determined such that $S(\omega_0)$ has a minimum distance to the shoulder point S_P .

If T_0 is not parallel to T_2 , we have

$$\begin{aligned} S(\omega) &= (S_x, S_y) = \frac{P_0 + 2\omega P_1 + P_2}{2(1 + \omega)} \\ &= \left(\frac{x_0 + 2\omega x_1 + x_2}{2(1 + \omega)}, \frac{y_0 + 2\omega y_1 + y_2}{2(1 + \omega)} \right). \end{aligned}$$

Solving the equation, $\frac{\partial d^2(P, S(\omega))}{\partial \omega} = 0$, where $d^2(S_P, S(\omega)) = (P_x - S_x)^2 + (P_y - S_y)^2$, we get

$$\omega_0 = \frac{1}{2} \cdot \frac{(x_0 + x_2 - 2P_x) + \alpha(y_0 + y_2 - 2P_y)}{(P_x - x_1) + \alpha(P_y - y_1)}$$

where $\alpha = \frac{y_0+y_2-2y_1}{x_0+x_2-2x_1}$. If T_0 is parallel to T_2 , we get in a similar way

$$\omega_0 = \frac{(2S_x - x_0 - x_2) + (2S_y - y_0 - y_2)}{2(t_x^2 + t_y^2)}.$$

5. Compute the approximate error e with equation (9).
6. If $e < \delta$, end this procedure. Otherwise, divide the segment into two parts at the point S_P and repeat the approximation method for them until the approximation error is less than δ .

Fig. 10. Error control

The algorithm is ensured to be terminable for any small positive number δ from the theorem below.

Theorem 3.4 *With Algorithm 3.3, the approximation error is convergent to zero.*

Proof. See also Figure 10. Suppose that we get the approximation curve $P(t)$ for a curve segment S after one step of approximation. We have that $P(t)$ and S are contained in the same triangles $P_0P_1P_2$. After one step of recursive segmentation, the curves are contained in triangle P_0Q_1M and P_2Q_2M respectively. The area between the two curves is restricted by the area of the control triangles. The distance error of the two curves must be less than the height of the triangle. After a sufficient number of segmentation, the error will become as small as possible. ■

Figure 11 shows an approximation spline for the curve $\mathcal{C}_1$ defined as follows:

$$\mathcal{C}_1 : (x^2 + y^2)^3 - 4x^2y^2 = 0.$$

The approximation process is as follows. $\mathcal{C}_1$ is first approximated with a segment $S_0(t)$, with the approximation error function e_0 (expressed as (9)) shown in Figure 12. $\mathcal{C}_1$ is then divided at its shoulder point V_1 and the two segments are approximated by $\mathcal{C}_{00}$ and $\mathcal{C}_{01}$ with approximation error functions e_{00} and e_{01} . Due to the symmetry of $\mathcal{C}_0$, we only show e_{00} in Figure 12. In a similar way, $e_{000}, e_{001}, e_{0010}, e_{0011}$ are obtained and shown in Figure 12.

Figure 13 shows the resulted approximations and error functions for the curve $\mathcal{C}_0$ defined in (2).

Fig. 11. Appr. segments of $\mathcal{C}_1$ Fig. 12. Approximation error of $\mathcal{C}_1$

4. Curve tracing

The following concepts from graph theory will be used [4]. A *trail* is a sequence of distinct edges $e_1, e_2, \dots, e_k$ such that e_i and e_{i+1} share a common vertex. A trail that traverses every edge of $\mathcal{G}$ is called an *Euler trail* of $\mathcal{G}$.

Theorem 4.1 [4] *In any graph, the number of vertices of odd degree is even.*

Theorem 4.2 [4] *A connected graph has an Euler trail if and only if it has at most two vertices of odd degree.*

To guarantee the G^1 continuity, two segments $S[P_1, T_1, P_2, T_2]$ and $S[P_3, T_3, P_4, T_4]$ can be connected in a trail only if $P_2 = P_3$ and $T_2 = \lambda T_3$ for a non-zero number λ . A trail in the topology graph of a curve is called a *branch* if the corresponding segments of every neighboring edges in the trail satisfy the above property.

By private communication with Guangxing Zeng, we know that²⁾

Lemma 4.3 *Every point on a real plane algebraic curve has an even number of segments originating from it.*

As a consequence of this result, the vertices in the graph $\mathcal{G}_{\tilde{\mathcal{C}}}$ with odd degrees must be the intersection of the curve with the boundaries of B , which are called *boundary points*. Thus, there are two kinds of trails for $\mathcal{G}_{\tilde{\mathcal{C}}}$: closed Euler trails for a closed component of $\mathcal{C}_B$ and trails from a boundary point to another boundary point.

²⁾This result makes our algorithm clearer, but is not necessary for our algorithm.

Fig. 13. Approximation of $\mathcal{C}_0$

Algorithm 4.4 *The input is a graph $\mathcal{G}_{\tilde{\mathcal{C}}}$ for a plane curve $\mathcal{C}_B$. The outputs are edge-disjoint branches $B_i, 1 \leq i \leq r$ in $\mathcal{G}_{\tilde{\mathcal{C}}}$ such that $\bigcup_{i=1}^r B_i = E(\mathcal{G}_{\tilde{\mathcal{C}}})$.*

1. Decompose $\mathcal{G}_{\tilde{\mathcal{C}}}$ into closed subgraphs $\bar{\mathcal{G}}_i, 1 \leq i \leq v$ and $\tilde{\mathcal{G}}_i, 1 \leq i \leq u$ such that there exists at least one vertex of odd degree in $\bar{\mathcal{G}}_i$.
2. For $1 \leq i \leq u$, try to generate a closed branch for $\bar{\mathcal{G}}_i$ as follows. If $\bar{\mathcal{G}}_i \cap V_S$ is not empty, select a point in it as v_0 . Otherwise, select an arbitrary vertex in $\bar{\mathcal{G}}_i$ as v_0 . Starting from v_0 , use a breadth-first way to search all the vertices of $\bar{\mathcal{G}}_i$ to generate a branch B with maximal length. Remove the edges in B from $\bar{\mathcal{G}}_i$ to obtain a new graph G' . If G' is empty, a closed branch is generated. Otherwise repeat step 1 for G' .
3. Let v_0 be a boundary vertex in $\tilde{\mathcal{G}}_i$ with odd degree. Starting from v_0 , generate a branch similar to step 2. Repeat this step until no boundary vertices exist.
4. Now there exist no vertices with odd degree in $\tilde{\mathcal{G}}_i$. If $\tilde{\mathcal{G}}_i$ is not empty, repeat step 1 for $\tilde{\mathcal{G}}_i$.

The tracing orders for the graphs in Figures 4 and 6 are $v_5v_9v_4v_2v_5v_7v_4v_0v_5$ and $v_5v_9v_2v_5v_7v_0v_5$ respectively.

5. Main algorithm and experimental results

Algorithm 5.1 *The inputs are a plane curve $\mathcal{C} : f(x, y) = 0$ with $f(x, y) \in Z[x, y]$ an irreducible polynomial of degree larger than two, a bounding box B , an error bound $\delta > 0$. The outputs are several rational quadratic B-spline curves $B_i(t), 1 \leq i \leq r$, such that they give a C^1 approximation to $\mathcal{C}$ within the bounding box B with the approximation error less than δ .*

1. Topology Determination. Determine the topology of $\mathcal{C}_B$ with Algorithm 2.2. Let the resulted segments be $\mathcal{S}_T$ and let $\mathcal{G}_T = \mathcal{U}(\mathcal{S}_T)$.
2. Flex Computation. Compute the set V_F of the flexes of $\mathcal{C}_B$ as shown in section 2.3.. Divide those segments in $\mathcal{S}_T$ containing flexes with Algorithm 2.5 to obtain a new set of segments $\mathcal{S}_F$ and let $\mathcal{G}_F = \mathcal{U}(\mathcal{S}_F)$.

Fig. 14. Approximation of $\mathcal{C}_2$ Fig. 15. Approximation of $\mathcal{C}_3$

3. Tangent Computation. Compute the tangent directions with Algorithm 2.7 to obtain $\mathcal{G}_{\vec{F}}$ and $\mathcal{S}_{\vec{F}}$.
4. Segment Combination. Combine some edges in the graph $\mathcal{G}_{\vec{F}}$ with Algorithm 2.9 to obtain a new graph $\mathcal{G}_{\vec{C}}$ and $\mathcal{S}_{\vec{C}}$.
5. Segment Approximation. Find a δ -approximation rational quadratic Bézier curve for each segment in $\mathcal{S}_{\vec{C}}$ with Algorithm 3.3.
6. Curve Tracing. Find r edge-disjoint branches $B_i, 1 \leq i \leq r$, with Algorithm 4.4. Let $E_i = \mathcal{U}^{-1}(B_i)$ be the corresponding curve branches in $\mathcal{C}_B$.
7. B-spline Conversion. These rational quadratic Bézier curves for the segments in E_i are converted into a B-spline curve C_i with a proper knot selection using the method in [3]. C_i provides a C^1 approximation to branch E_i .

The method reported is implemented in Maple. The benchmark curves $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are from [21]. Curves $\mathcal{C}_4$ and $\mathcal{C}_5$ are taken from [11].

$$\mathcal{C}_2 : x^4 + x^2y^2 - 2x^2y - xy^2 + y^2 = 0,$$

$$\mathcal{C}_3 : (x^2 + y^2)^3 - 4x^2y^2 = 0,$$

$$\mathcal{C}_4 : y^8 + y^7 - (8 + 7x)y^6 - (7 - 21x^2)y^5 - (-20 - 35x + 35x^3)y^4 - (-14 + 70x^2 - 35x^4)y^3 - (16 + 42x - 70x^3 + 21x^5)y^2 - (7 - 42x^2 + 35x^4 - 7x^6)y + 7x - 14x^3 + 7x^5 - x^7 = 0,$$

$$\mathcal{C}_5 : -3 + 12y^2 + 2y^4 - 12y^6 + y^8 + 12x^2 - 28y^2x^2 + 12y^4x^2 + 4y^6x^2 - 18x^4 + 20y^2x^4 + 2y^4x^4 + 12x^6 - 4x^6y^2 - 3x^8 = 0.$$

In the figures in this section, the left figures show the approximation results and the right figures show the approximation error functions. Specifically, the curve in the right figures defined in the interval $(i, i+1)$ corresponds to the $(i-1)$ -th curve segment in the left figures.

Tracing order for $\mathcal{C}_2$: $v_0v_1v_2v_3v_4v_0$.

Tracing order for $\mathcal{C}_3$: $v_0v_1v_2v_3v_1v_4v_5v_1v_6v_0$.

Tracing order for $\mathcal{C}_4$: $v_0v_1v_2v_3v_4v_5v_6v_7v_8$.

Tracing order for $\mathcal{C}_5$: $v_0v_1v_2v_3v_4v_5v_6v_7v_8$ and $v_9v_{10}v_{11}v_{12}v_9$. This curve consists of four branches and is symmetric with the x axis. We only show the result for two of the branches.

Fig. 16. Approximation of $\mathcal{C}_4$

Fig. 17. Approximation of $\mathcal{C}_5$

Fig. 18. Error function of $\mathcal{C}_5$

6. Extension to spatial curves

With the method we can also give a quadratic approximation to a spatial curve defined as the intersection of two surfaces in implicit form.

6.1. Approximation to implicitly defined spatial curves

Suppose that a spatial curve $SC := \text{Zero}(f, g)$, which is implicitly defined as the intersection of two surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$. We first have the following theorem about SC [8].

Theorem 6.1 [8] *We may find an inverse linear transformation $(x, y, z) = A(x', y', z')$ such that the new space curve C' defined by $\bar{f}(x', y', z') = 0$ and $\bar{g}(x', y', z') = 0$ is birational to a plane curve in the $x'y'$ plane $C : R(x', y') = 0$.*

We will first illustrate how to approximate a parametric function $f(t)$ with piecewise quadratic function.

Algorithm 6.2 *The input is a function $P(t), 0 \leq t \leq 1$ and an error bound δ . The output is a rational quadratic function spline $B(t)$ such that the approximation error between $B(t)$ and $P(t)$ is less than δ .*

This is a classic approximation problem. We will not give the details.

We can now give the following algorithm to approximate the space curve SC with rational quadratic space splines.

Algorithm 6.3 *The inputs are a space curve $SC : \text{Zero}(f, g)$ defined by $f(x, y, z) = 0$ and $g(x, y, z) = 0$ where $f(x, y, z), g(x, y, z) \in Z[x, y, z]$ are irreducible polynomials of degree larger than two, a bounding box $B = \{(x, y, z) : x_0 \leq x \leq x_1, y_0 \leq y \leq y_1, z_0 \leq z \leq z_1\}$, an error bound $\delta > 0$. The outputs are several rational quadratic space spline curves $B_i(t), 1 \leq i \leq r$, such that they give a C^1 approximation to SC within the bounding box B with the approximation error less than δ .*

1. Transform $SC : \text{Zero}(f, g)$ with a linear transformation A such that it is birational to a plane curve $C : R(x, y) = 0$ with Theorem 6.1. Rewrite the resulted functions as f, g, R and C .
2. Assume that the birational map between SC and C is defined as $x = x, y = y$ and $z = \frac{z_0(x, y)}{z_1(x, y)}$, where z_0 and z_1 are polynomials in x and y .
3. Approximate $R = 0$ with rational quadratic B-splines $(B_x(t), B_y(t))$ according to the main algorithm described in section 5.. We obtain $z(t) = \frac{z_0(x, y)}{z_1(x, y)} = \frac{z_0(B_x(t), B_y(t))}{z_1(B_x(t), B_y(t))}$.
4. Give a δ -approximation $B_z(t)$ for $z = z(t)$ with Algorithm 6.2. Subdivide the space curve into segments according to the knots of $B_z(t)$.
5. Re-approximate each segment resulted from the step above and approximate it as steps 3 and 4, ending with the quadratic approximation segments $\bar{B}(t) = (B_x(t), B_y(t), B_z(t))$. We conclude that the approximation error is less than δ .
6. Let $(x(t), y(t), z(t)) = A^{-1}(B_x(t), B_y(t), B_z(t))$ and $(x(t), y(t), z(t))$ is just what we need.

6.2. Approximation to parametric spatial curves

A similar algorithm can also be given to approximate a rational parametric space curve

$$SC : (x, y, z) = (x(t), y(t), z(t)). \quad (10)$$

$$(11)$$

We first introduce the following definition of a proper curve.

Definition 6.4 [9] (10) is called *proper* if, except a subset set of points on SC , for each (x_0, y_0, z_0) on SC , there exists only one t_0 such that $x_0 = x(t_0), y_0 = y(t_0), z_0 = z(t_0)$.

Theorem 6.5 [9] If (10) is not proper, we can find a new parameter $s = f(t)/g(t)$ where f, g are polynomials in t such that the re-parametrization of (10) in terms of s $x = x(s), y = y(s), z = z(s)$ are proper.

Theorem 6.6 If (10) is proper, there exist an algorithm to give the following inversion map $t = r(x, y, z)$ where r is a rational function in x, y, z .

With these results, we can approximate a rational parametric space curve with rational quadratic space splines.

Algorithm 6.7 The input is a spatial curve $SC : (x, y, z) = (x(t), y(t), z(t)), a \leq t \leq b$ and an error bound δ . The outputs are rational quadratic splines such that they approximate SC with the approximation error less than δ .

1. Transform SC into a proper form with Theorem 6.5 and rewrite them as $SC : (x, y, z) = (x(t), y(t), z(t)), a \leq t \leq b$.
2. From $x = x(t), y = y(t)$, find the inversion map $t = t(x, y)$ with Theorem 6.6.
3. Approximate $x = x(t), y = y(t)$ with the methods proposed in [10]. Write the resulted spline as $B(s) = (B_x(s), B_y(s))$.
4. Approximate $z(t) = z(t(x, y)) = z(t(B_x(s), B_y(s)))$ with Algorithm 6.2.
5. Subdivide each segment in SC according to the segments in $z(s)$.
6. Re-approximate each segment as steps 3 and 4, ending with the quadratic approximation segments $\bar{B}(t) = (B_x(t), B_y(t), B_z(t))$. We conclude that the approximation error is less than δ and $\bar{B}(t)$ is what we need.

7. Conclusion

In this paper, we try to give a simple and intuitive approximation of the plane algebraic curve with rational quadratic curves. The basic idea is to divide the curve into triangle convex segments which can be nicely approximated with quadratic Bézier curves and to connect the segments into certain maximal branches which can be globally approximated by quadratic B-splines. Some extension of the method to space curve is also illustrated. Experiments show that we can achieve high precision approximation with few segments. Instead of giving a power series for each approximated segment, the endpoint information and shoulder points are mainly used to express the segment. Since the geometric information is considered in the algorithm, the algorithm is easy to understand and many geometry characters of the approximated curve are kept. On the other hand, the algorithm involves the computation of the flexes of a plane curve, so it is especially efficient for a curve with no real inflection points. How to avoid the computation of flexes is an interesting work to do. It is also interesting to extend this algorithm to implicitly defined algebraic surfaces.

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