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Algorithmic Reduction and Rational Solutions of First Order Algebraic Differential Equations [1)]

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Abstract. Algebraic differential equations of the first order are considered. An effective necessary condition for an algebraic differential equation of the first order to have a general rational solution is given: the algebraic genus of the equation should be 0. Combining with Fuchs' conditions for algebraic differential equations without movable critical points, an algorithm is given for the computation of rational solutions of these equations. It is based on an algorithmic reduction of first order algebraic differential equations with genus 0 and without movable critical point to classical Riccati equations.

1. Introduction

The study of algebraic differential equations of the first order can be dated back to C. Briot and T. Bouquet [2], L. Fuchs [10] and H. Poincaré [18]. M. Matsuka [16] gave a modern interpretation of the results using the theory of differential algebraic function field of one variable.

From an algorithmic point of view, many authors have been interested in the constructions of elementary function solutions for differential equations (this problem can be traced

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back to the work of Liouville). In [19], Risch, gave an algorithm for finding closed forms for integration. In [12], Kovacic presented a method for solving second order linear homogeneous differential equations. In [23], Singer proposed a method for finding Liouvillian solutions of general linear differential equations. In [13], Li and Schwarz gave a method to find rational solutions for a class of partial differential equations. All these works are limited to linear cases.

For algebraic (nonlinear) differential equations there are some studies in this direction. For Riccati equations, polynomial solutions are considered in [4] and algorithms for the computation of rational solutions are given in [12, 3]. In [9, 15], algorithms for the computation of polynomial solutions and rational solutions are given for algebraic differential equations of the first order with constant coefficients. In [11], a method is given to compute a characterization of the general solutions and is applied to study the local behavior of the solutions in the neighborhood of a singular solution.

Another motivation of our work is differential algebraic geometry. In the series papers of Wu [25, 26, 27, 28, 29, 30, 31], the author studied algebraic differential geometry from several different points of view. In [31], the author presents an algorithmic method of solving arbitrary systems of algebrico-differential equations by extending the characteristic set method to differential case. The Devil's problem of Pommaret is solved in details as an illustration.

In this paper, we consider the computation of rational solutions of algebraic differential equations of the first order by using some results from algebraic geometry. We give an effective necessary condition for an algebraic differential equation to have a general rational solution: that is the algebraic genus of the equation should be 0. Combining with Fuchs' conditions for algebraic differential equations of the first order without movable critical points, we obtain an algorithm for the computation of (general) rational solutions. It is based on an algorithmic reduction of first order algebraic differential equation of genus 0 without movable critical point to classical Riccati equations.

2. Rational general solutions of algebraic differential equations of the first order

We first present some results from algebraic geometry, which is used in the following.

Let $f(x, y)$ be an irreducible polynomial over $\mathbf{C}$. We call that $f(x, y) = 0$ is a rational curve if there exist two rational functions $\phi(t)$, $\psi(t) \in \mathbf{C}(t)$ such that

(i) For all but a finite set of $t_0 \in \mathbf{C}$, $(\phi(t), \psi(t))$ is a point of f.

(ii) With a finite number of exceptions, for every point (x_0, y_0) of f there is a unique $t_0 \in \mathbf{C}$ such that $x_0 = \phi(t_0)$, $y_0 = \psi(t_0)$.

It is impossible to avoid having a finite number of exceptions in the above conditions. They arise from two sources. One is the fact that a rational function is not defined for some values of the variable, and the other is the presence of singular points on the curve.

The following results are well known in algebraic geometry.

Theorem 2.1 *An algebraic curve is rational if and only if its genus is* 0*.*

Theorem 2.2 *(a) Every rational transform of a rational curve is a rational curve.*
(b) If λ is transcendental over $\mathbf{C}$ and if $\mathbf{C} \subset F \subset \mathbf{C}(\lambda)$, $F \neq \mathbf{C}$, then there is an element $\mu \in F$, transcendental over $\mathbf{C}$, such that $F = \mathbf{C}(\mu)$.

(c) If a curve $f(x,y)=0$ satisfies (i) for rational functions $\phi(\lambda),\psi(\lambda)$ which are not both constants, then there exist rational functions $\tilde{\phi}(\lambda),\tilde{\psi}(\lambda)$ for which both (i) and (ii) are satisfied, and the curve is rational.

The three statements in Theorem 2.2 are all equivalent, and are often indiscriminately called Lüroth's theorem.

Now we consider an algebraic differential equation of the first order in the form

$$F(z,w,w')=0, \tag{1}$$

where F is a polynomial in w,w' with rational coefficients in $\mathbf{C}(z)$. By a rational general solution of (1), we mean a rational function $w(z,\lambda)$, depending on an arbitrary constant λ, satisfying equation (1). For algebraic differential equations of any order, Ritt gave a more precise definition to the concept of general solutions (see [20]). We shall use the above definition in this paper.

We now prove the following theorem on rational general solution of first-order algebraic differential equations.

Theorem 2.3 *If a first-order irreducible algebraic differential equation*

$$F(z,w,w')=0$$

owns a non-constant rational general solution, then the genus of $F=0$ with respect to w,w' is 0 for any z.

Proof. Let $w=r(z,\lambda)$ be the rational general solution of $F=0$ with the arbitrary constant λ. Then $w(z)=r(z,\lambda)$ and $w'(z)=\frac{\partial r}{\partial z}$ are rational functions and they satisfy equation $F(z,w,w')=0$.

Let z be fixed and consider the curve $f_z(x,y)=F(z,x,y)=0$. Denote $\phi_z(\lambda)=w(z)$ and $\psi_z(\lambda)=w'(z)$. In the transcendental extension field $\mathbf{C}(\lambda)$, consider the point $(\phi_z(\lambda),\psi_z(\lambda))$ of $f_z(x,y)=0$ for the parameter λ. It is clear that (i) is satisfied for all but finite λ. Hence $f_z(x,y)=0$ is a rational curve and its genus is 0 by Theorem 2.1 and Theorem 2.2. □

Motivated by this theorem, we present the following definition.

Definition 2.4 *The algebraic genus of a first order algebraic differential equation $F(z,w,w')=0$ is defined to be the genus of $F(z,w,w')=0$ with respect to w and w'.*

3. Reduction of algebraic differential equations with genus 0 and without movable critical point

For an algebraic differential equation $F(z,w,w')=0$, Fuchs presented a necessary condition for the equation $F=0$ to be free from movable *critical point* (or branch point). By the Painlevé Theorem, we know that Fuchs' condition is sufficient (see [10, 17, 18]).

3.1. Fuchs' criterion

The conditions, necessary and sufficient to secure that the differential equation

$$F(z,w,w')=0$$

of degree m, shall have no movable critical point, are:

1. The equation must have the form
$$w'^m + \psi_1 w'^{m-1} + \psi_2 w'^{m-2} + \cdots + \psi_m = 0,$$
where ψ_i (for $i = 1, \cdots, m$) is an algebraic polynomial in w of degree not higher than $2i$, the coefficients of the various powers of w being uniform function of z.

2. Let Δ denote the discriminant of F with respect to w'. If a root of the discriminant equation makes one or more roots of
$$F(z, \eta, w') = 0 \tag{2}$$
multiple, then η must be a solution of the original differential equation, so that we must have $w' = \frac{d\eta}{dz}$ for those multiple roots.

3. If the root $w' = \frac{d\eta}{dz} = \zeta$ of equation (2) is of multiplicity α, then $w = \eta$ must be a root of
$$F(z, w, \zeta) = 0$$
of multiplicity equal to, or greater than, $\alpha - 1$.

3.2. Reduction to classical Riccati equations

Consider now an algebraic differential equation $F(z, w, w') = 0$ of genus 0 and without movable critical point. One can find a parametrization of the algebraic curve of $F(z, x, y) = 0$ in the form $x = r_1(t, z)$ and $y = r_2(t, z)$. For algorithm on parametrization of rational curves we refer to [1, 22, 24]. One has
$$\frac{dt}{dz} = \Big(r_2(t, z) - \frac{\partial r_1}{\partial z}\Big)\Big/\frac{\partial r_1}{\partial t} = \frac{P(t, z)}{Q(t, z)}, \tag{3}$$
where P and Q are polynomials in t, z. Since the equation has no movable critical point, one knows from the Fuchs Theorem that equation (3) is a Riccati differential equation, that is
$$\frac{dt}{dz} = A(z)t^2 + B(z)t + C(z), \tag{4}$$
where A, B, C are rational functions of z. We distinguish two cases according to $A(z)$.

Case 1: If $A(z) \not\equiv 0$, we consider the change of variables $t(z) = -u(z)/A(z)$. One has
$$u'(z) + u^2 = (B(z) + A'(z)/A(z))u - C(z)A(z),$$
In which the coefficient $A(z)$ is reduced to -1. Next we make the change $u = v + \beta(z)$ to reduce the coefficient of u to zero by choosing an appropriate β. We obtain finally a classical Riccati equation in the form
$$v' + v^2 = r(z) \in \mathbf{C}(z). \tag{5}$$
Algorithms for the computations of rational solutions of classical Riccati equations are given in many literatures (see for example [12, 3]).

If $r(z) \not\equiv 0$ then a rational solution of equation (5) is equivalent to an exponential solution $e^{\int v(z)dz}$ of the linear differential equation
$$y'' = r(z)y. \tag{6}$$

Proposition 3.1 *If the Riccati equation* (5) *with* $r(z) \not\equiv 0$ *has a general rational solution, then* $r(z)$ *has the form*

$$r(z) = \sum_{i=1}^{m} \left(\frac{\beta_i}{(z-z_i)^2} + \frac{\gamma_i}{(z-z_i)} \right),$$

in which $4\beta_i = n_i^2 - 1$ *where* n_i *is an integer* ≥ 2.

Proof. Suppose that $v(z)$ is a rational solution of equation (5). Let $z_1, \cdots, z_m$ be the poles of r. According to Kovacic's algorithm (see [12]), $v(z)$ should be in the form

$$v(z) = \sum_{i=1}^{m} \sum_{j=1}^{\nu_i} \frac{a_{ij}}{(z-z_i)^j} + \sum_{k=1}^{d} \frac{1}{z-c_k} + f(z),$$

where the ν_i are known, a_{ij} are known up to a two choices each, d is known, and $f \in \mathbf{C}[z]$ is known up to two choices. Hence there may be arbitrary parameter only in the determination of the c_k.

Let

$$P(z) = \prod_{k=1}^{d} (z - c_k),$$

and

$$\omega(z) = \sum_{i=1}^{m} \sum_{j=1}^{\nu_i} \frac{a_{ij}}{(z-z_i)^j} + f(z).$$

Then $v = P'/P + \omega$ and $y = e^{\int v} = Pe^{\int \omega}$ is a solution of the linear differential equation (6). Hence P is a polynomial solutions of degree d of the following linear equation:

$$P'' + 2\omega P' + (\omega' + \omega^2 - r)P = 0. \quad (7)$$

One can determine whether it has a general polynomial solution or not.

Furthermore if (5) admits a rational general solution, then writing $r(z) = p(z)/q(z)$, according to [32], one has

(a) $\deg(p) - \deg(q) \leq -2$;

(b) $r(z)$ has only double poles, hence

$$r(z) = \sum_{i=1}^{m} \left(\frac{\beta_i}{(z-z_i)^2} + \frac{\gamma_i}{(z-z_i)} \right),$$

in which $4\beta_i = n_i^2 - 1$ where n_i is an integer ≥ 2. □

Therefore a possible rational solution of equation (5) with $r(z)$ as in the proposition should be

$$v(z) = \sum_{i=1}^{m} \frac{a_i}{z - z_i} + \sum_{k=1}^{d} \frac{1}{z - c_k},$$

where $a_i^2 - a_i - \beta_i = 0$. Hence to determine a rational general solution one needs to compute polynomial solutions of equation (7) in order to determine the c_k.

Case 2: If $A(z) \equiv 0$, then one can integrate easily the linear equation

$$t' = B(z)t + C(z)$$

to get the general solution

$$t(z) = \Big(\int C(z)dz + \lambda\Big)e^{\int B(z)dz},$$

where λ is an arbitrary constant. Effective algorithm is given in [19] for integration in closed forms. One may find rational solutions in this way. It is clear that in this case one may get rational general solutions only if both $\int C(z)dz$ and $e^{\int B(z)dz}$ are rational functions.

3.3. Algebraic differential equations of the first order with constant coefficients

As an application, we consider an algebraic differential equation of the first order with constant coefficients,

$$F(w, w') = 0. \tag{8}$$

This kind of equations is studied in [2].

When using the above reduction for equation (8), it is clear that z is not involved in the equation, we henceforth get a Riccati equation with constant coefficients.

As above there are two cases to be considered. In case 1, if equation (8) has a non constant rational solution $w(z)$ then $w(z+\lambda)$ is a general rational solution. Since the equation $u' + u^2 = c$ for a constant $c \neq 0$ does not have a rational solution, then one can reduce equation (8) to the equation $u' + u^2 = 0$ in which case we have the general solution

$$u = \frac{1}{z+\lambda}.$$

In case 2, the equation can be converted to $u' = bu + c$ with constant b, c. Then $u = \lambda e^{bz} - \frac{c}{b}$ if $b \neq 0$, and $u = cz + \lambda$ if $b = 0$, where λ is an arbitrary constant.

Summarizing the above, we then have the following

Corollary 3.2 *Consider the first order irreducible algebraic differential equation $F(w, w') = 0$ with constant coefficients. Then it has a non-constant rational solution if and only if it can be reduced either to a linear equation $u' = c$ for some constant c or to a Riccati equation of the form $u' + u^2 = 0$.*

Example 1. Consider

$$F(y, y') = {y'}^4 - 8\,{y'}^3 + (6 + 24\,y)\,{y'}^2 + 257 + 528\,y^2 - 256\,y^3 - 552\,y.$$

This example comes from [9]. One finds by computation in Maple that its algebraic genus is 0 and it has the following parametrization.

$$y = \frac{17}{16} - 27t + \frac{2187}{2}t^2 + 531441t^4, \quad y' = 78732t^3 + 81t - 1.$$

And the corresponding Riccati equation is $t'(z) = \frac{1}{27}$. Hence $t = \frac{1}{27}z + \lambda$ and the solution of the differential equation $F = 0$ is

$$y(z) = \frac{17}{16} - 27(\frac{1}{27}z + \lambda) + \frac{2187}{2}(\frac{1}{27}z + \lambda)^2 + 531441(\frac{1}{27}z + \lambda)^4,$$

where λ is an arbitrary parameter.

4. Algorithm

We can now give the following algorithm on seeking for the rational solutions (or rational general solutions) of an algebraic differential equation of the first order.

Algorithm:

Input: $F(z, w, w') = 0$ is an algebraic differential equation of the first order.
Output: The rational solutions of $F(z, w, w') = 0$ if it exists.

1. Determine the irreducibility of the equation.

   If $F(z, w, w') = 0$ is reducible, then factorize it and go to step (2) for each branch curve of $F(z, w, w') = 0$, else go to step 2 directly.

2. Compute the algebraic genus g of $F(z, w, w') = 0$. If $g \neq 0$, then there is no rational general solution by Theorem 2.3, else go to step 3.

3. Determine the Fuchs criterion of $F(z, w, w') = 0$. If it is not funchsian, the algorithm terminates, else go to step 4.

4. Find a parametrization of the algebraic curve of $F(z, w, w') = 0$ in the form $w = r_1(t, z)$ and $w' = r_2(t, z)$.

5. Compute the derivative $\frac{dt}{dz} = (r_2(t, z) - \frac{\partial r_1}{\partial z})/\frac{\partial r_1}{\partial t}$ which is a Riccati equation of the form (4) by the Fuchs Theorem.

6. Reduce the above Riccati equation to a classical Riccati equation (5) and compute a rational solution using the algorithm in [3, 12].

Example 2. Consider the following equation:

$$F(z, w, w') = w'^2 + \frac{2w}{z} w' - 4\, z w^3 + \frac{\left(1 + 12\, z^2\right) w^2}{z^2} - 12\, \frac{w}{z} + \frac{4}{z^2}.$$

Its genus with respect to w, w' is 0. One gets the following rational parametrization by Maple:

$$w = r_1 = \frac{t^2 z^2 + 4t^2 - 6tz + 1 + 4z^2}{4z(-z + t)^2},$$
$$w' = r_2 = -\frac{-4z^3 + 13tz^2 + t + 2t^2 z^3 - 10t^2 z + t^3 z^4 + t^3 z^2 + 4t^3}{4z^2(-z + t)^3}.$$

One obtains the following Riccati equation

$$t' = \frac{(z^2 + 2)}{2(z^2 + 1)} t^2 + \frac{z}{z^2 + 1} t + \frac{3}{2(z^2 + 1)}.$$

Continue the reduction procedure of Section 3.2. we obtain

$$u' + u^2 = Bu + C,$$

where

$$B = -\frac{z^3}{(z^2 + 1)(z^2 + 2)} \quad \text{and} \quad C = \frac{3(z^2 + 2)}{4(z^2 + 1)^2}.$$

And finally we obtain the following classical Riccati equation

$$v' + v^2 = -6(z^2+2)^{-2},$$

which has a rational general solution as follows

$$v(z) = -\frac{z}{z^2+2} + \frac{1}{z-\lambda} + \frac{1}{z+2/\lambda}$$

with arbitrary λ. One finally has by substitutions the following solution of equation $F(z, w, w') = 0$:

$$w(z) = \frac{z^2\lambda^2 - 2\,z\lambda^3 + 4\,z\lambda + 4 + \lambda^4 - 3\,\lambda^2}{(z\lambda + 2 - \lambda^2)^2\,z}.$$

5. Conclusion

In this paper, we present an algebraic geometry approach to the study of algebraic differential equations. The complexity of the algorithm and the necessary and sufficient conditions of the rational general solutions of the Riccati equations are studied in the monograph [14]. In [5] and [6], the algebraic geometry approach is also used to obtain bounds of the degree of rational solutions of a first order algebraic differential equation and of the enumeration of the rational solutions of a first order algebraic differential equations.

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