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On Miquel's Five-Circle Theorem

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Abstract. Miquel's Five-Circle Theorem is difficult to prove algebraically. In this paper, the details of the first algebraic proof of this theorem is provided. The proof is based on conformal geometric algebra and its accompanying invariant algebra called null bracket algebra, and is the outcome of the powerful computational techniques of null bracket algebra embodying the novel idea **breefs**.

Keywords: Miquel's Theorem, Mathematics mechanization, Conformal geometric algebra, Null bracket algebra, "breefs".

1. Introduction

Miquel's Five-Circle Theorem is among a sequence of wonderful theorems in plane geometry bearing his name. It states that by starting from a pentagon in the plane, one can construct a 5-star whose vertices are the intersections of the non-neighboring edges of the pentagon. If five circles are drawn such that each circle circumscribes a triangular corner of the star, then the neighboring circles intersect at five new points which are cocircular.

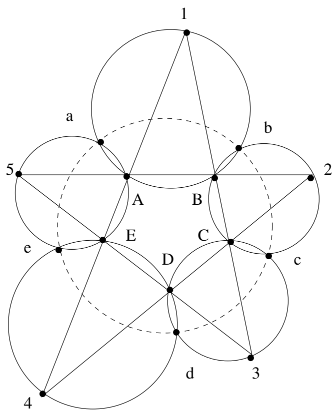

Fig. 1. Miquel's Five-Circle Theorem.

The sequence of Miquel's Theorems is as follows: Let there be an n-gon in the plane. When $n = 3$, the three vertices of a triangle are on a unique circle, which can be taken as the unique circle determined by the three edges of the triangle, called the *Miquel 3-circle*. When $n = 4$, the 4 edges of a quadrilateral form 4 distinct 3-tuples of edges, each determining a Miquel 3-circle, and Miquel's 4-Circle Theorem says that the 4 Miquel 3-circles pass through a common point (i.e., are concurrent), called the *Miquel 4-point*. Miquel's Five Circle Theorem says that the 5 Miquel 4-points of a pentagon are cocircular, and the unique circle is the *Miquel 5-circle* of the pentagon. In general, Miquel's $2m$-Circle Theorem says that the $2m$ Miquel $(2m-1)$-circles of an $(2m)$-gon are concurrent, and Miquel's $(2m+1)$-Circle Theorem says that the $2m+1$ Miquel $(2m)$-points of an $(2m+1)$-gon are cocircular.

The first proof of Miquel's n-Circle Theorem is given by Clifford in the 19th century in a purely geometric manner. By now, an algebraic (analytic) proof has not been found yet. While for $n = 4$ an algebraic proof is easy, for $n = 5$ it is extremely difficult. In fact, the first algebraic proof for $n = 5$ was found in 2001 [4]. In mechanical geometric theorem proving, the first machine proof for $n = 5$ was produced in 1994 [1], which is a purely geometric one.

Below let us analyze how to prove Miquel's 5-Circle Theorem algebraically. The geometric configuration has linear construction, i.e., the constrained objects can be constructed sequentially as unique geometric intersections, whose algebraic representations can be given explicitly by polynomial expressions. The following is a typical linear construction:

Free points in the plane: A, B, C, D, E.

Intersections:

$$\begin{array}{cccc} 1 = EA \cap BC, & 2 = AB \cap CD, & 3 = BC \cap DE, & 4 = CD \cap EA, \\ 5 = DE \cap AB, & a = AE5 \cap AB1, & b = BA1 \cap BC2, & c = CB2 \cap CD3, \\ d = DC3 \cap DE4, & e = ED4 \cap EA5. & & \end{array}$$

Conclusion: a, b, c, d, e are cocircular.

Here $1 = EA \cap BC$ denotes that point 1 is the intersection of lines EA and BC, and $a = AE5 \cap AB1$ denotes the second intersection of circles $AE5$ and $AB1$ other than A if they intersect, or A itself if the two circles are tangent to each other.

By symmetry, we only need to prove the cocircularity of a, b, c, d. Since the construction is linear, we only need to substitute the explicit expressions of a, b, c, d in terms of the free points A, B, C, D, E into the algebraic equality f representing the cocircularity of a, b, c, d, and then expand the result to get zero.

Indeed, the proof is strategically very easy: it is composed of a first procedure of substitution (elimination) and a second procedure of simplification. However, both procedures are too difficult to handle by either coordinates and any other classical invariants, because extremely complicated symbolic computations are involved.

In 2001, a new powerful algebraic tool for geometric computation, called *conformal geometric algebra*, was proposed [3]. Its accompanying invariant algebra called *null bracket algebra* was proposed at the same time [4]. The two algebras can provide elegant simplifications in symbolic computation of geometric problems. The idea leading to such simplifications is recently summarized as "**breefs**" (**b**racket-oriented **r**epresentation, **e**limi- nation and **e**xpansion for **f**actored and **s**hortest result) [6], [7], [8], [9], [10], [11].

In this paper, we are going to show the detailed procedure of the algebraic proof of Miquel's 5-Circle Theorem [4], which has never been shown before. We will show how the idea **breefs** is used in the proof, and how the simplifications are achieved by using conformal geometric algebra and its accompanying invariant algebra. Due to the limit of space we only cite the necessary theorems and formulas in both algebras needed in the proof. For details we refer to [3], [5], [9].

2. Conformal Geometric Algebra

We omit the introduction of *geometric algebra*, and recommend [2] for a clear reading.

Conformal geometric algebra is the geometric algebra set on the so-called "conformal model" of classical geometry: To study nD Euclidean (or hyperbolic, or spherical) geometry, we elevate the geometric space into the null cone of an $(n+2)$D Minkowski space $\mathcal{M}$. Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathcal{R}^n$, let $\mathbf{e}, \mathbf{e}_0$ be a standard null basis of the Minkowski plane orthogonal to $\mathcal{R}^n$, i.e., $\mathbf{e}^2 = \mathbf{e}_0^2 = 0$ and $\mathbf{e} \cdot \mathbf{e}_0 = -1$, then a point $a \in \mathcal{R}^n$ corresponds to the following null vector:

$$\mathbf{a} = \mathbf{e}_0 + a + \frac{a^2}{2}\mathbf{e}. \tag{2.1}$$

By setting $a = 0$ we see that the origin of $\mathcal{R}^n$ corresponds to vector $\mathbf{e}_0$. No point corresponds to $\mathbf{e}$: it represents the *point at infinity.* Vector $\mathbf{a}$ satisfies $\mathbf{a} \cdot \mathbf{e} = -1$. It is the *inhomogeneous* representation of point a. The *homogeneous* representation is any null vector collinear with $\mathbf{a}$, and is more convenient for symbolic computation. On the other hand, the inhomogeneous representation is more convenient for geometric interpretation. For example, in the inhomogeneous representation,

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = -\frac{d(a_1, a_2)^2}{2}. \tag{2.2}$$

i.e., the squared distance of two points becomes the inner product of the corresponding null vectors. The circles and spheres of various dimensions correspond to Minkowski subspaces, and can be computed homogeneously. The conformal transformations are realized by orthogonal transformations in $\mathcal{M}$ and can be computed by spin representations.

In conformal geometric algebra, geometric objects and relations often have universal and compact algebraic representations. For example, when 1 is the point at infinity, then $123 \cap 12'3'$ is exactly the intersection of two lines $23, 2'3'$. In fact, when any of the points is replaced by the points at infinity, the corresponding circle becomes a straight line, but the algebraic representation is still the same. The algebraic representation of $123 \cap 12'3'$, denoted by $\mathbf{123} \cap \mathbf{12'3'}$, has two different forms:

$$\begin{array}{rl} & \mathbf{123} \cap \mathbf{12'3'} \\ = & \mathbf{2} \cdot \mathbf{3}[\mathbf{122'3'}][\mathbf{132'3'}]\mathbf{1} - \frac{1}{2}[\mathbf{12312'3'}][\mathbf{132'3'}]\mathbf{2} + \frac{1}{2}[\mathbf{12312'3'}][\mathbf{122'3'}]\mathbf{3} \\ = & \mathbf{2'} \cdot \mathbf{3'}[\mathbf{1232'}][\mathbf{1233'}]\mathbf{1} + \frac{1}{2}[\mathbf{12312'3'}][\mathbf{1233'}]\mathbf{2'} - \frac{1}{2}[\mathbf{12312'3'}][\mathbf{1232'}]\mathbf{3'}. \end{array} \tag{2.3}$$

Obviously, $\mathbf{123} \cap \mathbf{12'3'} = \mathbf{132} \cap \mathbf{12'3'} = \mathbf{123} \cap \mathbf{13'2'} = \mathbf{12'3'} \cap \mathbf{123}$. Let $2'', 3''$ be points on circle $12'3'$. Then

$$\mathbf{1} \cdot (\mathbf{123} \cap \mathbf{12'3'}) = [\mathbf{12312'3'}]^2, \quad \mathbf{2} \cdot (\mathbf{123} \cap \mathbf{12'3'}) = (\mathbf{1} \cdot \mathbf{3})(\mathbf{2} \cdot \mathbf{3})[\mathbf{122'3'}]^2. \tag{2.4}$$

3. Null Bracket Algebra

Null bracket algebra is the accompanying invariant algebra of conformal geometric algebra when all geometric constructions are based on points. However, it can be defined and employed independent of conformal geometric algebra. The following is a self-contained definition of this algebra.

Definition 3.1 *Let $\mathcal{K}$ be a field of characteristic $\neq 2$. Let $n \leq m$ be two positive integers. The nD null bracket algebra generated by symbols $\mathbf{a}_1, \ldots, \mathbf{a}_m$, is the quotient of the polynomial ring over $\mathcal{K}$ with indeterminates $\langle \mathbf{a}_{i_1} \ldots \mathbf{a}_{i_{2p}} \rangle$ and $[\mathbf{a}_{j_1} \ldots \mathbf{a}_{j_{n+2q-2}}]$ for $p, q \geq 1$ and $1 \leq i_1, \ldots, i_{2p}, j_1, \ldots, j_{n+2q-2} \leq m$, modulo the ideal generated by the following 8 types of elements:*

B1. $[\mathbf{a}_{i_1} \ldots \mathbf{a}_{i_n}]$ *if $i_j = i_k$ for some $j \neq k$.*

B2. $[\mathbf{a}_{i_1} \ldots \mathbf{a}_{i_n}] - \mathrm{sign}(\sigma)[\mathbf{a}_{i_{\sigma(1)}} \ldots \mathbf{a}_{i_{\sigma(n)}}]$ *for a permutation σ of $1, \ldots, n$.*

B3. $\langle \mathbf{a}_i \mathbf{a}_j \rangle - \langle \mathbf{a}_j \mathbf{a}_i \rangle$ *for $i \neq j$.*

N. $\langle \mathbf{a}_i \mathbf{a}_i \rangle$ *for any i.*

GP1. $\displaystyle\sum_{k=1}^{n+1} (-1)^{k+1} \langle \mathbf{a}_j \mathbf{a}_{i_k} \rangle \, [\mathbf{a}_{i_1} \ldots \check{\mathbf{a}}_{i_k} \ldots \mathbf{a}_{i_{n+1}}]$.

GP2. $[\mathbf{a}_{i_1} \ldots \mathbf{a}_{i_n}][\mathbf{a}_{j_1} \ldots \mathbf{a}_{j_n}] - \det(\langle \mathbf{a}_{i_k} \mathbf{a}_{j_l} \rangle)_{k,l=1..n}$.

AB. $\displaystyle\langle \mathbf{a}_{i_1} \cdots \mathbf{a}_{i_{2l}} \rangle - \sum_{j=2}^{2l} (-1)^j \langle \mathbf{a}_{i_1} \mathbf{a}_{i_j} \rangle \, \langle \mathbf{a}_{i_2} \cdots \check{x}_{i_j} \cdots \mathbf{a}_{i_{2l}} \rangle$.

SB. $\displaystyle[\mathbf{a}_{i_1} \cdots \mathbf{a}_{i_{n+2l}}] - \sum_{\sigma} \mathrm{sign}(\sigma, \check{\sigma}) \langle x_{\sigma(1)} \cdots x_{\sigma(2l)} \rangle \times [x_{\check{\sigma}(1)} \cdots x_{\check{\sigma}(n)}]$ *for all partitions $\sigma, \check{\sigma}$ of $1, 2, \ldots, n+2l$ into two subsequences of length $2l$ and n respectively.*

In practice we usually use notations $\mathbf{a} \cdot \mathbf{a}_j$ and $\mathbf{a}_i^2$ instead of $\langle \mathbf{a}_i \mathbf{a}_j \rangle$ and $\langle \mathbf{a}_i \mathbf{a}_j \rangle$.

When $n = 4$, the following are some typical brackets and their geometric meanings:

1.

$$
\begin{array}{rclcl}
\langle \mathbf{e}\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 \rangle & = & (a_3 - a_2) \cdot (a_1 - a_2) & = & |a_1a_2||a_2a_3| \cos \angle(a_2a_3, a_2a_1); \\
[\mathbf{e}\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3] & = & 2S_{a_1a_2a_3} & = & |a_1a_2||a_2a_3| \sin \angle(a_2a_3, a_2a_1).
\end{array}
\tag{3.1}
$$

Here $S_{a_1a_2a_3}$ is the signed area of triangle $a_1a_2a_3$, and $\angle(a_2a_3, a_2a_1)$ is the oriented angle from vector a_2a_3 to vector a_2a_1. In particular, a_1, a_2, a_3 are collinear if and only if $[\mathbf{e}\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3] = 0$.

2.

$$
\begin{array}{rcl}
\langle \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4 \rangle & = & -8\dfrac{\rho_{a_1a_2a_3}\rho_{a_1a_3a_4}S_{a_1a_2a_3}S_{a_1a_3a_4}}{|a_1a_3|^2} \cos \angle(N^{a_1}_{a_1a_2a_3}, N^{a_1}_{a_1a_3a_4}) \\
[\mathbf{a}_1\mathbf{a}_3\mathbf{a}_2\mathbf{a}_4] & = & -8\dfrac{\rho_{a_1a_2a_3}\rho_{a_1a_3a_4}S_{a_1a_2a_3}S_{a_1a_3a_4}}{|a_1a_3|^2} \sin \angle(N^{a_1}_{a_1a_2a_3}, N^{a_1}_{a_1a_3a_4}) \\
 & = & (\rho^2_{a_1a_2a_3} - |o_{a_1a_2a_3}a_4|^2) S_{a_1a_2a_3}.
\end{array}
\tag{3.2}
$$

Here $o_{a_1a_2a_3}$ is the center of circle $a_1a_2a_3$; $\rho_{a_1a_2a_3}, \rho_{a_1a_3a_4}$ are the radii of circles $a_1a_2a_3$, $a_1a_3a_4$ respectively, and $N^{a_1}_{a_1a_2a_3}, N^{a_1}_{a_1a_3a_4}$ are the outward normal directions of circles $a_1a_2a_3$, $a_1a_3a_4$ at point a_1 respectively. In particular, points a_1, a_2, a_3, a_4 are cocircular if and only if $[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4] = 0$.

3.

$$
\begin{array}{rclcl}
[\mathbf{e}\mathbf{a}_2\mathbf{a}_3\mathbf{e}\mathbf{a}_4\mathbf{a}_5] & = & -4S_{a_2a_4a_3a_5} & = & 2|a_2a_3||a_4a_5| \sin \angle(a_4a_5, a_2a_3), \\
\langle \mathbf{e}\mathbf{a}_2\mathbf{a}_3\mathbf{e}\mathbf{a}_4\mathbf{a}_5 \rangle & = & 2(a_3 - a_2) \cdot (a_5 - a_4) & = & 2|a_2a_3||a_4a_5| \cos \angle(a_4a_5, a_2a_3).
\end{array}
\tag{3.3}
$$

Here $S_{a_2a_4a_3a_5}$ is the signed area of quadrilateral $a_2a_4a_3a_5$. In particular, $[\mathbf{ea}_2\mathbf{a}_3\mathbf{ea}_4\mathbf{a}_5] = 0$ if and only if $a_2a_3 \,||\, a_4a_5$; $\langle \mathbf{ea}_2\mathbf{a}_3\mathbf{ea}_4\mathbf{a}_5\rangle = 0$ if and only if $a_2a_3 \perp a_4a_5$.

In this paper, we often use the following shorthand notation:

$$[\mathbf{a}_1\mathbf{a}_2;\mathbf{a}_3\mathbf{a}_4] = -\frac{1}{2}[\mathbf{ea}_1\mathbf{a}_2\mathbf{ea}_3\mathbf{a}_4] = \mathbf{e}\cdot\mathbf{a}_1[\mathbf{ea}_2\mathbf{a}_3\mathbf{a}_4] - \mathbf{e}\cdot\mathbf{a}_2[\mathbf{ea}_1\mathbf{a}_3\mathbf{a}_4]. \tag{3.4}$$

4.

$$\begin{array}{rcl} \langle \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_4\mathbf{a}_5\rangle &=& 8\rho_{a_1a_2a_3}\rho_{a_1a_4a_5}S_{a_1a_2a_3}S_{a_1a_4a_5}\cos\angle(N^{a_1}_{a_1a_2a_3}, N^{a_1}_{a_1a_4a_5}),\\ {[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_4\mathbf{a}_5]} &=& 8\rho_{a_1a_2a_3}\rho_{a_1a_4a_5}S_{a_1a_2a_3}S_{a_1a_4a_5}\sin\angle(N^{a_1}_{a_1a_2a_3}, N^{a_1}_{a_1a_4a_5})\\ &=& 16S_{a_1a_2a_3}S_{a_1a_4a_5}S_{a_1o_{123}o_{145}}. \end{array} \tag{3.5}$$

Here o_{123}, o_{145} are the centers of circles $a_1a_2a_3, a_1a_4a_5$ respectively. In particular, $[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_4\mathbf{a}_5] = 0$ if and only if circles $a_1a_2a_3$, $a_1a_4a_5$ are tangent to each other; $\langle \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_4\mathbf{a}_5\rangle = 0$ if and only if they are perpendicular.

5.

$$[\mathbf{a}_1;\mathbf{a}_2\mathbf{a}_3;\mathbf{a}_4\mathbf{a}_5] = \mathbf{e}\cdot\mathbf{a}_4\,\mathbf{e}\cdot\mathbf{a}_5[\mathbf{ea}_1\mathbf{a}_2\mathbf{a}_3] - \mathbf{e}\cdot\mathbf{a}_2\,\mathbf{e}\cdot\mathbf{a}_3[\mathbf{ea}_1\mathbf{a}_4\mathbf{a}_5] + \mathbf{e}\cdot\mathbf{a}_2\,\mathbf{e}\cdot\mathbf{a}_4\,[\mathbf{ea}_1\mathbf{a}_3\mathbf{a}_5]$$

is twice the signed area of pentagon $a_1a_2a_3a_5a_4$. Furthermore,

$$\begin{array}{l} [\mathbf{a}_1;\mathbf{a}_2\mathbf{a}_3;\mathbf{a}_4\mathbf{a}_5] = -[\mathbf{a}_1;\mathbf{a}_4\mathbf{a}_5;\mathbf{a}_2\mathbf{a}_3];\\ {[\mathbf{a}_1;\mathbf{a}_2\mathbf{a}_3;\mathbf{a}_5\mathbf{a}_4]} = [\mathbf{a}_2;\mathbf{a}_3\mathbf{a}_4;\mathbf{a}_1\mathbf{a}_5] = \cdots = [\mathbf{a}_5;\mathbf{a}_1\mathbf{a}_2;\mathbf{a}_4\mathbf{a}_3]. \end{array} \tag{3.6}$$

The following are some typical formulas for algebraic computation in 4D null bracket algebra:

- **Fundamental formulas:**

$$\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_k\mathbf{a}_1 = 2\sum_{i=2}^{k}(-1)^i\mathbf{a}_1\cdot\mathbf{a}_i\,(\mathbf{a}_2\cdots\check{\mathbf{a}}_i\cdots\mathbf{a}_k\mathbf{a}_1). \tag{3.7}$$

In particular, $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1 = -\mathbf{a}_1\mathbf{a}_3\mathbf{a}_2\mathbf{a}_1$.

- **Expansion formulas:**

$$\begin{array}{l} \frac{1}{2}[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\mathbf{a}_1\mathbf{a}_5\cdots\mathbf{a}_{2l+5}]\\ \quad = \langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\rangle\,[\mathbf{a}_1\mathbf{a}_5\cdots\mathbf{a}_{2l+5}] + [\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4]\,\langle\mathbf{a}_1\mathbf{a}_5\cdots\mathbf{a}_{2l+5}\rangle,\\ \frac{1}{2}\langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\mathbf{a}_1\mathbf{a}_5\cdots\mathbf{a}_{2l+5}\rangle\\ \quad = \langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\rangle\,\langle\mathbf{a}_1\mathbf{a}_5\cdots\mathbf{a}_{2l+5}\rangle - [\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4]\,[\mathbf{a}_1\mathbf{a}_5\cdots\mathbf{a}_{2l+5}],\\ \frac{1}{2}[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_4\mathbf{a}_5\mathbf{a}_1\mathbf{a}_6\cdots\mathbf{a}_{2l+6}]\\ \quad = [\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_4\mathbf{a}_5]\langle\mathbf{a}_1\mathbf{a}_6\cdots\mathbf{a}_{2l+6}\rangle + \langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_4\mathbf{a}_5\rangle[\mathbf{a}_1\mathbf{a}_6\cdots\mathbf{a}_{2l+6}],\\ \frac{1}{2}\langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_4\mathbf{a}_5\mathbf{a}_1\mathbf{a}_6\cdots\mathbf{a}_{2l+6}\rangle\\ \quad = \langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_4\mathbf{a}_5\rangle\langle\mathbf{a}_1\mathbf{a}_6\cdots\mathbf{a}_{2l+6}\rangle - [\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_4\mathbf{a}_5][\mathbf{a}_1\mathbf{a}_6\cdots\mathbf{a}_{2l+6}]. \end{array} \tag{3.8}$$

- **Distribution formulas:**

$$\begin{aligned}\frac{1}{2}[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_4\mathbf{a}_5][\mathbf{a}_3\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\mathbf{a}_5] &= \mathbf{a}_1\cdot\mathbf{a}_2\,[\mathbf{a}_1\mathbf{a}_3\mathbf{a}_4\mathbf{a}_5][\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\mathbf{a}_5]\\ &\quad -\mathbf{a}_4\cdot\mathbf{a}_5\,[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4][\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_5],\\ \frac{1}{2}[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_4\mathbf{a}_5]\langle\mathbf{a}_3\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\mathbf{a}_5\rangle &= \mathbf{a}_1\cdot\mathbf{a}_2\,[\mathbf{a}_1\mathbf{a}_3\mathbf{a}_4\mathbf{a}_5]\langle\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\mathbf{a}_5\rangle\\ &\quad +\mathbf{a}_4\cdot\mathbf{a}_5\,\langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\rangle[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_5].\end{aligned} \tag{3.9}$$

- **Factorization formulas:**

Group 1.

$$\begin{aligned}&\mathbf{a}_2\cdot\mathbf{a}_3[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_5\mathbf{a}_6][\mathbf{a}_3\mathbf{a}_4\mathbf{a}_5\mathbf{a}_6]+\mathbf{a}_5\cdot\mathbf{a}_6[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_6][\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\mathbf{a}_5]\\ &\qquad = -\frac{1}{2}[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\mathbf{a}_5\mathbf{a}_6][\mathbf{a}_2\mathbf{a}_3\mathbf{a}_5\mathbf{a}_6],\\ &\mathbf{a}_2\cdot\mathbf{a}_3[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_5\mathbf{a}_6]\langle\mathbf{a}_3\mathbf{a}_4\mathbf{a}_5\mathbf{a}_6\rangle-\mathbf{a}_5\cdot\mathbf{a}_6[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_6]\langle\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\mathbf{a}_5\rangle\\ &\qquad = -\frac{1}{2}\langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\mathbf{a}_5\mathbf{a}_6\rangle[\mathbf{a}_2\mathbf{a}_3\mathbf{a}_5\mathbf{a}_6].\end{aligned} \tag{3.10}$$

Group 2.

$$\begin{aligned}&\langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\rangle[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_5\mathbf{a}_6]-\langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_5\mathbf{a}_6\rangle[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4]\\ &\qquad = -2\mathbf{a}_1\cdot\mathbf{a}_2\,\mathbf{a}_2\cdot\mathbf{a}_3[\mathbf{a}_1\mathbf{a}_3\mathbf{a}_4\mathbf{a}_1\mathbf{a}_5\mathbf{a}_6],\\ &\langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\rangle\langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_5\mathbf{a}_6\rangle+[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_5\mathbf{a}_6][\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4]\\ &\qquad = -2\mathbf{a}_1\cdot\mathbf{a}_2\,\mathbf{a}_2\cdot\mathbf{a}_3\langle\mathbf{a}_1\mathbf{a}_3\mathbf{a}_4\mathbf{a}_1\mathbf{a}_5\mathbf{a}_6\rangle.\end{aligned} \tag{3.11}$$

Group 3.

$$\begin{aligned}&\mathbf{a}_2\cdot\mathbf{a}_3[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_5\mathbf{a}_6][\mathbf{a}_1\mathbf{a}_2\mathbf{a}_5\mathbf{a}_1\mathbf{a}_3\mathbf{a}_4]-\mathbf{a}_2\cdot\mathbf{a}_5[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4][\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_5\mathbf{a}_6]\\ &\qquad = \frac{1}{2}[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\mathbf{a}_1\mathbf{a}_2\mathbf{a}_5\mathbf{a}_6][\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_5],\\ &\mathbf{a}_2\cdot\mathbf{a}_3\langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_5\mathbf{a}_6\rangle[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_5\mathbf{a}_1\mathbf{a}_3\mathbf{a}_4]-\mathbf{a}_2\cdot\mathbf{a}_5[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4]\langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_5\mathbf{a}_6\rangle\\ &\qquad = \frac{1}{2}\langle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\mathbf{a}_1\mathbf{a}_2\mathbf{a}_5\mathbf{a}_6\rangle[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_5].\end{aligned} \tag{3.12}$$

4. The Five-Circle Theorem: Elimination

We are now equipped with all necessary algebraic tools to prove the 5-circle theorem. In order to simplify the proof, we need to make some symmetry analysis of the geometric configuration of the theorem before the formal start of the proof.

- The conclusion that a, b, c, d are cocircular can be represented by $[\mathbf{abcd}] = 0$. We need to compute the expressions of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ by the 5 free points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}$. Since $[\mathbf{abcd}] = [(\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{c} \wedge \mathbf{d})]$, we need to compute $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{c} \wedge \mathbf{d}$.
- If $\mathbf{a} = f(\mathbf{A}, \mathbf{1}, \mathbf{5}, \mathbf{B}, \mathbf{E}) = g(\mathbf{A}, \mathbf{C}, \mathbf{D}, \mathbf{B}, \mathbf{E})$, then since points a, b are symmetric with respect to line $1D$ combinatorially, it must be that

$$\mathbf{b} = f(\mathbf{B}, \mathbf{1}, \mathbf{2}, \mathbf{A}, \mathbf{C}) = g(\mathbf{B}, \mathbf{E}, \mathbf{D}, \mathbf{A}, \mathbf{C}), \tag{4.1}$$

i.e., $\mathbf{b}$ can be derived from $\mathbf{a}$ by the interchanges $\mathbf{A} \longleftrightarrow \mathbf{B}$ and $\mathbf{C} \longleftrightarrow \mathbf{E}$.

- Since (d, c) and (a, b) are symmetric with respect to line $2E$ combinatorially, if we have obtained $\mathbf{a} \wedge \mathbf{b} = f(\mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}) = g(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E})$, then we can get directly

$$\mathbf{c} \wedge \mathbf{d} = -f(\mathbf{3}, \mathbf{2}, \mathbf{4}, \mathbf{D}, \mathbf{C}, \mathbf{B}, \mathbf{A}, \mathbf{E}) = -g(\mathbf{D}, \mathbf{C}, \mathbf{B}, \mathbf{A}, \mathbf{E}). \tag{4.2}$$

Thus, the first stage of the proof, i.e., elimination, should contain the following steps: (1) compute $\mathbf{a}$, (2) compute $\mathbf{b}$ by interchanges of symbols, then compute $a \wedge \mathbf{b}$, (3) compute $\mathbf{c} \cdot \mathbf{d}$ by interchanges of symbols, then compute $[\mathbf{abcd}]$. In this stage, computation means the elimination of $\mathbf{1}, \cdots, \mathbf{5}$.

Step 1. Compute $a = AB1 \cap AE5$: We express $\mathbf{a}$ as a linear combination of $\mathbf{A}, \mathbf{B}, \mathbf{1}$ instead of $\mathbf{A}, \mathbf{E}, \mathbf{5}$ in order to employ the symmetry between $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a} = \mathbf{B} \cdot \mathbf{1}\, [\mathbf{AE5B}][\mathbf{AE51}]\, \mathbf{A} - \frac{1}{2}[\mathbf{AE51}][\mathbf{AB1AE5}]\, \mathbf{B} + \frac{1}{2}[\mathbf{AE5B}][\mathbf{AB1AE5}]\, \mathbf{1}. \tag{4.3}$$

To eliminate $\mathbf{1}, \mathbf{5}$ from the brackets, in $\mathbf{A} \cdot \mathbf{1}$ we use $\mathbf{1} = \mathbf{1}(\mathbf{e}, \mathbf{A}, \mathbf{E})$ and (2.4), and in $\mathbf{B} \cdot \mathbf{1}$ we use $\mathbf{1} = \mathbf{1}(\mathbf{e}, \mathbf{B}, \mathbf{C})$. The results are two bracket monomials. In $[\mathbf{AE5B}]$ we use $\mathbf{5} = \mathbf{5}(\mathbf{e}, \mathbf{A}, \mathbf{B})$ so that $\mathbf{A}, \mathbf{B}$ in the bracket annihilate the same vector symbols in the representation of $\mathbf{5}$, leading to a bracket monomial result. Similarly, in $[\mathbf{AE51}]$ we use $\mathbf{1} = \mathbf{1}(\mathbf{e}, \mathbf{A}, \mathbf{E})$, and either of $\mathbf{5} = \mathbf{5}(\mathbf{e}, \mathbf{A}, \mathbf{B})$ and $\mathbf{5} = \mathbf{5}(\mathbf{e}, \mathbf{D}, \mathbf{E})$. We get

$$\begin{aligned}
[\mathbf{AE51}] &= -\mathbf{A} \cdot \mathbf{E}\, [\mathbf{eABC}][\mathbf{eABE}][\mathbf{eADE}][\mathbf{eBCE}][\mathbf{AB}; \mathbf{DE}], \\
[\mathbf{AE5B}] &= -\mathbf{A} \cdot \mathbf{B}\, [\mathbf{eABE}][\mathbf{eADE}][\mathbf{eBDE}], \\
\mathbf{A} \cdot \mathbf{1} &= \mathbf{e} \cdot \mathbf{E}\, \mathbf{A} \cdot \mathbf{E}\, [\mathbf{eABC}]^2, \\
\mathbf{B} \cdot \mathbf{1} &= \mathbf{e} \cdot \mathbf{C}\, \mathbf{B} \cdot \mathbf{C}\, [\mathbf{eABE}]^2,
\end{aligned}$$

and

$$\begin{aligned}
& -\frac{1}{2}[\mathbf{AB1AE5}] \\
=\ & \mathbf{A} \cdot \mathbf{B}[\mathbf{AE51}] - \mathbf{A} \cdot \mathbf{1}[\mathbf{AE5B}] \\
=\ & \mathbf{A} \cdot \mathbf{B}\, \mathbf{A} \cdot \mathbf{E}\, [\mathbf{eABC}][\mathbf{eABE}][\mathbf{eADE}] \\
& (-[\mathbf{eBCE}][\mathbf{AB}; \mathbf{DE}] + \mathbf{e} \cdot \mathbf{E}\, [\mathbf{eABC}][\mathbf{eBDE}]) \\
=\ & \mathbf{A} \cdot \mathbf{B}\, \mathbf{A} \cdot \mathbf{E}\, [\mathbf{eABC}][\mathbf{eABE}][\mathbf{eADE}] \\
& (\mathbf{e} \cdot \mathbf{D}[\mathbf{eABE}][\mathbf{eBCE}] - \mathbf{e} \cdot \mathbf{E}[\mathbf{eABD}][\mathbf{eBCE}] + \mathbf{e} \cdot \mathbf{E}\, [\mathbf{eABC}][\mathbf{eBDE}]) \\
=\ & \mathbf{A} \cdot \mathbf{B}\, \mathbf{A} \cdot \mathbf{E}\, [\mathbf{eABC}][\mathbf{eABE}][\mathbf{eADE}] \\
& (\mathbf{e} \cdot \mathbf{D}[\mathbf{eABE}][\mathbf{eBCE}] - \mathbf{e} \cdot \mathbf{E}[\mathbf{eABE}][\mathbf{eBCD}]) \\
=\ & \mathbf{A} \cdot \mathbf{B}\, \mathbf{A} \cdot \mathbf{E}\, [\mathbf{eABC}][\mathbf{eABE}]^2[\mathbf{eADE}][\mathbf{BC}; \mathbf{DE}].
\end{aligned}$$

The next to the last step is based on a *Grassmann-Plücker relation* [10]:

$$-\mathbf{e} \cdot \mathbf{E}[\mathbf{eABD}][\mathbf{eBCE}] + \mathbf{e} \cdot \mathbf{E}\, [\mathbf{eABC}][\mathbf{eBDE}] = -\mathbf{e} \cdot \mathbf{E}[\mathbf{eABE}][\mathbf{eBCD}]. \tag{4.4}$$

After removing 8 common bracket factors $\mathbf{A} \cdot \mathbf{B}\, \mathbf{A} \cdot \mathbf{E}\, [\mathbf{eABC}][\mathbf{eABE}]^3[\mathbf{eADE}]$, we get

$$\begin{aligned}
\mathbf{a} \;=\; & \mathbf{e} \cdot \mathbf{C}\, \mathbf{B} \cdot \mathbf{C}\, [\mathbf{eABE}][\mathbf{eBCE}][\mathbf{eBDE}][\mathbf{AB}; \mathbf{DE}]\, \mathbf{A} \\
& + \mathbf{A} \cdot \mathbf{E}\, [\mathbf{eABC}][\mathbf{eBCE}][\mathbf{AB}; \mathbf{DE}][\mathbf{BC}; \mathbf{DE}]\, \mathbf{B} \\
& - \mathbf{A} \cdot \mathbf{B}\, [\mathbf{eBDE}][\mathbf{BC}; \mathbf{DE}]\, \mathbf{1}.
\end{aligned} \tag{4.5}$$

Step 2. Compute $b = AB1 \cap BC2$ and $\mathbf{a} \wedge \mathbf{b}$:

$$\begin{array}{rcr} \mathbf{b} & = & \mathbf{B}\cdot\mathbf{C}\,[\mathbf{eABE}][\mathbf{eACE}][\mathbf{AB};\mathbf{CD}][\mathbf{EA};\mathbf{CD}]\,\mathbf{A} \\ & & -\,\mathbf{e}\cdot\mathbf{E}\,\mathbf{A}\cdot\mathbf{E}\,[\mathbf{eABC}][\mathbf{eACD}][\mathbf{eACE}][\mathbf{AB};\mathbf{CD}]\,\mathbf{B} \\ & & +\,\mathbf{A}\cdot\mathbf{B}\,[\mathbf{eACD}][\mathbf{EA};\mathbf{CD}]\,\mathbf{1} \end{array} \tag{4.6}$$

and $\mathbf{a} \wedge \mathbf{b} = \lambda_{AB}\mathbf{A} \wedge \mathbf{B} + \lambda_{A1}\mathbf{A} \wedge \mathbf{1} + \lambda_{B1}\mathbf{B} \wedge \mathbf{1}$, where

$$\begin{array}{rcl} \lambda_{AB} & = & -\mathbf{A}\cdot\mathbf{E}\,\mathbf{B}\cdot\mathbf{C}\,[\mathbf{eABC}][\mathbf{eABE}][\mathbf{eACE}][\mathbf{eBCE}][\mathbf{eCDE}][\mathbf{AB};\mathbf{CD}] \\ & & \qquad [\mathbf{AB};\mathbf{DE}][\mathbf{D};\mathbf{AC};\mathbf{BE}], \\ \lambda_{A1} & = & -\mathbf{A}\cdot\mathbf{B}\,\mathbf{B}\cdot\mathbf{C}\,[\mathbf{eABC}][\mathbf{eABE}][\mathbf{eBDE}][\mathbf{eCDE}][\mathbf{EA};\mathbf{CD}][\mathbf{A};\mathbf{CE};\mathbf{DB}], \\ \lambda_{B1} & = & \mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,[\mathbf{eABC}][\mathbf{eABE}][\mathbf{eACD}][\mathbf{eCDE}][\mathbf{BC};\mathbf{DE}][\mathbf{B};\mathbf{DA};\mathbf{EC}]. \end{array} \tag{4.7}$$

After removing 4 common factors $[\mathbf{eABC}][\mathbf{eABE}][\mathbf{eCDE}][\mathbf{D};\mathbf{AC};\mathbf{BE}]$, we get

$$\begin{array}{rcr} \mathbf{a}\wedge\mathbf{b} & = & -\mathbf{A}\cdot\mathbf{E}\,\mathbf{B}\cdot\mathbf{C}\,[\mathbf{eACE}][\mathbf{eBCE}][\mathbf{AB};\mathbf{CD}][\mathbf{AB};\mathbf{DE}]\,\mathbf{A}\wedge\mathbf{B} \\ & & +\mathbf{A}\cdot\mathbf{B}\,\mathbf{B}\cdot\mathbf{C}\,[\mathbf{eBDE}][\mathbf{EA};\mathbf{CD}]\,\mathbf{A}\wedge\mathbf{1} \\ & & +\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,[\mathbf{eACD}][\mathbf{BC};\mathbf{DE}]\,\mathbf{B}\wedge\mathbf{1}. \end{array} \tag{4.8}$$

Step 3. Compute $\mathbf{c} \wedge \mathbf{d}$ and$[\mathbf{abcd}]$:

$$\begin{array}{rcl} \mathbf{c}\wedge\mathbf{d} & = & \lambda_{CD}\mathbf{C}\wedge\mathbf{D} + \lambda_{C3}\mathbf{C}\wedge\mathbf{3} + \lambda_{D3}\mathbf{D}\wedge\mathbf{3} \\ & = & \;\;\mathbf{B}\cdot\mathbf{C}\,\mathbf{D}\cdot\mathbf{E}\,[\mathbf{eBCE}][\mathbf{eBDE}][\mathbf{AB};\mathbf{CD}][\mathbf{EA};\mathbf{CD}]\,\mathbf{C}\wedge\mathbf{D} \\ & & +\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}\,[\mathbf{eABD}][\mathbf{EA};\mathbf{BC}]\,\mathbf{C}\wedge\mathbf{3} \\ & & +\mathbf{B}\cdot\mathbf{C}\,\mathbf{C}\cdot\mathbf{D}\,[\mathbf{eACE}][\mathbf{AB};\mathbf{DE}]\,\mathbf{D}\wedge\mathbf{3} \end{array} \tag{4.9}$$

and

$$\begin{array}{rcl} [\mathbf{abcd}] & = & \lambda_{AB}\lambda_{CD}[\mathbf{ABCD}] + \lambda_{AB}\lambda_{C3}[\mathbf{ABC3}] + \lambda_{AB}\lambda_{D3}[\mathbf{ABD3}] \\ & & +\lambda_{A1}\lambda_{CD}[\mathbf{A1CD}] + \lambda_{A1}\lambda_{C3}[\mathbf{A1C3}] + \lambda_{A1}\lambda_{D3}[\mathbf{A1D3}] \\ & & +\lambda_{B1}\lambda_{CD}[\mathbf{B1CD}] + \lambda_{B1}\lambda_{C3}[\mathbf{B1C3}] + \lambda_{B1}\lambda_{D3}[\mathbf{B1D3}]. \end{array} \tag{4.10}$$

Here $\lambda_{AB}, \lambda_{A1}, \lambda_{B1}$ denote the coefficients of $\mathbf{A} \wedge \mathbf{B}, \mathbf{A} \wedge \mathbf{1}, \mathbf{B} \wedge \mathbf{1}$ in (4.8).

After eliminating $\mathbf{1}, \mathbf{3}$ from (4.10), using the first of the distribution formulas (3.9) to get rid of the square brackets not involving $\mathbf{e}$, then removing 5 common bracket factors

$(\mathbf{B}\cdot\mathbf{C})^2\,[\mathbf{eACE}][\mathbf{eBCE}][\mathbf{eBDE}]$, we get an expression of 14 terms:

$$\begin{aligned}
&[\mathbf{abcd}]\\
=\;&\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{AB};\mathbf{DE}][\mathbf{EA};\mathbf{CD}][\mathbf{eBCE}][\mathbf{eABC}][\mathbf{eABD}]\\
&-\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{AB};\mathbf{DE}][\mathbf{EA};\mathbf{CD}][\mathbf{eACD}][\mathbf{eBCD}][\mathbf{eBCE}]\\
&-\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{AB};\mathbf{DE}][\mathbf{EA};\mathbf{BC}][\mathbf{eABC}][\mathbf{eABD}][\mathbf{eCDE}]\\
&-\mathbf{A}\cdot\mathbf{E}\,\mathbf{B}\cdot\mathbf{C}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{CD}][\mathbf{AB};\mathbf{DE}]^2[\mathbf{eABD}][\mathbf{eACE}][\mathbf{eCDE}]\\
&+\mathbf{A}\cdot\mathbf{E}\,(\mathbf{C}\cdot\mathbf{D})^2[\mathbf{AB};\mathbf{DE}]^2[\mathbf{BC};\mathbf{DE}][\mathbf{eABC}][\mathbf{eABD}][\mathbf{eACE}]\\
&-\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}]^2[\mathbf{BC};\mathbf{DE}][\mathbf{eACD}][\mathbf{eACE}][\mathbf{eBCD}]\\
&+\mathbf{A}\cdot\mathbf{B}\,\mathbf{B}\cdot\mathbf{C}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{EA};\mathbf{CD}]^2[\mathbf{eABE}][\mathbf{eACD}][\mathbf{eBDE}]\\
&+\mathbf{A}\cdot\mathbf{B}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{EA};\mathbf{BC}][\mathbf{EA};\mathbf{CD}]^2[\mathbf{eABC}][\mathbf{eABD}][\mathbf{eBDE}]\\
&-(\mathbf{A}\cdot\mathbf{B})^2\,\mathbf{D}\cdot\mathbf{E}[\mathbf{EA};\mathbf{BC}][\mathbf{EA};\mathbf{CD}]^2[\mathbf{eACD}][\mathbf{eBCD}][\mathbf{eBDE}]\\
&+\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}][\mathbf{EA};\mathbf{CD}][\mathbf{BC};\mathbf{DE}][\mathbf{eABC}][\mathbf{eADE}][\mathbf{eBCD}]\\
&-\mathbf{A}\cdot\mathbf{B}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{DE}][\mathbf{EA};\mathbf{BC}][\mathbf{EA};\mathbf{CD}][\mathbf{eABC}][\mathbf{eADE}][\mathbf{eBCD}]\\
&+\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{EA};\mathbf{CD}][\mathbf{BC};\mathbf{DE}][\mathbf{eABE}][\mathbf{eACD}][\mathbf{eBCD}]\\
&+\mathbf{e}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{B}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{EA};\mathbf{BC}][\mathbf{EA};\mathbf{CD}][\mathbf{eABC}][\mathbf{eABD}][\mathbf{eADE}][\mathbf{eCDE}]\\
&-\mathbf{e}\cdot\mathbf{C}\,\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}][\mathbf{BC};\mathbf{DE}][\mathbf{eABE}][\mathbf{eACD}][\mathbf{eADE}][\mathbf{eBCD}].
\end{aligned}\tag{4.11}$$

5. The Five-Circle Theorem: Simplification

On a Pentium III/500MHz, setting $A=(0,0)$, $B=(1,0)$, we get zero from (4.11) in 0.315 seconds. This is pretty satisfactory for a machine proof. However, we shall show that by means of the term reduction and factorization techniques in null bracket algebra, we can get zero from (4.11) in two steps of term collection without resorting to either coordinate representation or computer program.

Step 4. Grouping terms on the right side of (4.11) according to the inner products not involving $\mathbf{e}$, we can reduce the number of terms to 6 by factorization within each group of terms:

$$
\begin{array}{rl}
(1) & \mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{AB};\mathbf{DE}][\mathbf{EA};\mathbf{CD}][\mathbf{eBCE}][\mathbf{eABC}][\mathbf{eABD}] \\
 & -\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{AB};\mathbf{DE}][\mathbf{EA};\mathbf{BC}][\mathbf{eABC}][\mathbf{eABD}][\mathbf{eCDE}] \\
= & \mathbf{e}\cdot\mathbf{E}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{AB};\mathbf{DE}][\mathbf{eABC}][\mathbf{eABD}][\mathbf{eACE}][\mathbf{eBCD}];
\end{array}
$$

$$
\begin{array}{rl}
(2) & -\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{AB};\mathbf{DE}][\mathbf{EA};\mathbf{CD}][\mathbf{eACD}][\mathbf{eBCD}][\mathbf{eBCE}] \\
 & +\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{EA};\mathbf{CD}][\mathbf{BC};\mathbf{DE}][\mathbf{eABE}][\mathbf{eACD}][\mathbf{eBCD}] \\
= & -\mathbf{e}\cdot\mathbf{E}\,\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{EA};\mathbf{CD}][\mathbf{eABC}][\mathbf{eACD}][\mathbf{eBCD}][\mathbf{eBDE}];
\end{array}
$$

$$
\begin{array}{rl}
(3) & -\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}][\mathbf{BC};\mathbf{DE}][\mathbf{AB};\mathbf{DE}][\mathbf{eACD}][\mathbf{eACE}][\mathbf{eBCD}] \\
 & +\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}][\mathbf{EA};\mathbf{CD}][\mathbf{BC};\mathbf{DE}][\mathbf{eABC}][\mathbf{eADE}][\mathbf{eBCD}] \\
 & -\mathbf{e}\cdot\mathbf{C}\,\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}][\mathbf{BC};\mathbf{DE}][\mathbf{eABE}][\mathbf{eACD}][\mathbf{eADE}][\mathbf{eBCD}] \\
= & \mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}][\mathbf{BC};\mathbf{DE}][\mathbf{eBCD}] \\
 & ([\mathbf{AB};\mathbf{CD}][\mathbf{eACE}][\mathbf{eADE}]-[\mathbf{AB};\mathbf{DE}][\mathbf{eACD}][\mathbf{eACE}]) \\
= & -\mathbf{e}\cdot\mathbf{A}\,\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}][\mathbf{BC};\mathbf{DE}][\mathbf{eABD}][\mathbf{eACE}][\mathbf{eBCD}][\mathbf{eCDE}];
\end{array}
$$

$$
\begin{array}{rl}
(4) & \mathbf{A}\cdot\mathbf{B}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{EA};\mathbf{BC}][\mathbf{EA};\mathbf{CD}][\mathbf{EA};\mathbf{CD}][\mathbf{eABC}][\mathbf{eABD}][\mathbf{eBDE}] \\
 & -\mathbf{A}\cdot\mathbf{B}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{DE}][\mathbf{EA};\mathbf{BC}][\mathbf{EA};\mathbf{CD}][\mathbf{eABC}][\mathbf{eADE}][\mathbf{eBCD}] \\
 & +\mathbf{e}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{B}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{EA};\mathbf{BC}][\mathbf{EA};\mathbf{CD}][\mathbf{eABC}][\mathbf{eABD}][\mathbf{eADE}][\mathbf{eCDE}] \\
= & \mathbf{A}\cdot\mathbf{B}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{EA};\mathbf{BC}][\mathbf{EA};\mathbf{CD}][\mathbf{eABC}] \\
 & (-[\mathbf{AB};\mathbf{CD}][\mathbf{eADE}][\mathbf{eBDE}]+[\mathbf{EA};\mathbf{CD}][\mathbf{eABD}][\mathbf{eBDE}]) \\
= & \mathbf{e}\cdot\mathbf{D}\,\mathbf{A}\cdot\mathbf{B}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{EA};\mathbf{BC}][\mathbf{EA};\mathbf{CD}][\mathbf{eABC}][\mathbf{eABE}][\mathbf{eACD}][\mathbf{eBDE}];
\end{array}
$$

$$
\begin{array}{rl}
(5) & -\mathbf{A}\cdot\mathbf{E}\,\mathbf{B}\cdot\mathbf{C}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{CD}][\mathbf{AB};\mathbf{DE}]^2[\mathbf{eABD}][\mathbf{eACE}][\mathbf{eCDE}] \\
 & +\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}]^2[\mathbf{BC};\mathbf{DE}][\mathbf{eABC}][\mathbf{eABD}][\mathbf{eACE}] \\
= & -\dfrac{1}{4}\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}]^2[\mathbf{eCBAeCDE}][\mathbf{eABD}][\mathbf{eACE}][\mathbf{eBCD}];
\end{array}
$$

$$
\begin{array}{rl}
(6) & \mathbf{A}\cdot\mathbf{B}\,\mathbf{B}\cdot\mathbf{C}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{EA};\mathbf{CD}]^2[\mathbf{eABE}][\mathbf{eACD}][\mathbf{eBDE}] \\
 & -\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{B}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{EA};\mathbf{BC}][\mathbf{EA};\mathbf{CD}]^2[\mathbf{eACD}][\mathbf{eBCD}][\mathbf{eBDE}] \\
= & -\dfrac{1}{4}\mathbf{A}\cdot\mathbf{B}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{EA};\mathbf{CD}]^2[\mathbf{eCBAeCDE}][\mathbf{eABC}][\mathbf{eACD}][\mathbf{eBDE}].
\end{array}
$$

In factorization (5) and (6), we have used the first of formulas (3.12). The result after the term reduction is

$$
\begin{aligned}
& [\mathbf{abcd}] \\
= \ & \mathbf{e}\cdot\mathbf{E}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{AB};\mathbf{DE}][\mathbf{eABC}][\mathbf{eABD}][\mathbf{eACE}][\mathbf{eBCD}] \\
& -\mathbf{e}\cdot\mathbf{E}\,\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{EA};\mathbf{CD}][\mathbf{eABC}][\mathbf{eACD}][\mathbf{eBCD}][\mathbf{eBDE}] \\
& -\mathbf{e}\cdot\mathbf{A}\,\mathbf{A}\cdot\mathbf{B}\,\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}][\mathbf{BC};\mathbf{DE}][\mathbf{eABD}][\mathbf{eACE}][\mathbf{eBCD}][\mathbf{eCDE}] \\
& +\mathbf{e}\cdot\mathbf{D}\,\mathbf{A}\cdot\mathbf{B}\,\mathbf{C}\cdot\mathbf{D}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{EA};\mathbf{BC}][\mathbf{EA};\mathbf{CD}][\mathbf{eABC}][\mathbf{eABE}][\mathbf{eACD}][\mathbf{eBDE}] \\
& -\frac{1}{4}\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}]^2[\mathbf{eCBAeCDE}][\mathbf{eABD}][\mathbf{eACE}][\mathbf{eBCD}] \\
& -\frac{1}{4}\mathbf{A}\cdot\mathbf{B}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{EA};\mathbf{CD}]^2[\mathbf{eCBAeCDE}][\mathbf{eABC}][\mathbf{eACD}][\mathbf{eBDE}].
\end{aligned}
\tag{5.12}
$$

Step 5. Grouping terms on the right side of (5.12) according to the inner products not involving $\mathbf{e}$ and doing factorization within each group of terms, using the first of formulas (3.8) and (3.11), we get zero from each group, thus finishing the proof of the theorem.

$$
\begin{aligned}
(a)\quad & \mathbf{e}\cdot\mathbf{E}\,\underline{\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}]\underline{[\mathbf{AB};\mathbf{DE}]}[\mathbf{eABC}]\underline{[\mathbf{eABD}][\mathbf{eACE}][\mathbf{eBCD}]} \\
& -\mathbf{e}\cdot\mathbf{A}\,\mathbf{A}\cdot\mathbf{B}\,\underline{\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}]}[\mathbf{BC};\mathbf{DE}]\underline{[\mathbf{eABD}][\mathbf{eACE}][\mathbf{eBCD}]}[\mathbf{eCDE}] \\
& -\frac{1}{4}\underline{\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}]}[\mathbf{AB};\mathbf{DE}][\mathbf{eCBAeCDE}]\underline{[\mathbf{eABD}][\mathbf{eACE}][\mathbf{eBCD}]} \\
= \ & \frac{1}{2}\mathbf{A}\cdot\mathbf{E}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{AB};\mathbf{DE}][\mathbf{eABD}][\mathbf{eACE}][\mathbf{eBCD}] \\
& \quad (2\ \mathbf{e}\cdot\mathbf{E}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{eABC}]+\langle\mathbf{eCDE}\rangle[\mathbf{AB};\mathbf{DE}][\mathbf{eABC}] \\
& \quad -2\ \mathbf{e}\cdot\mathbf{A}\,\mathbf{A}\cdot\mathbf{B}[\mathbf{BC};\mathbf{DE}][\mathbf{eCDE}]-\langle\mathbf{eABC}\rangle[\mathbf{AB};\mathbf{DE}][\mathbf{eCDE}]) \\
= \ & 0;
\end{aligned}
$$

$$
\begin{aligned}
(b)\quad & -\mathbf{e}\cdot\mathbf{E}\,\underline{\mathbf{A}\cdot\mathbf{B}}\,\mathbf{A}\cdot\mathbf{E}\,\underline{\mathbf{D}\cdot\mathbf{E}}[\mathbf{AB};\mathbf{CD}]\underline{[\mathbf{EA};\mathbf{CD}][\mathbf{eABC}][\mathbf{eACD}]}[\mathbf{eBCD}]\underline{[\mathbf{eBDE}]} \\
& +\mathbf{e}\cdot\mathbf{D}\,\underline{\mathbf{A}\cdot\mathbf{B}}\,\mathbf{C}\cdot\mathbf{D}\,\underline{\mathbf{D}\cdot\mathbf{E}}[\mathbf{EA};\mathbf{BC}]\underline{[\mathbf{EA};\mathbf{CD}][\mathbf{eABC}]}[\mathbf{eABE}]\underline{[\mathbf{eACD}][\mathbf{eBDE}]} \\
& -\frac{1}{4}\underline{\mathbf{A}\cdot\mathbf{B}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{EA};\mathbf{CD}]}[\mathbf{EA};\mathbf{CD}][\mathbf{eCBAeCDE}]\underline{[\mathbf{eABC}][\mathbf{eACD}][\mathbf{eBDE}]} \\
= \ & \frac{1}{2}\mathbf{A}\cdot\mathbf{B}\,\mathbf{D}\cdot\mathbf{E}[\mathbf{EA};\mathbf{CD}][\mathbf{eABC}][\mathbf{eACD}][\mathbf{eBDE}] \\
& \quad (-2\ \mathbf{e}\cdot\mathbf{E}\,\mathbf{A}\cdot\mathbf{E}[\mathbf{AB};\mathbf{CD}][\mathbf{eBCD}]-\langle\mathbf{eEAB}\rangle[\mathbf{EA};\mathbf{CD}][\mathbf{eBCD}] \\
& \quad +2\ \mathbf{e}\cdot\mathbf{D}\,\mathbf{C}\cdot\mathbf{D}[\mathbf{EA};\mathbf{BC}][\mathbf{eABE}]+\langle\mathbf{eBCD}\rangle[\mathbf{EA};\mathbf{CD}][\mathbf{eABE}]) \\
= \ & 0.
\end{aligned}
$$

Step 6. Nondegeneracy conditions: The above algebraic proof is valid under the assumption that ABC, BCD, CDE, DEA, EAB are triplets of noncollinear points, i.e., the pentagon is nondegenerate.

6. Conclusion

In this paper, we present the details of an algebraic proof of Miquel's 5-circle theorem. The proof shows that tremendous simplification by using conformal geometric algebra and null bracket algebra, in that the expressions can be more easily factored, and the number of terms can be more easily reduced. The major idea behind the simplification is **breefs**, which in this example is:

- Eliminate constrained points within each bracket.
- Choose suitable elimination rules to make the number of terms after the elimination as small as possible.
- Group terms by inner products, preferably those not involving **e**, before doing factorization.

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