Hyperexponential Solutions of Finite-rank Ideals in Orthogonal Ore Algebras (the General Case)

George Labahn¹⁾ and Ziming Li²⁾

Abstract. An orthogonal Ore algebra is an abstraction of common properties of linear partial differential, shift and q -shift operators. Using orthogonal Ore algebras, we present an algorithm for finding hyperexponential solutions of a system of linear differential, shift and q -shift operators, or any mixture thereof, whose solution space is finite-dimensional. The algorithm is applicable to factoring modules over an orthogonal Ore algebra when the modules are also finite-dimensional vector spaces over the field of rational functions.

1. Introduction

This report is an extensive version of [15], which describes an algorithm for computing hyperexponential solutions of finite-rank ideals of a bivariate orthogonal Ore ring. We deal with the case in which several continuous and discrete variables may occur.

A (partial) linear functional system consists of linear partial differential, shift, and q -shift operators. In this paper we consider those linear functional systems which have finite-dimensional solution spaces. In recent years there has been work on decomposing these systems into “subsystems” whose solution spaces are of lower dimension. This has been done following either ideal-theoretical [16, 17] or module-theoretical [2, 21] approaches. In both cases the methods require the computation of hyperexponential solutions of some linear functional systems obtained from either the associated equations or the integrable systems [20]. This observation motivates us to develop a general algorithm for hyperexponential solutions of linear functional systems.

Recall that a system of algebraic polynomials with a finite number of solutions can be solved as follows: compute a Gröbner basis G with respect to an elimination ordering; find the roots of the univariate polynomial in G , and finally perform back-substitution. Unfortunately, this approach does not seem to work well for linear functional systems, because a solution of an ordinary linear functional equation may contain several unspecified constants with respect to one operator. These constants are, however, not constants with respect to other operators. As such, the back-substitution introduces new unknowns in addition to introducing complicated irrational function coefficients (possibly involving integral signs). This means that we do not know what closed-form solutions of the new unknowns should be looked for in order to find hyperexponential solutions.

¹⁾Department of Mathematics, Simon Fraser University, Burnaby, British Columbia, V5A 1S6, Canada, Email: glabahn@scg.uwaterloo.ca

²⁾Key Lab of Mathematics-Mechanization, Academy of Mathematics and System Sciences, Zhong Guan Cun, Beijing (100080), China, Email: zmli@mmrc.iss.ac.cn

The method presented in this paper needs only operations on rational functions to combine univariate hyperexponential functions, although it has to solve several ordinary linear functional equations. However, there has been significant work done in recent years for computing hyperexponential solutions of linear ordinary functional equations. We refer the reader to [18, 7, 13, 14, 4] for algorithms and their implementations. In addition the algorithm in [17] computes hyperexponential solutions for a linear partial differential system with finite-dimensional solution space.

Guided by the work of [9, 10] we introduce the notion of orthogonal Ore algebras. These are general enough to include algebras of partial differential, shift, and q -shift operators over the field of multivariate rational functions but also simple enough to analyze the nonlinear compatibility conditions of a left ideal of rank one. Analysis of the compatibility conditions is both a key technique used in this paper and a basic tool for studying multivariate functions and sequences (see for example [5, 17]). Based on orthogonal Ore algebras and the associated linear action of their elements, we can understand a hyperexponential function as a function whose partial “logarithmic derivatives” are rational. In differential algebra these are of course exponential functions while in difference algebra these are usually called hypergeometric terms. Finding hyperexponential solutions of a linear functional system is closely related to finding rank-one ideals containing the left ideal generated by the operators in the system.

The main result of this paper is summarized in Theorem 9. Roughly speaking this says that in order to find hyperexponential solutions, we need to find univariate hyperexponential solutions of several ordinary linear functional equations, to compute rational solutions of some special nonlinear functional systems which can be decoupled by simple transformations and then finally to compute rational solutions of some linear functional systems.

The remainder of this paper is organized as follows: Section 2. specifies the problem to be solved and presents necessary preliminaries. Section 3. defines the notion of orthogonal Ore algebra while the next section connects elements of an orthogonal Ore algebra with some special pseudo-linear operators. Section 5. defines hyperexponential elements with the notion of similarity of such elements appearing in the next section. Section 6. illustrates how to construct hyperexponential elements by their rational certificates. Section 8. briefly describes an algorithm for rational solutions of a linear functional system. Section 9. outlines the basic idea with the main algorithm given in Section 10..

2. Preliminaries

Let C be a field of characteristic zero and $\mathbf{F}$ the field of rational functions in $x_1, \dots, x_n$ over C . Denote by $\mathbf{1}$ the identity of $\mathbf{F}$, and by $\mathbf{0}$ the mapping that sends everything to zero. For $i = 1, \dots, n$, let σ_i be an automorphism of $\mathbf{F}$ and δ_i be an additive mapping of $\mathbf{F}$ with the property that

$$\delta_i(ab) = \sigma_i(a)\delta_i(b) + \delta_i(a)b \quad \text{for all } a, b \in \mathbf{F}. \quad (1)$$

Set

$$\Phi = \{(\sigma_1, \delta_1), \dots, (\sigma_n, \delta_n)\}. \quad (2)$$

The set Φ is said to be *commutative* if

$$\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i, \quad \sigma_i \circ \delta_j = \delta_j \circ \sigma_i, \quad \delta_i \circ \delta_j = \delta_j \circ \delta_i,$$

where $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$. With the commutative set Φ , we can construct a well-defined Ore algebra $[\partial_1, \dots, \partial_n]$ over whose multiplicative rules are

$$\partial_i \partial_j = \partial_j \partial_i \quad \text{and} \quad \partial_i a = \sigma_i(a) \partial_i + \delta_i(a), \quad (3)$$

for all $i, j \in \{1, 2, \dots, n\}$ and $a \in$ (see [10, §1.2]). We denote the noncommutative ring $[\partial_1, \dots, \partial_n]$ by $\mathbf{A}$. Note that the first multiplicative rule hinges on the commutativity of Φ .

Example 1 Let $\mathbf{F} = \mathbf{Q}(x, k)$ and $\Phi = \left\{ \left(\mathbf{1}, \frac{\partial}{\partial x} \right), (\sigma_2, \mathbf{0}) \right\}$, where σ_2 sends $a(x, k) \in$ to $a(x, k+1)$. The set Φ is commutative. Then $\mathbf{A} = (\mathbf{x}, \mathbf{k})[\partial_x, E_k]$ is an Ore algebra with multiplicative rules $\partial_x E_k = E_k \partial_x$, $\partial_x a = a \partial_x + \frac{\partial a}{\partial x}$ and $E_k a = a(x, k+1) E_k$. $\square$

All ideals of $\mathbf{A}$ considered in this paper will be left ideals. Since $\mathbf{F}$ is a field, we can use a noncommutative version of Buchberger's algorithm to compute Gröbner bases (see Thm. 1.2 in [10]). An ideal I of $\mathbf{A}$ is of *finite rank* if the $\mathbf{F}$ -linear space of $\mathbf{A}/I$ is finite-dimensional with the *rank* of I then being defined to be the dimension of the vector space $\mathbf{A}/I$ over $\mathbf{F}$. Proposition 2.1 in [10] states that an ideal I is of finite rank if and only if it contains $g_i \in [\partial_i]$, for $i = 1, \dots, n$. The set $\{g_1, \dots, g_n\}$ is called a *rectangular system* of I . We are concerned with

Problem H. Given a basis of an ideal I of finite rank, compute all ideals of rank one that contain I .

We will see later in Section 5. that Problem H is slightly more general than that of finding hyperexponential solutions of an ideal with finite rank. Note, however, that the statement of Problem H does not involve any field extension of $\mathbf{F}$.

Example 2 Consider $\mathbf{A}$ from Example 1. Let

$$L_1 = \partial_x^2 - \frac{2((k-x)^2 + k)}{x(k-x)} \partial_x + \frac{((k-x)^3 - 3xk + 3k^2 + 2k)}{x^2(k-x)},$$

$$L_2 = E_k^2 - \frac{2x(x-k-1)}{x-k-2} E_k + \frac{x^2(x-k)}{x-k-2},$$

and I the ideal generated by L_1 and L_2 in $\mathbf{A}$. One can verify that L_1 and L_2 is a Gröbner basis of I by, for example, the Maple packages `Ore_algebra` and `Groebner`. Thus, $\mathbf{A}/I$ has rank four with a $\mathbf{F}$ -linear basis $\{1, \partial_x, E_k, \partial_x E_k\}$. One ideal of rank one which contains I is generated by

$$\partial_x + \frac{x-k-1}{x} \quad \text{and} \quad E_k - x.$$

We show later in Example 8 that there are infinitely many rank-one ideals containing I . $\square$

An ideal J of rank one in $\mathbf{A}$ can be generated by a basis

$$f_1 = \partial_1 - r_1, \dots, f_n = \partial_n - r_n, \quad \text{with } r_i \in \mathbf{F}. \quad (4)$$

Since the S -polynomial $g_{ij} = (\partial_i f_j - \partial_j f_i)$ belongs to J , reducing g_{ij} with respect to the basis (4) yields

$$(\sigma_j(r_i) r_j - \sigma_i(r_j) r_i + \delta_j(r_i) - \delta_i(r_j)) \in J.$$

As J is not the entire ring $\mathbf{A}$, we have

$$\sigma_j(r_i)r_j + \delta_j(r_i) = \sigma_i(r_j)r_i + \delta_i(r_j), \quad (1 \leq i < j \leq n). \quad (5)$$

We call (5) the *compatibility conditions* of J . If both σ_i and σ_j are $\mathbf{1}$, (5) becomes $\delta_j(r_i) = \delta_i(r_j)$, which is the usual compatibility condition for first-order linear PDE's. If both δ_i and δ_j are $\mathbf{0}$, (5) becomes $\sigma_j(r_i)r_j = \sigma_i(r_j)r_i$ (see Definition 3 of [5]). If A is the algebra in Example 1, then, (5) becomes $r_1(x, k+1)r_2(x, k) = r_2(x, k)r_1(x, k) + \frac{\partial r_2}{\partial x}$.

If we are given $r_1, \dots, r_n$ satisfying (5) then (4) is a Gröbner basis and hence J is of rank one. A solution of Problem H is an algorithm to find elements $r_1, \dots, r_n \in \mathbf{F}$ such that (5) holds, and such that every element of the given basis of I can be reduced to zero by the basis (4).

3. Orthogonal Ore algebras

In this section, we define the notion of orthogonal Ore algebras, which are a special case of the Ore algebras defined in [10]. Our motivation is from two observations: (i) the presence of arbitrary σ_i and δ_j would cause very complicated calculations; (ii) an operator in a linear functional system usually acts non-trivially on only one variable. Let $\mathbf{F}_i$ be the field $C(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, for $i = 1, \dots, n$.

Definition 1 *The set Φ in (2) is said to be orthogonal if, for all i with $1 \leq i \leq n$, the following conditions are satisfied: (1) $\sigma_i(x_i) \in C[x_i]$ and $\sigma_i(a) = a$ if and only if $a \in \mathbf{F}_i$; (2) $\delta_i(x_i) \in C(x_i)$ and $\delta_i(a) = 0$ if and only if $a \in \mathbf{F}_i$; (3) $\delta_i(x_i)$ is not $\mathbf{0}$ if $\sigma_i = \mathbf{1}$. If Φ is orthogonal, the Ore algebra $\mathbf{A}$ with multiplicative rules (3) is said to be orthogonal.*

If $\mathbf{A}$ is orthogonal, we have, for $i, j \in \{1, \dots, n\}$ with $i \neq j$,

$$\partial_j x_i = x_i \partial_j, \quad \partial_j \sigma_i(x_i) = \sigma_i(x_i) \partial_j, \quad \partial_j \delta_i(x_i) = \delta_i(x_i) \partial_j, \quad (6)$$

The goal of this paper is to provide a solution to Problem H when $\mathbf{A}$ is orthogonal.

Both the ring of usual partial differential operators and that of partial shift operators are instances of orthogonal Ore algebras as is the algebra in Example 1. One can observe that the pair (σ_i, δ_i) in these examples have the property that either $\sigma_i = \mathbf{1}$ or $\delta_i = \mathbf{0}$. The next theorem shows that this observation is not a coincidence.

Theorem 1 *Let $\mathbf{A}$ be an orthogonal algebra with Φ given in (2). Then $\mathbf{A}$ is isomorphic to $\mathbf{B} = \mathbf{F}[\Delta_1, \dots, \Delta_n]$ whose multiplicative rules are: (1) $\Delta_i \Delta_j = \Delta_j \Delta_i$, (2) $\Delta_i a = \sigma_i(a) \Delta_i$ if $\sigma_i \neq \mathbf{1}$, (3) $\Delta_j a = a \Delta_j + \delta_j(a)$ if $\sigma_j = \mathbf{1}$, where i, j are in $\{1, \dots, n\}$ and a in $\mathbf{F}$.*

Proof The Ore algebra $\mathbf{B}$ is well-defined and orthogonal because the commutative set $\{(\sigma_1, \epsilon_1), \dots, (\sigma_n, \epsilon_n)\}$, in which $\epsilon_i = \mathbf{0}$ if $\sigma_i \neq \mathbf{1}$, and $\epsilon_i = \delta_i$ if $\sigma_i = \mathbf{1}$, is orthogonal.

Note that $\sigma_i(x_i) \neq x_i$ and $\sigma_i(x_i) \in C[x_i]$ if $\sigma_i \neq \mathbf{1}$. Define

$$D_i = \Delta_i \text{ if } \sigma_i = \mathbf{1}, \quad D_i = \frac{\Delta_i + \delta_i(x_i)}{x_i - \sigma_i(x_i)} \text{ if } \sigma_i \neq \mathbf{1}.$$

Let ϕ be the $\mathbf{F}$ -linear mapping sending ∂_i to D_i , for all i with $1 \leq i \leq n$, and $\partial_1^{m_1} \dots \partial_n^{m_n}$ to $D_1^{m_1} \dots D_n^{m_n}$. To see that ϕ is well-defined, we have to show $D_i D_j = D_j D_i$. Assume that $\sigma_i \neq \mathbf{1}$ and set $y_i = \frac{1}{x_i - \sigma_i(x_i)} \in C(x_i)$. If $\sigma_j = \mathbf{1}$, a repeated use of (6) shows that

$$D_j D_i = \Delta_j y_i (\Delta_i + \delta_i(x_i)) = y_i (\Delta_i + \delta_i(x_i)) \Delta_j = D_i D_j.$$

If σ_j is not $\mathbf{1}$ then we set $y_j = \frac{1}{x_j - \sigma_j(x_j)}$. A similar calculation shows

$$D_j D_i = y_j y_i (\Delta_i \Delta_j + \delta_i(x_j) \Delta_i + \delta_j(x_i) \Delta_j + \delta_j(x_j) \delta_i(x_i)).$$

As the right hand side of the above equality is invariant under the permutation of i and j , $D_i D_j = D_j D_i$. Therefore, ϕ is a well-defined $\mathbf{F}$ -linear bijection from $\mathbf{A}$ to $\mathbf{B}$.

To see that ϕ is a ring homomorphism, we need to verify

$$\phi_i(\partial_i a) = D_i a \quad \text{for all } a \in \mathbf{F} \text{ and } i \in \{1, \dots, n\}, \quad (7)$$

which follows directly from Proposition 3.1 in [11, §8.3] or Theorem 9 in [9]. By the identity $D_i D_j = D_j D_i$ and (7), an easy induction proves that ϕ is a ring homomorphism. $\square$

From this point on we assume that $\mathbf{A}$ is an orthogonal Ore algebra, in which, for i with $1 \leq i \leq n$, $\delta_i = \mathbf{0}$ if $\sigma_i \neq \mathbf{1}$.

4. Special linear action

To solve Problem H for orthogonal Ore algebras, we shall regard indeterminates $\partial_1, \dots, \partial_n$ as operators acting on some field extensions of $\mathbf{F}$. This view enables us to connect ideals of rank one with certain “solutions” of Ore polynomials.

A field extension $\mathbf{K}$ of $\mathbf{F}$ is said to be Φ -compatible if all the σ_i 's and δ_j 's can be extended on $\mathbf{K}$ in such a way that all the σ_i 's are injective field endomorphisms of $\mathbf{K}$, and (1) holds for all elements of $\mathbf{K}$. An element c of $\mathbf{K}$ is called a *constant with respect to σ_i and δ_i* if $\sigma_i(c) = c$ and $\delta_i(c) = 0$. The element c is called a *constant* if it is a constant with respect to all σ_i and δ_i , for $i = 1, \dots, n$. The reader is referred to [9] for a general way to define the action of Ore polynomials on $\mathbf{K}$. We need only a special kind of pseudo-linear operators.

Definition 2 Let $\mathbf{K}$ be a Φ -compatible field extension of $\mathbf{F}$. Define θ_i to be the mapping sending an element a of $\mathbf{K}$ to $\sigma_i(a)$ if $\sigma_i \neq \mathbf{1}$, and to $\delta_i(a)$ if $\sigma_i = \mathbf{1}$, $i = 1, \dots, n$.

By Lemma 3 in [9] we conclude that θ_i is $\mathbf{K}$ -pseudo-linear with respect to σ_i and δ_i , that is, θ_i is additive and

$$\theta_i(ab) = \sigma_i(a)\theta_i(b) + \delta_i(a)b, \quad a, b \in \mathbf{K}. \quad (8)$$

Definition 2 and the commutativity of Φ imply $\theta_i \circ \theta_j = \theta_j \circ \theta_i$. This commutativity allows us to define the action of

$$P = \sum_{m_1, \dots, m_n} p_{m_1, \dots, m_n} \partial_1^{m_1} \dots \partial_n^{m_n} \quad (P \in \mathbf{A})$$

on $a \in \mathbf{K}$ as $P \bullet a = \sum_{m_1, \dots, m_n} p_{m_1, \dots, m_n} \theta_1^{m_1} \circ \dots \circ \theta_n^{m_n}(a)$. In this way the ∂_i 's can be regarded as $\mathbf{K}$ -pseudo-linear operators with respect to σ_i and δ_i on $\mathbf{K}$. If c is a constant with respect to σ_i and δ_i , then $\partial_i \bullet (ca) = c \partial_i \bullet a$ by (8). So a constant with respect to σ_i and δ_i is also simply called a ∂_i -constant. An element of $\mathbf{A}$ is a linear operator over the field of all constants in $\mathbf{K}$.

Example 3 In applications we are mainly interested in the following three operators: If σ_i is $\mathbf{1}$, then ∂_i is called a differential operator. If $\sigma_i(x_i) = x_i + 1$ and $\delta_i = 0$, then ∂_i is called a shift operator. If $\sigma_i(x_i) = qx_i$ and $\delta_i = 0$, where q is a constant in C with $q^m \neq 1$

for all $m \in \mathbf{Z}^+$, then ∂_i is called a q -shift operator. If σ_i is not $\mathbf{1}$ and unspecified, then ∂_i is called a σ -shift operator. Consider the algebra $\mathbf{A}$ in Example 1. By Definition 2, the application of ∂_x on a function $a(x, k)$ is $\frac{\partial a}{\partial x}$, and that of E_k is $a(x, k+1)$. Thus ∂_x and E_k can be viewed as differential and shift operators, respectively. $\square$

5. Hyperexponential elements

A nonzero element h of some Φ -compatible extension of $\mathbf{F}$ is *hyperexponential* over $\mathbf{F}$ with respect to ∂_i if $\partial_i \bullet h = r_i h$, where $r_i \in \mathbf{F}$. The rational function r_i is called the ∂_i -certificate of h . Note that r_i is assumed to be nonzero in [9]. We can remove this restriction because the θ_i 's in Definition 2 are very special. An element is *hyperexponential* over $\mathbf{F}$ (with respect to $\partial_1, \dots, \partial_n$) if it is hyperexponential over $\mathbf{F}$ with respect to all the ∂_i 's. We note that all nonzero elements of $\mathbf{F}$ are hyperexponential and that the product of two hyperexponential elements is hyperexponential.

Example 4 Let $\mathbf{A} = C(m, k)[E_m, E_k]$, where E_m and E_k are shift operators with respect to m and k , respectively. A hyperexponential element over $C(m, k)$ is usually called a *hypergeometric term*. For example: $\binom{m}{k}$, $(m+k)!$, 3^{m+k} are all bivariate hypergeometric terms. $\square$

The next proposition describes a relation between hyperexponential solutions and rank-one ideals.

Proposition 2 *If an ideal I of $\mathbf{A}$ has a hyperexponential solution h in some Φ -compatible field extension of $\mathbf{F}$, then the rank-one ideal*

$$J = (\partial_1 - r_1, \dots, \partial_n - r_n) \quad (9)$$

contains I , where $r_i = \frac{\partial_i \bullet h}{h}$, $i = 1, \dots, n$.

Proof The ideal J is of rank one since h is a nonzero solution J . Moreover, $I \subset J$ since h is a solution of I . $\square$

Example 5 *The rank-one ideal in Example 2 corresponds to the hyperexponential element $\exp(-x)x^{k+1}$ whose ∂_x and E_k -certificates are $(k+1-x)/x$ and x , respectively. As pointed out in Example 2, there are infinitely many rank-one ideals containing I . Consequently, the ideal I may have infinitely many different hyperexponential solutions, although two hyperexponential elements with constant ratio are usually considered to be equal.* $\square$

Unfortunately, the converse of Proposition 2 is not always true. For example, the only solution of J given in (9) is zero if ∂_n is a shift operator and r_n is zero. However, we may compute such rank-one ideals J with $r_n = 0$ by computing the rank-one ideals containing $\tilde{I} = I + (\partial_n)$. If $\tilde{I}$ is trivial, then we are done. Otherwise, we need only to compute rank-one ideals containing $\tilde{I} \cap \mathbf{F}[\partial_1, \dots, \partial_{n-1}]$. Such a reduction is unnecessary if ∂_n is a differential operator.

The reader is referred to [19, 20] for the existence of field extensions that contain hyperexponential solutions of linear ordinary differential and difference equations. In this paper, we merely treat hyperexponential elements as a convenient way to represent rank-one ideals J in (9), in which r_i is nonzero if ∂_i is a σ -shift operator. By “given (computing) a hyperexponential element”, we mean that we are given (computing) its certificates. Note

that, if h is hyperexponential with respect to all the ∂_i 's, then its certificates $r_1, \dots, r_n$ must satisfy (5).

6. Constructions of hyperexponential solutions

One may wonder how the hyperexponential solutions of a rank-one ideal looks like. In this section, we construct their explicit expressions in terms of their certificates in the differential, shift and mixed cases. The results of this section will not be used in the remainder of the paper. Let a rank-one ideal J be generated by operators in (4). Hence, (5) holds. In addition, we assume that $r_i \neq 0$ if ∂_i is a σ -shift operator.

First, we consider the case in which all the ∂_i 's are differential operators. Let $\partial_i \bullet x_i = f_i$, which is a nonzero element in $C(x_i)$. Then $\partial_i = f_i \frac{\partial}{\partial x_i}$, where $\frac{\partial}{\partial x_i}$ is the usual partial differential operator with respect to ∂_i . The ideal J is then generated by

$$f_1 \frac{\partial}{\partial x_1} - r_1, \dots, f_n \frac{\partial}{\partial x_n} - r_n.$$

The compatibility condition (5) becomes

$$f_i \frac{\partial r_j}{\partial x_i} = f_j \frac{\partial r_i}{\partial x_j} \quad \text{for all } 1 \leq i < j \leq n.$$

It follows that

$$\frac{\partial \left(\frac{r_j}{f_j} \right)}{\partial x_i} = \frac{\partial \left(\frac{r_i}{f_i} \right)}{\partial x_j},$$

because f_j is a constant with respect to $\frac{\partial}{\partial x_i}$. Thus, a hyperexponential solution of I can be written as:

$$\exp \left(\int \frac{r_1}{f_1} dx_1 + \dots + \frac{r_n}{f_n} dx_n \right).$$

Second, we consider the case in which all the ∂_i 's are shift operators. Then the compatibility condition (5) becomes

$$\frac{r_j(x_1, \dots, x_i + 1, \dots, x_j, \dots, x_n)}{r_j(x_1, \dots, x_i, \dots, x_j, \dots, x_n)} = \frac{r_i(x_1, \dots, x_i, \dots, x_j + 1, \dots, x_n)}{r_i(x_1, \dots, x_i, \dots, x_j, \dots, x_n)} \quad (10)$$

for all i, j with $1 \leq i < j \leq n$.

Lemma 3 *Let $p_1, \dots, p_n$ be nonzero polynomials in $C[x_1, \dots, x_n]$. Then there are integers $k_1, \dots, k_n$ such that*

$$p_i(x_1, \dots, x_{i-1}, t_i, k_{i+1}, \dots, k_n) \neq 0$$

for all i with $1 \leq i \leq n$ and for all integers t_i with $t_i \geq k_i$.

Proof We proceed by induction. The lemma clearly holds for $n = 1$. Let l_i be the leading coefficient of p_i with respect to x_i for all i with $1 \leq i \leq n$. Let k_n be an integer greater than any integer root of the product $p_n l_1 \cdots l_n$, viewed as a univariate polynomial in $\mathbf{F}_n[x_n]$. Let

$\tilde{p}_i$ be the polynomial obtained by substituting k_n for x_n in p_i . Then all the $\tilde{p}_i$'s are nonzero. By induction we can find integers $k_1, \dots, k_{n-1}$ such that

$$\tilde{p}_i(x_1, \dots, x_{i-1}, t_i, k_{i+1}, \dots, k_{n-1}) \neq 0$$

for all i with $1 \leq i \leq (n-1)$ and for all integers $t_i \geq k_i$. Then the integral tuple $(k_1, \dots, k_n)$ is what we seek. $\square$

The integral tuple $(k_1, \dots, k_n)$ given in Lemma 3 will be called an *initial tuple* for the nonzero polynomials of $p_1, \dots, p_n$.

Let p_i be the product of the numerator and denominator of r_i for $i = 1, \dots, n$. Let $(k_1, \dots, k_n)$ be an initial tuple of $p_1, \dots, p_n$, and let h be the product

$$\prod_{j_1=k_1}^{x_1-1} r_1(j_1, k_2, \dots, k_n) \prod_{j_2=k_2}^{x_2-1} r_2(x_1, j_2, k_3, \dots, k_n) \cdots \prod_{j_n=k_n}^{x_n-1} r_n(x_1, \dots, x_{n-1}, j_n).$$

We verify that h is a solution of the ideal I . It suffices to show that, for all i with $1 \leq i \leq n$, $\sigma_i(h) = r_i h$. Denote the first $(i-1)$ products by c_i , which is a ∂_i -constant, and the last $(n-i)$ product by b_i . Then

$$h = c_i b_i \left(\prod_{j_i=k_i}^{x_i-1} r_i(x_1, \dots, x_{i-1}, j_i, k_{i+1}, \dots, k_n) \right).$$

We compute

$$\begin{aligned} \sigma_i(h) &= c_i \prod_{j_i=k_i}^{x_i} r_i(x_1, \dots, x_{i-1}, j_i, k_{i+1}, \dots, k_n) \\ &\quad \prod_{j_{i+1}=k_{i+1}}^{x_{i+1}} r_{i+1}(x_1, \dots, x_{i-1}, x_i + 1, j_{i+1}, k_{i+2}, \dots, k_n) \\ &\quad \dots \\ &\quad \prod_{j_n=k_n}^{x_n-1} r_n(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_{n-1}, j_n). \end{aligned}$$

By (10) we get

$$\begin{aligned} \sigma_i(h) &= c_i b_i \prod_{j_i=k_i}^{x_i} r_i(x_1, \dots, x_{i-1}, j_i, k_{i+1}, \dots, k_n) \\ &\quad \prod_{j_{i+1}=k_{i+1}}^{x_{i+1}-1} \frac{r_i(x_1, \dots, x_i, j_{i+1}+1, k_{i+2}, \dots, k_n)}{r_i(x_1, \dots, x_i, j_{i+1}, k_{i+2}, \dots, k_n)} \\ &\quad \dots \\ &\quad \prod_{j_n=k_n}^{x_n-1} \frac{r_i(x_1, \dots, x_i, x_{i+1}, \dots, x_{n-1}, j_n+1)}{r_i(x_1, \dots, x_i, x_{i+1}, \dots, x_{n-1}, j_n)}. \end{aligned}$$

Canceling the like terms appearing yields

$$\sigma_i(h) = c_i r_i b_i \prod_{j_i=k_i}^{x_i} r_i(x_1, \dots, x_{i-1}, j_i, k_{i+1}, \dots, k_n) = r_i h.$$

The verification is completed.

At last we consider the mixed case. Assume that $\partial_1, \dots, \partial_m$ are differential operators with $\partial_i = f_i(x_i)\delta_i$, and that $\partial_{m+1}, \dots, \partial_n$ are shift operators. Let

$$\begin{aligned} v &= \prod_{j_{m+1}=k_{m+1}}^{x_{m+1}-1} r_{m+1}(x_1, \dots, x_m, j_{m+1}, k_{m+2}, \dots, k_n) \\ &\dots \\ &\prod_{j_n=k_n}^{x_n-1} r_n(x_1, \dots, x_{n-1}, j_n), \end{aligned}$$

where $(k_{m+1}, \dots, k_n)$ is an initial tuple of the product of $f_1, \dots, f_m$, the numerators and denominators of $r_{m+1}, \dots, r_n$. Let u equal

$$\exp \left(\int \frac{r_1(x_1, \dots, x_m, k_{m+1}, \dots, k_n)}{f_1(x_1, \dots, x_m, k_{m+1}, \dots, k_n)} dx_1 + \dots + \frac{r_m(x_1, \dots, x_m, k_{m+1}, \dots, k_n)}{f_m(x_1, \dots, x_m, k_{m+1}, \dots, k_n)} dx_m \right).$$

We verify that uv is a solution of I . Since u is a constant with respect to $\partial_{m+1}, \dots, \partial_n$, uv is a solution of the last $(n - m)$ equations. It remains to show that uv solves the first m equations. Note that the compatibility conditions (5) becomes

$$\begin{aligned} \frac{\delta_i(r_j)}{r_j} &= r_i(x_1, \dots, x_m, x_{m+1}, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_n) \\ &\quad - r_i(x_1, \dots, x_m, x_{m+1}, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n), \end{aligned} \tag{11}$$

where $1 \leq i \leq m$ and $(m + 1) \leq j \leq n$. To simplify the computation, let

$$v_s = \prod_{j_s=k_s}^{x_s-1} r_s(x_1, \dots, x_{s-1}, j_s, k_{s+1}, \dots, k_n), \quad s = m + 1, \dots, n.$$

So $V = V_{m+1} \cdots V_n$. For $i = 1, \dots, m$, we compute

$$\begin{aligned} \delta_i(v_s) &= v_s \sum_{j_s=k_s}^{x_s-1} \frac{\delta_i(r_s(x_1, \dots, x_i, \dots, x_{s-1}, j_s, k_{s+1}, \dots, k_n))}{r_s(x_1, \dots, x_i, \dots, x_{s-1}, j_s, k_{s+1}, \dots, k_n)} \\ &= V_s \sum_{j_s=k_s}^{x_s-1} (r_i(x_1, \dots, x_{s-1}, j_s + 1, k_{s+1}, \dots, k_n) \\ &\quad - r_i(x_1, \dots, x_{s-1}, j_s, k_{s+1}, \dots, k_n)) \quad \text{by (11)} \\ &= v_s (r_i(x_1, \dots, x_{s-1}, x_s, k_{s+1}, \dots, k_n) \\ &\quad - r_i(x_1, \dots, x_{s-1}, k_s, k_{s+1}, \dots, k_n)). \end{aligned}$$

Therefore

$$\delta_i(v) = v(r_i(x_1, \dots, r_n) - r_i(x_1, \dots, x_m, k_{m+1}, \dots, k_n)).$$

It follows that

$$\begin{aligned}
\delta_i(uv) &= \delta_i(u)v + u\delta_i(v) \\
&= uv(r_i(x_1, \dots, x_m, k_{m+1}, \dots, k_n) \\
&\quad + r_i(x_1, \dots, r_n) - r_i(x_1, \dots, x_m, k_{m+1}, \dots, k_n)) \\
&= uvr_i(x_1, \dots, r_n).
\end{aligned}$$

The last verification is completed.

7. Similarity

In this section, we introduce the notion of similarity among hyperexponential solutions. This notion helps us organize possibly infinitely many different hyperexponential solutions into finitely many equivalence classes. The notion of similarity is well known in linear ordinary differential and shift equations [18] and has been extended to the partial differential case in [17].

In the following we first characterize the condition that an ideal of rank one can have a nonzero rational solution.

Proposition 4 *Let J be an ideal of $\mathbf{A}$ of rank one generated by $(\partial_1 - q_1), \dots, (\partial_n - q_n)$. Then J has a nonzero solution in $\mathbf{F}$ if and only if $q_i = \frac{\partial_i \bullet s_i}{s_i}$ for some nonzero $s_i \in \mathbf{F}$, and (5) is satisfied when r_i is replaced by $\frac{\partial_i \bullet s_i}{s_i}$, for all i with $1 \leq i \leq n$.*

Proof The necessity is trivial. The case in which all the σ_i 's are equal to $\mathbf{1}$, is a special case of Lemma 3.1 in [17]. Thus we assume that $\sigma_n \neq \mathbf{1}$ and hence that $\delta_n = \mathbf{0}$.

We proceed by induction on n . If $n = 1$, then s_1 is the solution. Assume that the proposition holds for $(n - 1)$. If $\sigma_i \neq \mathbf{1}$ for some i with $1 \leq i \leq (n - 1)$, then (5) becomes

$$\sigma_n \left(\frac{\sigma_i(s_i)}{s_i} \right) \frac{\sigma_n(s_n)}{s_n} = \sigma_i \left(\frac{\sigma_n(s_n)}{s_n} \right) \frac{\sigma_i(s_i)}{s_i}.$$

It follows that

$$\sigma_n \left(\frac{\sigma_i(t_i)}{t_i} \right) = \frac{\sigma_i(t_i)}{t_i}, \quad \text{where } t_i = \frac{s_i}{s_n} \quad (12)$$

so that $\frac{\sigma_i(t_i)}{t_i}$ is a ∂_n -constant. Since $\mathbf{A}$ is orthogonal, we have $\frac{\sigma_i(t_i)}{t_i} \in \mathbf{F}_n$. If $\sigma_j = \mathbf{1}$, then (5) becomes

$$\sigma_n \left(\frac{\delta_j(s_j)}{s_j} \right) \frac{\sigma_n(s_n)}{s_n} = \frac{\sigma_n(s_n)}{s_n} \frac{\delta_j(s_j)}{s_j} + \delta_j \left(\frac{\sigma_n(s_n)}{s_n} \right). \quad (13)$$

Since $\sigma_j = \mathbf{1}$, expanding the last term of (13) yields

$$\delta_j \left(\frac{\sigma_n(s_n)}{s_n} \right) = \frac{s_n \delta_j \circ \sigma_n(s_n) - \sigma_n(s_n) \delta_j(s_n)}{s_n^2}. \quad (14)$$

Replacing the last term of (13) by the right hand side of (14), and moving $\frac{\sigma_n(s_n)}{s_n}$ from the left hand side of (13) to the right hand side, we get

$$\sigma_n \left(\frac{\delta_j(s_j)}{s_j} \right) = \frac{\delta_j(s_j)}{s_j} + \frac{\delta_j \circ \sigma_n(s_n)}{\sigma_n(s_n)} - \frac{\delta_j(s_n)}{s_n}.$$

Using the identity on logarithmic derivatives:

$$\frac{\delta_j(a)}{a} - \frac{\delta_j(b)}{b} = \frac{\delta_j(f)}{f}, \quad \text{where } f = \frac{a}{b}, \quad (15)$$

and $\sigma_n \left(\frac{\delta_j(s_j)}{s_j} \right) = \frac{\delta_j \circ \sigma_n(s_j)}{\sigma_n(s_j)}$, we see that

$$\sigma_n \left(\frac{\delta_j(t_j)}{t_j} \right) = \frac{\delta_j(t_j)}{t_j}, \quad \text{where } t_j = \frac{s_j}{s_n}. \quad (16)$$

Hence, $\frac{\delta_j(t_j)}{t_j}$ belongs to $\mathbf{F}_n$ by the same reason. A tedious but straightforward calculation verifies that, for $i, j = 1, \dots, n-1$, (5) is satisfied when r_k is replaced by $\frac{\partial_k \bullet t_k}{t_k}$ for $k = 1, \dots, n-1$ (One would see that such a verification is just a repetition of the calculation to get (12) or (16)). Hence, by induction, the rank-one ideal in $\mathbf{F}_n[\partial_1, \dots, \partial_{n-1}]$ generated by

$$\partial_1 - \frac{\partial_1 \bullet t_1}{t_1}, \dots, \partial_{n-1} - \frac{\partial_{n-1} \bullet t_{n-1}}{t_{n-1}},$$

has a nonzero solution r in $\mathbf{F}_n$. We claim that rs_n is a solution of I . The operator $\left(\partial_n - \frac{\partial_n \bullet s_n}{s_n} \right)$ annihilates rs_n , because r is a ∂_n -constant. If $\sigma_i \neq 1$, then $\frac{\sigma_i(rs_n)}{rs_n} = \frac{\sigma_i(s_i)}{s_i}$ since $\left(\partial_i - \frac{\partial_i \bullet t_i}{t_i} \right)$ annihilates r . It follows that $\left(\partial_i - \frac{\partial_i \bullet s_i}{s_i} \right)$ annihilates rs_n . If $\sigma_j = 1$, $\left(\frac{\delta_j(r)}{r} - \frac{\delta_j \bullet t_j}{t_j} \right) = 0$. By (15) $\frac{\delta_j(rs_n)}{rs_n} = \frac{\delta_j(s_j)}{s_j}$ and hence, $\left(\partial_j - \frac{\partial_j \bullet s_j}{s_j} \right)$ annihilates rs_n . $\square$

Let h_1 and h_2 be hyperexponential over $\mathbf{F}$ with respect to ∂_i . We say that h_1 and h_2 are *similar* with respect to ∂_i (denoted by $h_1 \sim_i h_2$) if their ratio is the product of a ∂_i -constant and a rational function of $\mathbf{F}$. If h_1 and h_2 are hyperexponential over $\mathbf{F}$, then h_1 and h_2 are *similar* (denoted by $h_1 \sim h_2$) if their ratio is the product of a constant and a rational function of $\mathbf{F}$. Both $\sim_i$ and $\sim$ are equivalence relations. The ∂_i -constant in the definition of the similarity $\sim_i$ should be explicitly stated, because it may be not a ∂_j -constant when $j \neq i$. In the univariate case such a constant may be ignored. Clearly, $h_1 \sim_i h_2$ ($1 \leq i \leq n$) if $h_1 \sim h_2$. The converse follows from the next lemma.

Lemma 5 *Two hyperexponential elements h_1 and h_2 are similar if and only if $h_1 \sim_i h_2$ for all i with $1 \leq i \leq n$.*

Proof Assume that $h_1 \sim_i h_2$, for $i = 1, \dots, n$ and denote by h the ratio of h_1 and h_2 . Then we have

$$h = c_1 s_1, \dots, h = c_n s_n,$$

where c_i is a ∂_i -constant and $s_i \in \mathbf{F}$. Thus

$$\partial_1 \bullet h = c_1 \partial_1 \bullet s_1, \dots, \partial_n \bullet h = c_n \partial_n \bullet s_n.$$

These two equations imply

$$\frac{\partial_1 \bullet h}{h} = \frac{\partial_1 \bullet s_1}{s_1}, \dots, \frac{\partial_n \bullet h}{h} = \frac{\partial_n \bullet s_n}{s_n}. \quad (17)$$

It follows that $\frac{\partial_1 \bullet s_1}{s_1}, \dots, \frac{\partial_n \bullet s_n}{s_n}$ satisfies (5) when r_i is replaced by $\frac{\partial_i \bullet s_i}{s_i}$. By Proposition 4, there exists $s \in \mathbf{F}$ such that $\frac{\partial_i \bullet s}{s} = \frac{\partial_i \bullet s_i}{s_i}$, for $i = 1, \dots, n$. Equation (17) implies that both h and s are nonzero solutions of the ideal generated by $(\partial_1 - \frac{\partial_1 s_1}{s_1}), \dots, (\partial_n - \frac{\partial_n s_n}{s_n})$. One can then easily verify $h = cs$ for some constant c by a case distinction that ∂_i is either differential or σ -shift. $\square$

The next lemma extends Theorem 5.1 in [18].

Lemma 6 *Let $h_1, \dots, h_m$ be hyperexponential elements with respect to ∂_k in a Φ -compatible field. If there exist nonzero ∂_k -constants $c_1, \dots, c_m$, and nonzero rational functions $r_1, \dots, r_m$ such that*

$$c_1 r_1 h_1 + \dots + c_m r_m h_m = 0, \quad (18)$$

then $h_i \sim_k h_j$ for some $1 \leq i < j \leq m$.

Proof If ∂_k is a differential operator, then the lemma follows from Proposition 3.3 in [17].

Consider the case in which $\partial_k = \sigma_k$. We proceed by induction on m . The lemma clearly holds if $m = 2$. Assume that the lemma is true for $(m-1)$ and the product $c_m r_m$ is nonzero. Apply σ_k to (18) to get

$$c_1 \sigma_k(r_1) \sigma_k(h_1) + \dots + c_m \sigma_k(r_m) \sigma_k(h_m) = 0. \quad (19)$$

Since h_i is hyperexponential with respect to ∂_i , there exists a nonzero rational function u_i such that $\sigma_k(h_i) = u_i h_i$. It follows from (19) that

$$c_1 q_1 h_1 + \dots + c_m q_m h_m = 0 \quad (20)$$

where $q_i = \sigma_k(r_i) u_i$, for $i = 1, \dots, m$. Equations (18) and (20) imply that

$$c_1 (q_m r_1 - q_1 r_m) h_1 + \dots + c_{m-1} (q_m r_{m-1} - q_{m-1} r_m) h_{m-1} = 0.$$

If one of the coefficients is nonzero, then the lemma follows from the induction hypothesis. Otherwise, we must have $q_m r_1 = q_1 r_m$, and hence,

$$\frac{u_m}{u_1} = \frac{\sigma_k(r)}{r} \quad \text{where } r = \frac{r_1}{r_m}. \quad (21)$$

Compute

$$\sigma_k \left(\frac{r h_1}{h_m} \right) = \sigma_k(r) \frac{u_1}{u_m} \frac{h_1}{h_m} = \frac{r h_1}{h_m},$$

in which the last equality follows from (21). This implies that the ratio of $r h_m$ and h_1 is a ∂_k -constant. Hence, $q_m r_1 = q_1 r_m$. $\square$

Due to the existence of Wronskian and Casoratian determinants (see [12, p. 271]), each P in $\mathbf{F}[\partial_i]$ has at most $\deg_{\partial_i} P$ linear independent solutions over a field of ∂_i -constants. This fact combined with Lemma 6 implies that

Corollary 7 *If $h_1, \dots, h_m$ are pairwise dissimilar hyperexponential solutions of P in $\mathbf{F}[\partial_i]$ with respect to ∂_i , then m is no more than the degree of P in ∂_i .*

8. Rational solutions

In this section, we briefly describe an algorithm for rational solutions of a rectangular system $\mathcal{P} = \{P_1, \dots, P_n\}$, where $P_i \in \mathbf{F}[\partial_i]$. Such an algorithm is needed in Section 10.

First, we consider polynomial solutions. Assume that we can find polynomial solutions for each of the P_i 's. Let $p_{i1}, \dots, p_{im_i}$ be a basis of the polynomial solutions of P_i . Then

$$d_i = \max(\deg_{x_i} p_{i1}, \dots, \deg_{x_i} p_{im_i}).$$

is a degree bound for all the polynomial solutions of $\mathcal{P}$ in x_i , for $i = 1, \dots, n$. Using these degree bounds, we can then compute all polynomial solutions of $\mathcal{P}$ by solving a linear system over C .

Second, we deal with rational solutions. Assume that we are able to find rational solutions for each of the P_i 's. Let $q_{i1}, \dots, q_{im_i}$ be a basis of the rational solutions of P_i . Let g be the least common multiple of the denominators of all the q_{ij} 's, where $i = 1, \dots, n$ and $j = 1, \dots, m_i$. We claim that g is a common denominator of all rational solutions of $\mathcal{P}$. For, otherwise, suppose that a rational solution r can be written as $\frac{p}{ug}$, where $p, u \in C[x_1, \dots, x_n]$ and $\gcd(u, p) = 1$. If $\deg_{x_i} u > 0$, then $r = \frac{\bar{p}}{\bar{u}g}$, where $\bar{p}$ and $\bar{u}$ are in $C[x_1, \dots, x_n]$ and the degree of $\bar{u}$ in x_i equals zero, because r is a rational solution of P_i . These two expressions of r imply that u divides $\bar{u}$, a contradiction. Thus, each rational solution of P can be written as $\frac{f}{g}$, in which the polynomial f is yet to be determined. Regarding f as a new unknown and applying the P_i 's to $\frac{f}{g}$ yields another rectangular system whose polynomial solutions are the numerators of the rational solutions of $\mathcal{P}$.

Example 6 Let $\mathbf{A}$ be given in Example 1. We compute the rational solutions of the rectangular system:

$$(x - k)\partial_x^2 + 2\partial_x, \text{ and} \\ (x - k - 2)(k + 2)E_k^2 - 2(x - k - 1)(k + 1)E_k + (x - k)k.$$

The first equation has rational solutions $\frac{c_1 + c_2 x}{x - k}$, where c_1 and c_2 are constants with respect to ∂_x . The second equation has rational solutions $\frac{c_3 + c_4 k}{(x - k)xk}$, where c_3 and c_4 are E_k -constants. Thus, rational solutions of the system have a common denominator $(x - k)xk$. Applying the two operators to $\frac{p}{(x - k)xk}$, where p is an indeterminate, we see that p satisfies

$$x^2 \partial_x^2 - 2x \partial_x + 2, E_k^2 - 2E_k + 1.$$

The first operator has polynomial solutions $c_5 x + c_6 x^2$, where c_5 and c_6 are ∂_x -constants while the second has polynomial solutions $c_7 + c_8 k$, where c_7 and c_8 are E_k -constants. Thus the polynomial solutions of the second system are of the form

$$g = c_9 + c_{10}x + c_{11}k + c_{12}xk + c_{13}x^2,$$

where $c_9, \dots, c_{13}$ are constants. Substituting g into the second system gives $c_9 = 0$ and $c_{11} = 0$. Thus, the rational solutions of the first system are $\frac{c_{10} + c_{12}k + c_{13}x}{k(x - k)}$. $\square$

In our algorithm for computing hyperexponential solutions, rectangular systems are all consistent and of minimal degrees. Therefore, there is no need to compute Gröbner basis

to check the consistency of the system. The reader is referred to [6, 3, 8] for algorithms to compute polynomial and rational solutions of ordinary linear functional equations. Relevant information may also be found in the papers on ordinary hyperexponential solutions, as cited in Section 1..

We need to solve the following problem later:

Problem C. Let ∂_i be a σ -shift operator and r is a nonzero rational function in $\mathbf{F}$. Find a nonzero ∂_i -constant c such that

$$L_i = \partial_i - cr$$

has a solution in $\mathbf{F}$.

We remark that Problem C can be solved when σ_i is a shift or a q -shift operator. This is because the method for computing a denominator for all rational solutions of L_i in [1] does not depend on c and because the method for bounding the degree of all polynomial solutions of L_i in [3] does not depend on c either. Therefore, a ∂_i -constant c can be determined.

9. Common Associates

The basic idea we use to solve Problem H is to compute hyperexponential solutions of an ideal I of finite rank, and then use Proposition 2 to get the rank-one ideals containing I .

Let H be the set of hyperexponential solutions of I , and let $\{P_1, \dots, P_n\}$ be a rectangular system contained in I , in which $P_i \in \mathbf{F}[\partial_i]$ for all i with $1 \leq i \leq n$. Denote by H_i the set of hyperexponential solutions of P_i with respect to ∂_i . Since $H \subset H_i$, H is empty if one of the H_i 's is empty. Assume that all the H_i 's are nonempty. Although each of the H_i 's is infinite, the set $\bar{H}_i = H_i / \sim_i$ is finite by Corollary 7. Since an element h of H is similar to an element of H_i with respect to ∂_i , h must belong to an equivalence class in $\bar{H}_i$ for all i with $1 \leq i \leq n$. This observation motivates us to introduce

Definition 3 For $i = 1, \dots, n$, let h_i be hyperexponential with respect to ∂_i . A hyperexponential element h is called a common associate of h_i if $h \sim_i h_i$, for $i = 1, \dots, n$.

Our approach involves two basic steps. First, we search for dissimilar common associates of elements in $H_1 \times \dots \times H_n$, a finite number modulo $\sim$ since $\bar{H}_1 \times \dots \times \bar{H}_n$ is finite. Let G be the set of dissimilar common associates. Then every element h of H is similar to one and only one $g \in G$ by Lemma 5 and so $h = crg$ for some rational function r and a constant c . The second step is to determine r by applying the P_j 's to rg with r treated as an unknown. Since g is hyperexponential, the applications of the P_j 's result in another rectangular system, and r is one of its rational solutions.

10. Computing common associates

In this section, we present an algorithm to construct common associates. Common associates reduce Problem H to finding rational solutions of some rectangular systems. We deal with the bivariate case in Section 10.1.. The results obtained in Section 10.1. form the inductive base for the general case handled in Section 10.2..

For nonzero $a \in \mathbf{F}$, we can decompose a as a product of two rational functions a_i and q such that $a = a_i q$, where a_i is in $C(x_i)$ whose denominator and numerator are both monic, and both numerator and denominator of q , which are relatively prime polynomials, have no

nontrivial factors in $C[x_i]$. It is easy to verify that a_i (denoted by $uv_i(a)$) and q (denoted by $mv_i(a)$) are unique.

Lemma 8 *If nonzero $a \in \mathbf{F}$, then $\sigma_i(uv_i(a)) = \alpha uv_i(\sigma_i(a))$ and $\sigma_i(mv_i(a)) = \beta mv_i(\sigma_i(a))$ for some nonzero α, β in C . Moreover, if nonzero $b \in \mathbf{F}$, then $uv_i(ab) = uv_i(a)uv_i(b)$ and $mv_i(ab) = mv_i(a)mv_i(b)$.*

Proof Recall that $\sigma_i(x_i)$ is a polynomial in x_i with positive degree and that $\sigma_i(x_j) = x_j$, for $i \neq j$, since Φ is orthogonal. Clearly, $\sigma_i(uv_i(a)) \in C(x_i)$. Write the numerator of $mv_i(a)$ as a polynomial over $C[x_i]$ with coefficients $c_1, \dots, c_m$ in $C[x_i]$. Since $c_1, \dots, c_m$ are relatively prime, so are $\sigma_i(c_1), \dots, \sigma_i(c_m)$. The same argument applies to the denominator of $mv_i(a)$. The last two equations follow from Gauss's lemma. $\square$

10.1. Bivariate case

Let h_1 and h_2 be hyperexponential with respect to ∂_1 and ∂_2 , respectively. We want to find a common associate of h_1 and h_2 . Assume that $\partial_1 h_1 = u_1 h_1$ and $\partial_2 h_2 = u_2 h_2$, where u_1 and u_2 belong to $\mathbf{F}$. Note that $u_i \neq 0$ if $\sigma_i \neq \mathbf{1}$. If h is a common associate of h_1 and h_2 , then

$$h = c_1 r_1 h_1 \quad \text{and} \quad h = c_2 r_2 h_2, \quad (22)$$

where nonzero r_1, r_2 belong to $\mathbf{F}$, and c_1, c_2 are ∂_1 and ∂_2 -constants, respectively.

Applying operators ∂_1 to the first equation and ∂_2 to the second in (22) then gives

$$\begin{cases} \partial_1 \bullet h = c_1 (\sigma_1(r_1) \partial_1 \bullet h_1 + \delta_1(r_1) h_1) \\ \partial_2 \bullet h = c_2 (\sigma_2(r_2) \partial_2 \bullet h_2 + \delta_2(r_2) h_2). \end{cases} \quad (23)$$

Equations (22) and (23) imply

$$\partial_1 \bullet h = \underbrace{\left(\frac{\sigma_1(r_1)}{r_1} u_1 + \frac{\delta_1(r_1)}{r_1} \right)}_{w_1} h \quad (24)$$

and

$$\partial_2 \bullet h = \underbrace{\left(\frac{\sigma_2(r_2)}{r_2} u_2 + \frac{\delta_2(r_2)}{r_2} \right)}_{w_2} h. \quad (25)$$

Let J be the ideal generated by $(\partial_1 - w_1)$ and $(\partial_2 - w_2)$. Since h is a solution of J , the ideal is of rank one, and, hence, the compatibility condition (5) implies that

$$\sigma_2(w_1)w_2 + \delta_2(w_1) = \sigma_1(w_2)w_1 + \delta_1(w_2). \quad (26)$$

Set z to be $\frac{r_1}{r_2}$. We will construct a system satisfied by z .

Differential Case. If both σ_1 and σ_2 are $\mathbf{1}$, then (26) becomes

$$\delta_2 \left(\frac{\delta_1 z}{z} \right) = \delta_1 u_2 - \delta_2 u_1 \quad (27)$$

(see equation (7) in [17]). A rational solution of this equation can be found by the algorithm **PolynomialRatio** [17]. Thus, we will only consider σ -shift and mixed cases. From this point on, let $\sigma_2 \neq \mathbf{1}$ and so $\delta_2 = \mathbf{0}$. Hence $u_2 \neq 0$.

σ -Shift Case. Since both δ_1 and δ_2 are $\mathbf{0}$, (26) becomes

$$\frac{\sigma_2 \circ \sigma_1(r_1)}{\sigma_2(r_1)} \frac{\sigma_2(r_2)}{r_2} \sigma_2(u_1)u_2 = \frac{\sigma_1 \circ \sigma_2(r_2)}{\sigma_1(r_2)} \frac{\sigma_1(r_1)}{r_1} \sigma_1(u_2)u_1.$$

Since σ_1 and σ_2 commute it follows that

$$\frac{z(\sigma_1 \circ \sigma_2)(z)}{\sigma_1(z)\sigma_2(z)} = \frac{u_1\sigma_1(u_2)}{u_2\sigma_2(u_1)}. \quad (28)$$

Mixed Case. Assume that $\sigma_1 = \mathbf{1}$. Then $\delta_1 \neq \mathbf{0}$ and

$$w_1 = u_1 + \frac{\delta_1(r_1)}{r_1} \quad \text{and} \quad w_2 = \frac{\sigma_2(r_2)}{r_2} u_2.$$

Denote $\frac{\sigma_2(r_2)}{r_2}$ by f . Then the left hand side L of (26) becomes:

$$L = u_2\sigma_2(u_1)f + \underbrace{\frac{\sigma_2 \circ \delta_1(r_1)}{\sigma_2(r_1)}}_{a_1} fu_2,$$

and the right hand side R of (26) becomes:

$$u_1u_2f + \underbrace{\frac{\delta_1(r_1)}{r_1}}_{a_2} fu_2 + \delta_1(u_2)f + \underbrace{\frac{\delta_1 \circ \sigma_2(r_2)}{r_2}}_{a_3} u_2 - \underbrace{\frac{\delta_1(r_2)}{r_2}}_{a_4} fu_2.$$

Set

$$v = \frac{u_1u_2 + \delta_1(u_2) - \sigma_2(u_1)u_2}{u_2}. \quad (29)$$

It follows from $L = R$ that

$$(a_4 - a_2)f + a_1f - a_3 = vf.$$

Hence, $(a_4 - a_2) + \left(a_1 - \frac{a_3}{f}\right) = v$. Since $(a_4 - a_2) = -\frac{\delta_1(z)}{z}$ and $a_1 - \frac{a_3}{f} = \frac{\delta_1(\sigma_2(z))}{\sigma_2(z)}$, we find by logarithmic derivatives (see (15)) that

$$\delta_1\left(\frac{\sigma_2(z)}{z}\right) = v\frac{\sigma_2(z)}{z}. \quad (30)$$

We can compute $\text{mv}_1(z)$ by (28), and $\text{mv}_2(z)$ by (30), respectively, and use them to construct common associates. For the σ -shift case, set y to be the ratio of $\sigma_1(z)$ and z , then (28) becomes $\sigma_2(y) = \frac{u_1\sigma_1(u_2)}{u_2\sigma_2(u_1)}y$. If this equation has a nonzero rational solution f , then all its rational solutions are in form $c(x_1)f$, where $c(x_1) \in \mathbf{F}_2$. From the definition of y we get $\sigma_1(z) = c(x_1)fz$, for some $c(x_1) \in \mathbf{F}_2$. Lemma 8 implies that $\sigma_1(\text{mv}_1(z)) = \beta\text{mv}_1(f)\text{mv}_1(z)$

for some nonzero β in C . Regarding $\text{mv}_1(z)$ as an unknown, we determine that $\text{mv}_1(z)$ satisfies

$$\begin{cases} \sigma_2(Y) = \frac{u_1\sigma_1(u_2)}{u_2\sigma_2(u_1)} Y \\ \sigma_1(Z) = \beta \text{mv}_1(y) Z, \end{cases} \quad (31)$$

where y is any nonzero rational solution of the first equation, and β is an undetermined nonzero element in C .

Now consider the mixed case. Set $y = \frac{\sigma_2(z)}{z}$ so that (30) becomes $\delta_1(y) = vy$ and $\sigma_2(z) = yz$, where v is defined in (29). From the first equation we find its general rational solutions in the form $d(x_2)g$, where $d(x_2)$ is an unspecified nonzero element in $\mathbf{F}_1$ and g belongs to $\mathbf{F}$. By Lemma 8 we have $\sigma_2(\text{mv}_2(z)) = \beta \text{mv}_2(g) \text{mv}_2(z)$. As in the shift case, we determine that $\text{mv}_2(z)$ satisfies

$$\begin{cases} \delta_1(Y) = \frac{u_1u_2 + \delta_1(u_2) - \sigma_2(u_1)u_2}{u_2} Y \\ \sigma_2(Z) = \beta \text{mv}_2(y) Z, \end{cases} \quad (32)$$

where y is any nonzero rational solution of the first equation, and β is an undetermined constant in C .

If we find a constant β such that the second equation in (31) (resp. in (32)) has a rational solution Z , then $\frac{\sigma_2(Z)}{Z}$ is a rational solution of the first equation in (31) (resp. in (32)). Hence, Z is a rational solution of (28) (resp. of (30)). We can determine whether there exists a nonzero β such that the second equation in (31) (resp. in (32)) has a rational solution, and find one solution if such β exists (see the remark at the end of Section 8.).

Our main results are summarized in

Theorem 9 *Let $\mathbf{A}_2 = C(x_1, x_2)[\partial_1, \partial_2]$ be an orthogonal Ore algebra, in which $\sigma_1 = \mathbf{1}$ if $\delta_1 \neq \mathbf{0}$, $\sigma_1 \neq \mathbf{1}$ if $\delta_1 = \mathbf{0}$, $\sigma_2 \neq \mathbf{1}$ and $\delta_2 = \mathbf{0}$. Let h_i be hyperexponential with respect to ∂_i with certificate u_i , $i = 1, 2$. Then h_1 and h_2 have a common associate if and only if one of the following conditions holds:*

1. *If neither σ_1 nor σ_2 are $\mathbf{1}$, then system (31) has a rational solution (y, z) with $yz \neq 0$.*
2. *If $\delta_1 \neq \mathbf{0}$ and $\sigma_2 \neq \mathbf{1}$, then system (32) has a rational solution (y, z) with $yz \neq 0$.*

Moreover, suppose that such y and z exist. Let p_1 and p_2 be the numerator and denominator of z , respectively. Then

$$R_1 = \frac{\sigma_1(p_1)}{p_1} u_1 + \frac{\delta_1(p_1)}{p_1}, \quad R_2 = \frac{\sigma_2(p_2)}{p_2} u_2 + \frac{\delta_2(p_2)}{p_2} \quad (33)$$

are respective ∂_1 and ∂_2 -certificates of a common associate of h_1 and h_2 .

Proof Sufficiency has been proven by the argument before the theorem. We shall prove the necessity for the mixed case with the σ -shift case handled in a similar manner.

Assume that $\delta_1 \neq \mathbf{0}$ and $\sigma_2 \neq \mathbf{1}$ so that $\sigma_1 = \mathbf{1}$ and $\delta_2 = \mathbf{0}$. Suppose that the pair (y, z) is a rational solution of (32) with $yz \neq 0$ for some $\beta \in C$. Since $\text{mv}_2(y)$ is also a solution of the first equation, z is a solution of (30). Reversing the argument to get (30), we see

that (26) holds for any nonzero r_1 and r_2 with $z = r_1/r_2$. In particular, R_1 and R_2 in (33) are respective ∂_1 and ∂_2 -certificates of a hyperexponential element h .

It remains to show that h is a common associate of h_1 and h_2 . Recall that $u_i = \frac{\partial_i \bullet h_i}{h_i}$, for $i = 1, 2$. Thus, (24) becomes

$$\frac{\delta_1(h)}{h} = \frac{\delta_1(h_1)}{h_1} + \frac{\delta_1(p_1)}{p_1} = \frac{\delta_1(p_1 h)}{p_1 h},$$

since $\sigma_1 = \mathbf{1}$. The ratio of h and $p_1 h_1$ is a ∂_1 -constant, because of the logarithmic derivatives (see (15)) and $\partial_1 = \delta_1$. Similarly, (25) becomes

$$\frac{\sigma_2(h)}{h} = \frac{\sigma_2(p_2)}{p_2} \frac{\sigma_2(h_2)}{h_2},$$

since $\delta_2 = 0$. Thus, the ratio of h and $p_2 h_2$ is a ∂_2 -constant, since $\partial_2 = \sigma_2$. $\square$

Example 7 Let $\mathbf{A}$ be as in Example 1. We compute the common associate of $h_1 = \exp(-x)x^{k+1}$ and $h_2 = \frac{kx^k}{x-k}$. The system (32) becomes

$$\begin{cases} \frac{\partial Y(x, k)}{\partial x} = \frac{-1}{(x-k)(x-k-1)} Y(x, k) \\ Z(x, k+1) = \beta \text{mv}_k(Y) Z(x, k). \end{cases}$$

The first equation has a rational solution $q = \frac{k-x}{k-x+1}$ with $\text{mv}_k(q) = q$ and $\beta = 1$. The second equation then has a rational solution $\frac{1}{k-x}$. By Theorem 9, a common associate h of h_1 and h_2 has ∂_x -certificate $\frac{1+k-x}{x}$ and E_k -certificate $\frac{(k+1)x}{k}$. One may write h as $k \exp(-x)x^{k+1}$. $\square$

We can now outline an algorithm for hyperexponential solutions of an ideal I with finite rank. Assume that we are given a finite basis of I . First, we compute a Gröbner basis of I and then use linear algebra to construct a rectangular system $\mathcal{P} = \{P_1, P_2\}$ contained in I . In fact, the rectangular system will be of minimal degrees. We then find the set H_i of hyperexponential solutions of P_i with respect to ∂_i , for $i = 1, 2$. If one of the H_i 's is empty, then I has no hyperexponential solution. Otherwise we choose dissimilar elements $h_{i1}, \dots, h_{im_i}$ such that any element of H_i is similar to one and only one h_{ij_i} . For each pair (h_{1j_1}, h_{2j_2}) , we then use Theorem 9 to compute a common associate, discarding pairs without common associates. If no common associate is found then we are done. Otherwise assume that the common associates are $h_1, \dots, h_m$. Then, for each $k = 1, \dots, m$, apply the P_i 's to rh_k to obtain a new rectangular system $\mathcal{Q}_k$ over $\mathbf{F}$, since $\partial_i \bullet h_k = u_{ik} h_k$ with $u_{ik} \in \mathbf{F}$. Find rational solutions of $\mathcal{Q}_k$. If no rational solution is found, discard h_k . Write the general rational solution r_k of $\mathcal{Q}_k$ as a linear combination of rational functions, in which unspecified constants appear linearly. Apply each operator in the given basis to $r_k h_k$ to get a linear algebraic system in these constants and solve these linear systems.

Example 8 Let $\mathbf{A}$ be as in Example 1 and let us find hyperexponential solutions of the rank-four ideal given in Example 2. We use the Maple command `DEtools[expsols]` to find that all hyperexponential solutions of L_1 are similar to the hyperexponential function $h_1 = \exp(-x)x^{k+1}$ with respect to x . Likewise, the command `LREtools[hypergeomsols]`

finds that all hyperexponential (hypergeometric) solutions of L_2 are similar to the term $h_2 = \frac{kx^k}{x-k}$ with respect to k . As shown in Example 7, a common associate of h_1 and h_2 is $h = k \exp(-x)x^{k+1}$. Hence, all hyperexponential solutions of I are similar to h . Apply L_1 and L_2 to rh to get a new rectangular system (in r), which is the first system in Example 6. Taking the rational solution of the system from Example 6, we conclude that the ideal I has hyperexponential solutions in the form

$$\frac{c_1 + c_2k + c_3x}{x-k} \exp(-x)x^{k+1}. \quad \square$$

Example 9 Consider the shift algebra given in Example 4. We compute the hyperexponential (hypergeometric) solutions of the ideal I generated by

$$L_1 = E_m^2 + \frac{18+(m-k)(9-(m+k)(m+k+1))}{(m-k)(m+k-3)-3} E_m \\ + \frac{3(4k^2-10k+(m-k)(2k-5+(m+k)(m+k-1)))}{(m-k)(m+k-3)-3}$$

and

$$L_2 = E_k^2 - \frac{2+(m-k)(4(m+k)(m+k+1)-1)}{(m-k)(2m+2k-1)+1} E_k \\ + \frac{(m-k)((m+k)(4m+4k-2)+-8k)-16k^2}{(m-k)(2m+2k-1)+1}.$$

The operators L_1 and L_2 form a Gröbner basis of I . The hyperexponential solutions of L_1 with respect to E_m are either similar to $h_{11}=3^m$ or to $h_{12}=\Gamma(m+k)$, while the hyperexponential solutions of L_2 are either similar to $h_{21}=1$ or to $h_{22}=2^k\Gamma(m+k)$. The pairs (h_{11}, h_{22}) and (h_{21}, h_{22}) have no common associates, because the system (31) for each of the pairs has no rational solution. On the other hand, the pair (h_{11}, h_{21}) has a common associate $h_1 = 3^m$, and (h_{12}, h_{22}) has a common associate $h_2 = 2^k\Gamma(m+k)$. Hence, the hyperexponential solutions of I are in the form of either r_1h_1 and r_2h_2 , where r_1, r_2 are in $C(m, k)$. Using the algorithm given in Section 8., we get $r_1=(m-k)$ and $r_2=1$. The hyperexponential solutions of I are then $c_1(m-k)3^k$ and $c_22^k\Gamma(m+k)$, where c_1 and c_2 are constants. $\square$

10.2. General case

In this section, we describe an algorithm for computing common associates of n elements $h_1, \dots, h_n$, where h_i is hyperexponential with respect to ∂_i , for $i = 1, \dots, n$. Let $\partial_1, \dots, \partial_k$ be differential operators, that is, $\sigma_i = \mathbf{1}$ and $\delta_i \neq \mathbf{0}$ for $i = 1, \dots, n$, and $\partial_{k+1}, \dots, \partial_n$ be σ -shift operators, that is, $\sigma_j \neq \mathbf{1}$ and $\delta_j = \mathbf{0}$, for $j = k+1, \dots, n$.

Lemma 10 If $h_1, \dots, h_n$ have a common associate, then there exists a common associate h such that $h = c_i p_i h_i$ for all i with $1 \leq i \leq n$, where c_i is a ∂_i -constant and p_i is a polynomial in $C[x_1, \dots, x_n]$.

Proof Let g be a common associate of the h_i 's. Then there are rational functions $r_1, \dots, r_n$ in $\mathbf{F}$ such that

$$g = c_1 r_1 h_1, \dots, g = c_n r_n h_n$$

where c_i is a ∂_i -constant for $i = 1, \dots, n$. Let q be a common denominator of the r_i 's. Then we can set $h = qg$ and $p_i = q r_i$ for $i = 1, \dots, n$. $\square$

By Lemma 10 we need to find a hyperexponential element h and polynomials $p_1, \dots, p_n$ in $C[x_1, \dots, x_n]$ such that

$$h = c_i p_i h_i \quad \text{where } c_i \text{ is some } \partial_i \text{ constant, for } i = 1, \dots, n. \quad (34)$$

Let u_i be the certificate of h_i with respect to ∂_i for all i with $1 \leq i \leq n$. Then the certificate of h with respect to ∂_i is

$$\frac{\partial_i \bullet h}{h} = \underbrace{\frac{\sigma_i(p_i)}{p_i} u_i + \frac{\delta_i(p_i)}{p_i}}_{w_i}. \quad (35)$$

By (5) we need only to find nonzero polynomials $p_1, \dots, p_n$ such that

$$\sigma_j(w_i)w_j + \delta_j(w_i) = \sigma_i(w_j)w_i + \delta_i(w_j), \quad (1 \leq i < j \leq n). \quad (36)$$

Let $z_{ij} = \frac{p_i}{p_j}$. By (27), (28) and (30), equation (36) holds if and only if

$$\delta_j \left(\frac{\delta_i z_{ij}}{z_{ij}} \right) = \delta_j(u_i) - \delta_i(u_j) \quad (1 \leq i < j \leq k), \quad (37)$$

$$\delta_i \left(\frac{\sigma_j(z_{ij})}{z_{ij}} \right) = \frac{u_i u_j + \delta_i(u_j) - \sigma_j(u_i) u_j}{u_j} \frac{\sigma_j(z_{ij})}{z_{ij}} \quad (1 \leq i \leq k \text{ and } k < j \leq n), \quad (38)$$

and

$$\sigma_j \left(\frac{\sigma_i(z_{ij})}{z_{ij}} \right) = \frac{u_i \sigma_i(u_j)}{u_j \sigma_j(u_i)} \left(\frac{\sigma_i(z_{ij})}{z_{ij}} \right) \quad (k \leq i < j \leq n). \quad (39)$$

Denote by $\mathcal{E}_n$ the system consisting of all equations in (37), (39) and (38). We say that a vector $(r_1, \dots, r_n)$, where $r_1, \dots, r_n$ are nonzero rational functions (resp. nonzero polynomials) in $\mathbf{F}$, is a rational (resp. polynomial) solution of the system $\mathcal{E}_n$ if all equations in $\mathcal{E}_n$ hold when z_{ij} is replaced by $\frac{r_i}{r_j}$ for all i, j with $1 \leq i < j \leq n$. Lemma 10 implies that $\mathcal{E}_n$ has a rational solution if and only if it has a polynomial solution. Hence, $h_1, \dots, h_n$ has a common associate if and only if $\mathcal{E}_n$ has a polynomial solution. The next lemma shows how two solutions of $\mathcal{E}_n$ are related.

Lemma 11 *If $(f_1, \dots, f_n)$ and $(g_1, \dots, g_n)$ be rational solutions of $\mathcal{E}_n$ in $\mathbf{F}$, then there exists a rational function r in $\mathbf{F}$ such that*

$$\frac{\delta_i(f_i)}{f_i} = \frac{\delta_i(g_i)}{g_i} + \frac{\delta_i(r)}{r} \quad \text{for } i = 1, \dots, k, \quad (40)$$

and

$$\frac{\sigma_j(f_j)}{f_j} = \frac{\sigma_j(g_j)}{g_j} \frac{\sigma_j(r)}{r} \quad \text{for } i = k+1, \dots, n. \quad (41)$$

In particular, we have $f_i = c_i r g_i$ where c_i is a ∂_i -constant, $i = 1, \dots, n$.

Proof Since both the f_i 's and g_i 's satisfy (37), (38) and (39), both

$$I_f = \left(\partial_1 - \left(\frac{\sigma_1(f_1)}{f_1} u_1 + \frac{\delta_1(f_1)}{f_1} \right), \dots, \partial_n - \left(\frac{\sigma_n(f_n)}{f_n} u_n + \frac{\delta_n(f_n)}{f_n} \right) \right) \quad (42)$$

and

$$I_g = \left(\partial_1 - \left(\frac{\sigma_1(g_1)}{g_1} u_1 + \frac{\delta_1(g_1)}{g_1} \right), \dots, \partial_n - \left(\frac{\sigma_n(g_n)}{g_n} u_n + \frac{\delta_n(g_n)}{g_n} \right) \right) \quad (43)$$

are rank-one ideals. Let f and g be nonzero solutions of I_f and I_g , respectively. Using the same method described in the proof of Theorem 9, we find $f \sim_i h_i$ and $g \sim_i h_i$, for all i with $1 \leq i \leq n$. Therefore, $f \sim g$ by Lemma 6. There then exist a nonzero $r \in \mathbf{F}$ and a constant c such that $f = crg$. Applying ∂_i to both sides of $f = crg$ yields

$$\partial_i \bullet (f) = c\sigma_i(r)\partial_i \bullet (g) + \delta_i(r)g.$$

It follows from (42) and (43) that

$$\frac{\sigma_i(f_i)}{f_i} u_i + \frac{\delta_i(f_i)}{f_i} = \frac{\sigma_i(r)}{r} \left(\frac{\sigma_i(g_i)}{g_i} u_i + \frac{\delta_i(g_i)}{g_i} \right) + \frac{\delta_i(r)}{r}. \quad (44)$$

For every i with $1 \leq i \leq k$, (44) becomes (40), and for every i with $k+1 \leq i \leq m$, (44) becomes (41). $\square$

We describe an algorithm which determines whether $\mathcal{E}_n$ has a polynomial solution and computes such a solution when it exists. As a convenience, we denote by $\mathcal{E}_m$ the set of all equations in (37), (38) and (39), where $1 \leq i < j \leq m$. Note that the coefficients of $\mathcal{E}_m$ are in $\mathbf{F}$.

If $m = 2$, then we can apply the method described in Section 10.1. to find $p_1, p_2 \in C(x_3, \dots, x_n)[x_1, x_2]$ such that (p_1, p_2) is a polynomial $\mathcal{E}_2$. If such p_1 and p_2 do not exist, then $\mathcal{E}_2$ has no polynomial solutions, neither does $\mathcal{E}_n$. Assume that such p_1 and p_2 do exist. We may then further assume that p_1 and p_2 are polynomials in $x_1, \dots, x_n$, because the denominators of p_1 and p_2 are polynomials in $x_3, \dots, x_n$, which are constants with respect to ∂_1 and ∂_2 . If k , the number of differential operators, is greater than one. We can apply the algorithm **PolynomialRatio** in [17] to find a polynomial solution $(p_1, \dots, p_k)$ of $\mathcal{E}_k$, where the p_i 's are in $C(x_{k+1}, \dots, x_n)[x_1, \dots, x_k]$. If such $p_1, \dots, p_k$ do not exist, then $\mathcal{E}_k$ has no polynomial solutions, neither does $\mathcal{E}_n$. Otherwise, we may also assume that $p_1, \dots, p_k$ are polynomials in $x_1, \dots, x_n$ by the same reason. Hence, the induction hypothesis is that we have found a polynomial solution $(p_1, \dots, p_{m-1})$ of $\mathcal{E}_{m-1}$ and ∂_m is a σ -shift operator. We want to find a polynomial solution of $\mathcal{E}_m$.

The equations in $(\mathcal{E}_m \setminus \mathcal{E}_{m-1})$ are

$$\delta_i \left(\frac{\sigma_m(z_{im})}{z_{im}} \right) = v_i \frac{\sigma_m(z_{im})}{z_{im}} \quad (45)$$

where $i = 1, \dots, k$, and

$$\sigma_j \left(\frac{\sigma_m(z_{jm})}{z_{jm}} \right) = v_j \frac{\sigma_m(z_{jm})}{z_{jm}} \quad (46)$$

where $j = k+1, \dots, m-1$. Note that the v_j 's are given in equations (39) and (38). With the assumption that $(p_1, \dots, p_{m-1})$ is a polynomial solution of $\mathcal{E}_{m-1}$, we claim that $\mathcal{E}_m$ has a polynomial solution if and only if there exists a nonzero rational function q such that

$$\left(\frac{p_1}{q}, \dots, \frac{p_{m-1}}{q} \right) \quad (47)$$

is a rational solution of (45) and (46).

Proof of the claim. First, we assume that $(r_1, \dots, r_m)$ is a polynomial solution of $\mathcal{E}_{m-1}$. Then $(r_1, \dots, r_{m-1})$ is a polynomial solution of $\mathcal{E}_{m-1}$. It follows from Lemma 11 that there exists a rational function s such that $p_i = c_i s r_i$, where c_i is a ∂_i -constant, $i = 1, \dots, (m-1)$. Since equations (45) and (46) hold when z_{im} is replaced by $\frac{r_i}{r_m}$, for $i = 1, \dots, (m-1)$, they hold when z_{im} is replaced by $\frac{p_i}{c_i s r_m}$. Note that $\sigma_m(c_i)$ is again a ∂_i -constant because $\mathbf{A}$ is orthogonal. Therefore, (47) is a solution of (45) and (46) with $q = s r_m$. Conversely, assume that (47) is a rational solution of (45) and (46). Then $(s p_1, \dots, s p_{m-1}, t)$ is a polynomial solution of $\mathcal{E}_m$, where s and t are the denominator and numerator of q , respectively. The claim is proved.

Now, the problem is reduced to that of finding a rational solution of (45) and (46) in form (47). Substituting $\frac{p_i}{q}$ for z_{im} into these equations and setting $\sigma_m(q)/q$ to be r , we get

$$\begin{cases} \delta_i(r) = t_i r & i = 1, \dots, k \\ \sigma_j(r) = t_j r & j = k+1, \dots, m-1 \\ \sigma_m(q) = r q \end{cases} \quad (48)$$

where $t_1, \dots, t_{m-1}$ are rational functions. First, we compute a rational solution of the first $(m-1)$ equations in (48). If no rational solution is found, then such q does not exist. So $\mathcal{E}_m$ has no polynomial solution by the claim. Otherwise, let $\bar{r}$ be a rational solution. Then all the rational solutions of these equations can be written as $c\bar{r}$ where c is a nonzero element in $B(x_m)$, where $B = C(x_{m+1}, \dots, x_n)$. The field B plays the same role as C in Section 10.1. Using the same method described in Section 10.1, we can find a rational solution (r, q) of (48). This completes an algorithm that computes a common associate of $h_1, \dots, h_n$. It is easy to adapt the description of the algorithm for hyperexponential solutions to the general case.

Acknowledgement

We thank M. Bronstein, R.F. Burger and M. Petkovšek for valuable discussions.

References

- [1] S. Abramov. Rational solutions of linear difference and q -difference equations with polynomial coefficients. In *Proc. Int. Symp. on Symbolic Algebraic Comput. (ISSAC'95)*, pages 285–289. A.H.M. Levelt (ed.), ACM Press, 1995.
- [2] S. Abramov and M. Bronstein. Multivariate skew polynomial rings. Invited talk on the monthly seminar of Ontario Research Center for Computer Algebra (ORCCA), Dec. 2003.
- [3] S. Abramov, M. Bronstein, and M. Petkovšek. On polynomial solutions of linear operator equations. In *Proc. Int. Symp. on Symbolic Algebraic Comput. (ISSAC'95)*, pages 290–296. A.H.M. Levelt (ed.), ACM Press, 1995.
- [4] S. Abramov, P. Paule, and M. Petkovšek. q -Hypergeometric solutions of q -difference equations. *Discrete Mathematics*, 180:3-22, 1998.
- [5] S. Abramov and M. Petkovšek. On the structure of multivariate hypergeometric terms. *Advances in Applied Math.*, 29:386–411, 2002.

- [6] M. Bronstein. On solutions of linear differential equations in their coefficient fields. *J. Symb. Comput.*, 13:413–440, 1992.
- [7] M. Bronstein. Linear ordinary differential equations: breaking through the order 2 barrier. In *Proc. Int. Symp. on Symbolic Algebraic Comput. (ISSAC'92)*, pages 42–48. P. Wang (ed.), ACM Press, 1992.
- [8] M. Bronstein. Solutions of linear ordinary difference equations in their coefficient field. *J. Symb. Comput.*, 29:841–877, 2000.
- [9] M. Bronstein, M. Petkovšek. An introduction to pseudo-linear algebra. *Theoretical Computer Science*, 157:3–33, 1996.
- [10] F. Chyzak and B. Salvy. Non-commutative elimination in Ore algebras proves multivariate identities. *J. Symb. Comput.*, 26:187–227, 1998.
- [11] P.M. Cohn. *Free rings and their relations*. Academic Press, London - New York, 1971.
- [12] R. Cohn. *Difference Algebra*. Interscience Publishers, New York - London - Sydney, 1965.
- [13] M. van Hoeij. Factorization of differential operators with rational function coefficients. *J. Symb. Comput.*, 24:537–561, 1997.
- [14] M. van Hoeij. Finite singularities and hypergeometric solutions of linear recurrence equations. *J. Pure Appl. Algebra*, 139:109–131, 1999.
- [15] G. Labahn, Z. Li. Hyperexponential solutions of finite-rank ideals in orthogonal Ore rings. *ACM International Symposium on Symbolic and Algebraic Computation*, 213–220. ACM Press, 2004.
- [16] Z. Li, F. Schwarz and S. Tsarev. Factoring zero-dimensional ideals of linear partial differential operators. In *Proc. Int. Symp. on Symbolic Algebraic Comput. (ISSAC'02)*, pages 168–175. T. Mora (ed.), ACM Press, 2002.
- [17] Z. Li, F. Schwarz, and S. Tsarev. Factoring systems of linear PDE's with finite-dimensional solution spaces *J. Symb. Comput.*, 36:443–471, 2003.
- [18] M. Petkovšek. Hypergeometric solutions of linear recurrences with polynomial coefficients. *J. Symb. Comput.*, 14:243–264, 1992.
- [19] M. van der Put and M. Singer. *Galois Theory of Difference Equations*. V. 1666 in Lecture Notes in Mathematics, Springer-Verlag, 1997.
- [20] M. van der Put and M. Singer. *Galois Theory of Linear Differential Equations*. A series of comprehensive studies in Mathematics, 328, Springer, 2003.
- [21] M. Wu. Notes on factoring partial differential modules. Manuscript.