Non-Travelling Wave Solutions for 
(2+1)-Dimensional Integrable Equation Via a 
Generalized Trilinear Form and Symbolic 
Computation

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Abstract.
In this letter, with the aid of symbolic computation, the generalized trilinear form (GTF) is derived for the (2+1)-dimensional integrable equation
\[ 4u_{xt} + u_{xxxz} + 4u_xu_{xz} + 2u_{xx}u_z + \partial_x^{-1}u_{zzz} = 0. \]
And then some non-travelling wave solutions are obtained by the means of the GTF and some ansatze. These solutions involve shock-like wave solutions, singular soliton-like solutions, shock wave solution, singular soliton solution and rational solutions. Moreover we show that the potential variable \( u_z \), admits the exponentially localized solution (dromion solution) rather than the physical field \( u(x, z, t) \) itself.

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1. Introduction

Since the ‘soliton’ concept was presented, more attention has been paid to investigate travelling wave (soliton) solutions of nonlinear evolution equations in nonlinear science[1]. In fact, the relationship between independent variables, say \( x, t \), in many nonlinear evolution equations is not always linear, i.e., \( kx + \lambda t + c \). Maybe they have nonlinear relationships. Therefore the investigation of travelling wave (soliton) solutions of nonlinear evolution equations is not enough to research the rules of these nonlinear evolution equations. Recently, some papers have reported a variety of approaches to seek non-travelling wave solutions of some nonlinear evolution equations[2-10], which are important because they involve not only known travelling wave(soliton) solutions but also many new types of solutions.

The (2+1)-dimensional integrable equation
\[ 4u_{xt} + u_{xxxz} + 4u_xu_{xz} + 2u_{xx}u_z + \partial_x^{-1}u_{zzz} = 0, \tag{1} \]
was presented by different ways[11-13,16], where \( \partial_x^{-1} = \frac{1}{2}(\int_{-\infty}^{x} - \int_{x}^{\infty}), \partial_x \partial_x^{-1} = \partial_x^{-1}\partial_x = 1. \)
When one misses the last term, (1) reduces to the equation
\[ 4u_{xt} + u_{xxxz} + 4u_xu_{xz} + 2u_{xx}u_z = 0, \tag{2} \]
which was derived by Schiff from the reduction of self-dual Yang-Mills equations\cite{17}. Some exact solutions and other properties of (2) have been obtained (see \cite{3} and referees therein). Equation (1) has been shown to possess the Lax pair\cite{11,12,14}

\begin{equation}
\begin{cases}
(\partial_x^2 + u_x + \partial_z)^2 \psi = \lambda \psi, \\
\psi_1 = (\partial_x^2 \partial_z + \frac{1}{2} \partial_z^2 + \frac{1}{2} u_x \partial_x + u_x \partial_z + \frac{3}{4} u_{xx} - \frac{1}{4} \partial_x^{-1} u_{zz}) \psi.
\end{cases}
\end{equation}

It is shown that (1) possesses the Painlevé property\cite{14}. Recently, Yu et al\cite{11} gave N-soliton solutions of (1) in terms of the trilinear form related to (1)

\begin{equation}
\begin{cases}
u = 2 \partial_x (\log \tau), \\
(36 T^2 T + 4 T^4 T^* + 8 T^3 T^* T_x + 9 T^3) \tau \cdot \tau = 0,
\end{cases}
\end{equation}

where operators $T$ and $T^*$ are defined by

\[T^n f(z) \cdot g(z) \cdot h(z) = (\partial_z + j \partial_x + j^2 \partial_x^2)^n f(z_1) g(z_2) h(z_3) |_{z_1 = z_2 = z_3 = z},\]

$j$ is the cubic root of unity, $j = \exp(2i\pi/3)$ and $T^*$ is the complex conjugate operator of $T_z$.

As far as we know, no work on non-travelling wave solutions of (1) seems to have been reported. In the following we will consider this problem.

2. The generalized trilinear form and non-travelling wave solutions

To deduce non-travelling wave solutions of (1), we need to derive its generalized trilinear form. We truncate the Laurent series expansions related to (1) at $O(\phi^0)$ as\cite{15}

\[u(x, z, t) = \frac{u_0}{\phi} + u_1.\]  

The substitution of (5) into (1) leads to $u_0 = 2 \phi_x$. If we assume that $u_1$ is a solution of (1), then (5) along with $u_0 = 2 \phi_x$ makes (1) become the trilinear form in $\phi$

\[2(4 \phi_z^2 \phi_t - \phi_z^2 \phi_x - 2 \phi_x \phi_{xx} \phi_{zz} + 2 \phi_z \phi_{xx} + 2 \phi_x \phi_{xx}) + 2 u_{1z} \phi_z^2 + 4 u_{1z} \phi_x \phi_z^2 + \phi_z^3 \]

\[\phi(8 \phi_x \phi_{xx} + 4 \phi_t \phi_{xx} - 2 \phi_x \phi_{xx} \phi_{zz} + 4 \phi_x \phi_{xx} \phi_{xx} + \phi_z \phi_{xx} + 4 u_{1z} \phi_x^2 + 8 u_{1z} \phi_x \phi_x + 8 u_{1z} \phi_x \phi_x)
\]

\[+ 4 u_{1z} \phi_{xx} + 2 u_{1z} \phi_{xx} + 2 u_{1z} \phi_{xx} + \phi_{xxx} + \phi_{zz} = 0,\]  

which can further reduce to the special system of differential equations:

\[4 \phi_x^2 \phi_t - \phi_x^2 \phi_x - 2 \phi_x \phi_{xx} \phi_{zz} + 2 \phi_x \phi_x \phi_{xx} + 2 \phi_x \phi_z \phi_{xx} + 2 u_{1z} \phi_x^2 + 4 u_{1z} \phi_x \phi_x + \phi_z^3 = 0,\]  

\[8 \phi_x \phi_{xx} + 4 \phi_t \phi_{xx} - 2 \phi_x \phi_{xx} \phi_{zz} + 4 \phi_x \phi_{xx} \phi_{xx} + \phi_z \phi_{xx} + 4 u_{1z} \phi_x^2 + 8 u_{1z} \phi_x \phi_x
\]

\[+ 4 u_{1z} \phi_{xx} + 2 u_{1z} \phi_{xx} + 6 u_{1z} \phi_{xx} + 3 \phi_z \phi_{zz} = 0,\]  

\[4 \phi_x \phi_{xx} + 4 u_{1z} \phi_x \phi_{xx} + 2 u_{1z} \phi_{xx} + 2 u_{1z} \phi_{xx} + \phi_{xxx} + \phi_{zzz} = 0.\]
It follows that for a given solution $u_1$ of (1), if one obtains nontrivial function $\phi$ from (6) or (7)-(9), then one can use (5) to derive another (new) solution of (1). Therefore (5) and (6) or (7)-(9) compose an auto-Backlund transformation of the (2+1)-dimensional integrable equation (1).

**Remark 1:** Let $u_1 = 0$. Then (5) and (6) is equivalent to the known trilinear form (3). Therefore we call (5) and (6) the generalized trilinear form of (1).

In fact, it is easy to see that the set of $\phi$ in (6) is greater than (7)-(9) for any given initial solution $u_1$ of (1). In the following we do not restrict the initial condition $u_1 = 0$. We will seek the generalized soliton solutions (non-travelling wave solutions) of (1) from (5) and (6). By inspection, we choose a proper initial solution of (1) as

$$u_1(x, z, t) = kx + f(t)z^2 + g(t)z + h(t),$$

(10)

where $k$ is a constant and $f(t), g(t), h(t)$ are arbitrary smooth functions of $t$ only.

**Type 1:** (Soliton-like solution)

To obtain non-travelling wave solutions of (1), following these ideas[2-4, 6, 8, 9], we suppose $\phi$ in (6) is expressed by the generalized form

$$\phi(x, z, t) = P(z, t) + \exp[\Theta(z, t)x + \Psi(z, t)].$$

(11)

where $P(z, t) \neq 0, \Theta(z, t)$ and $\Psi(z, t)$ are differentiable functions w.r.t. $z$ and $t$ only to be determined later. In fact, we can replace $P(z, t)$ by $\mu = \pm 1$, and then (11) simplifies

$$\phi(x, z, t) = \mu + \exp[\Theta(z, t)x + \Psi(z, t)].$$

(12)

With the aid of symbolic computation, we substitute (10) and (12) into (6) and get a polynomial in $x^i e^{j(\Theta x + \Psi)}$. Setting to zero their coefficients gives rise to the system of nonlinear partial differential equations

$$\begin{cases}
\Theta_z = \Theta_t = 0, \\
\Psi_{zzz} = 0,
\end{cases}$$

(13)

$$4\Theta^2 \phi_t + 4k\Theta^2 \phi_z + \Theta^4 \phi_z + 2\Theta^3(2fz + g) + \phi_z^3 + 3\phi_z \phi_{zz} = 0.$$ 

From the first two equations of (13), we have

$$\Theta(z, t) = \theta = \text{const.} \neq 0, \quad \psi(z, t) = A(t)z^2 + B(t)z + C(t),$$

(14)

where $A(t), B(t), C(t)$ are functions of $t$.

The substitution of (14) into the last one of (13) gives rise to

$$\begin{cases}
A(t) = 0, \\
4\theta^2 B'(t) + 4\theta^3 f(t) = 0, \\
4\theta^2 C'(t) + B(t)(\theta^4 + 4k\theta^2) + 2\theta^3 g(t) + B^3(t) = 0,
\end{cases}$$

(15)
which reads
\[ B(t) = c_0 - \theta \int_0^t f(t')dt', \] (16)\n\[ C(t) = -\int_0^t \left[ \left( \frac{1}{4} \theta^2 + k \right) \left( c_0 - \theta \int_0^{t'} f(s)ds \right) + \frac{1}{2} \theta \int_0^{t'} g(s)ds \right] \] \[ + \left( \frac{1}{4} \theta^2 \right) \left( c_0 - \theta \int_0^{t'} f(s)ds \right)^3 \] \[ dt'. \] (17)\nwhere \( c_0 \) is a integration constant.

Therefore we get the shock-like wave solution of (1) with \( \mu = 1 \)
\[ u = \theta \tanh \frac{1}{2} \theta x + \left( c_0 - \theta \int_0^t f(t')dt' \right) z - \int_0^t \left[ \left( \frac{1}{4} \theta^2 + k \right) \left( c_0 - \theta \int_0^{t'} f(s)ds \right) \right] \] \[ + \left( \frac{1}{2} \theta \right) \int_0^{t'} g(s)ds \] \[ + \left( \frac{1}{4} \theta^2 \right) \left( c_0 - \theta \int_0^{t'} f(s)ds \right)^3 dt' + \theta + kx + f(t)\theta x^2 + g(t)z + h(t), \] (18)\nand singular soliton-like solution of (1) with \( \mu = -1 \)
\[ u = \theta \coth \frac{1}{2} \theta x + \left( c_0 - \theta \int_0^t f(t')dt' \right) z - \int_0^t \left[ \left( \frac{1}{4} \theta^2 + k \right) \left( c_0 - \theta \int_0^{t'} f(s)ds \right) \right] \] \[ + \left( \frac{1}{2} \theta \right) \int_0^{t'} g(s)ds \] \[ + \left( \frac{1}{4} \theta^2 \right) \left( c_0 - \theta \int_0^{t'} f(s)ds \right)^3 dt' + \theta + kx + f(t)\theta x^2 + g(t)z + h(t), \] (19)\nSince the solutions (18) and (19) involve three arbitrary smooth functions \( f(t), g(t), h(t) \) of \( t \), one can obtain abundant structures from (18) and (19).

**Remark 2:** When we take \( k = f(t) = g(t) = 0, h(t) = c_1 \), the solution (18) reduces to shock wave solution of (1)
\[ u = \theta \tanh \frac{1}{2} \left[ \theta x + c_0 z - \left( \frac{1}{4} \theta^2 c_0 - \frac{1}{4} \theta^2 c_0^3 \right) t + c_2 \right] + \theta + c_1. \] (20)\nwhich is equivalent to the known one-soliton (12) in Ref.[11] when \( c_1 = 0 \). Moreover we also gain the singular soliton solution of (1)
\[ u = \theta \coth \frac{1}{2} \left[ \theta x + c_0 z - \left( \frac{1}{4} \theta^2 c_0 - \frac{1}{4} \theta^2 c_0^3 \right) t + c_2 \right] + \theta + c_1, \] (21)\nwhich shows that this type of wave solution will below up at the plane \( \theta x + c_0 z - \left( \frac{1}{4} \theta^2 c_0 - \frac{1}{4} \theta^2 c_0^3 \right) t_0 + c_2 = 0 \) for a certain time \( t = t_0 \).

**Remark 3:** It is clearly seen that the field \( u \) is not exponentially localized in all directions. However for the two potentials \( v_1 = u_x \) and \( v_2 = u_z \), that is
\[ v_1 = u_x = \frac{1}{2} \theta^2 \text{sech}^2 \frac{1}{2} \left[ \theta x + c_0 z - \left( \frac{1}{4} \theta^2 c_0 - \frac{1}{4} \theta^2 c_0^3 \right) t + c_2 \right], \] (22)
\[ v_2 = u_z = \frac{1}{2} \theta c_0 \text{sech}^2 \left( \frac{1}{2} \theta x + c_0 z - \left( \frac{1}{4} \theta^2 c_0 - \frac{1}{4 \theta^2 c_0^3} \right) t + c_2 \right), \] (23)

It is easy to know that when \( \theta \to 0 \) and \( c_0 \to 0 \), \( v_2 \to 0 \). Therefore the solution \( v_2 = u_z \) is the dromion solution[18]. But the solution \( v_2 \) is finite on the camber \( \theta x + c_0 z - \left( \frac{1}{4} \theta^2 c_0 - \frac{1}{4 \theta^2 c_0^3} \right) t + c_2 = 0 \) and decays exponentially away for the camber.

Type 2: (Rational solution)

We assume that \( \phi \) in (6) is expressed by another \( x \)-linear form

\[ \phi(x, z, t) = M(z, t)x + N(z, t), \] (24)

where \( M(z, t) \neq 0 \) and \( N(z, t) \) are differentiable functions w.r.t. \( z \) and \( t \) only to be determined later. Substituting (10) and (24) into (6), we have

\[
\begin{align*}
2M_z^3 - 3MM_zM_{zz} + M^2M_{zzz} &= 0, \\
3M_z^2 N_z - 3M(M_zN_{zz} + M_{zz}N_z) + M^2N_{zz} + 2MN_{zzz} &= 0, \\
2(4M^2M_t + 4kM^2M_z + 3M_zN_z^2) - M(8MM_t + 8kMM_z + 3N_zN_{zz}) &= 0, \\
-3N(N_zM_z + N_zM_{zz}) + 2MNN_{zz} + N^2M_{zzz} &= 0, \\
2(4M^2N_t + 4kM^2N_z + 2(2f(t)z + g(t))M^3 + N_z^3) + N^2N_{zz} &= 0.
\end{align*}
\] (25)

Therefore we get the rational solution of (1)

\[ u = \frac{2M(z, t)}{M(z, t)x + N(z, t)} + f(t)z^2 + g(t)z + h(t), \] (26)

where \( M(z, t), N(z, t) \) satisfy (25).

In particular, we give the special solutions of (25) as

\[ M(z, t) = M(t), \quad N(z, t) = p(t)z + q(t), \] (27)

with

\[ p(t) = M^2(t) \left( C_1 - \int t \frac{f(t')}{M(t')} dt' \right), \] (28)

\[ q(t) = M^2(t) \left\{ C_2 - \int t \frac{1}{M(t')} \left[ 2g(t') + 4kM(t') \left( C_1 - \int t' \frac{f(s)}{M(s)} ds \right) \right. \right. \]

\[ \left. \left. + M^3(t') \left( C_1 - \int t' \frac{f(s)}{M(s)} ds \right)^3 \right] dt' \right\}. \] (29)

where \( C_1, C_2 \) are constants.
Therefore we get the rational solution of (1)

\[ u = \frac{2M(t)}{M(t)x + p(t)z + q(t)} + f(t)z^2 + g(t)z + h(t), \]  

where \( p(t), q(t) \) are given by (28) and (29).

In summary, we have found many types of non-travelling wave solutions of the (2+1)-dimensional integrable equations (1). These solutions contain shock-like wave solutions, singular soliton-like solutions, shock wave solution, singular soliton solution and rational solutions, which may be useful to explain some physical phenomena. Moreover it should be pointed out that when using Backlund transformation to seek solutions, the chosen initial solution is important to obtain new types of solutions. Study is needed further to see whether (1) possesses other types of solutions from (6).

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References