Numerical Doubly-Periodic Solutions of
(2+1)-Dimensional Boussinesq Equation Via an
Improved Decomposition Scheme

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Abstract.
In this paper, an improved scheme is presented to investigate the approximate solutions of (2+1)-dimensional Boussinesq equation with initial conditions based on the decomposition method. The improve scheme makes all components $u_i$ of $u = \sum_{i=0}^{\infty} u_i$ be of the same degree in $t$. We choose the (2+1)-dimensional Boussinesq equation with initial conditions of doubly-periodic functions to illustrate the scheme. As a result, the approximate and exact doubly-periodic solutions are obtained. For differential modulus $m = 0.1, 0.9$, we compare the approximate solution with the exact solution by their graphs. Moreover we analyze the absolute error and relative error, and give the contour and density plots of the approximate doubly-periodic solutions.

Keywords: (2+1)-dimensional Boussinesq equation; the decomposition method; doubly-periodic solutions; symbolic computation; error analysis

1. Introduction

The Adomian decomposition method[1,2] is a powerful to investigate approximate solutions, or even closed-form analytical solutions of nonlinear differential equations. It provides more realistic solutions by solving the nonlinear problem without simplification and series solutions which generally converge very rapidly in real physical models. Moreover no linearization or perturbation is required in the method. Recently, the method has been applied to investigate many nonlinear differential equations with differential initial conditions such that solitary wave solutions, rational solutions, compacton solutions and other types of solutions were found[3-8,11].

Recently, Wazwaz[3] applied the modified decomposition method to investigate the Boussinesq equation

$$u_{tt} = u_{xx} + 3(u^2)_{xx} + u_{xxxx}, \quad (1.11)$$

with the two kinds of initial conditions

$$u(x, 0) = \frac{2ak^2 e^{kx}}{(1 + ae^{kx})^2}, \quad u_t(x, 0) = -\frac{2ak^2 \sqrt{1 + k^2 e^{kx}} (ae^{kx} - 1)}{(1 + ae^{kx})^2}, \quad (1.2)$$
such that the solitary wave solution and polynomial solution were given by

\[
  u(x, t) = \frac{2ak^2 \exp(kx + k\sqrt{1+k^2}t)}{1 + a \exp(kx + k\sqrt{1+k^2}t)^2},
\]

\[
  u(x, t) = a + bx + (A + bx)t + 3b^2t^2 + 2Bbt^3 + \frac{1}{2}B^4t^4,
\]

Subsequently, we\cite{6} applied the modified decomposition method to investigate a family of fully Boussinesq equation\(B(m, n)\)

\[
  u_{tt} = (u^n)_{xx} + (u^m)_{xxxx}, \quad m, n \in R.
\]

with the following initial conditions

\[
  n = m = 2, \quad u(x, 0) = \frac{4}{3}v^2 \sin^2\left(\frac{1}{4}x\right), \quad u_t(x, 0) = -\frac{1}{3}v^3 \sin\left(\frac{1}{2}x\right),
\]

\[
  n = m = 2, \quad u(x, 0) = \frac{4}{3}v^2 \cos^2\left(\frac{1}{4}x\right), \quad u_t(x, 0) = \frac{1}{3}v^3 \sin\left(\frac{1}{2}x\right).
\]

\[
  n = m = 3, \quad u(x, 0) = \frac{\sqrt{6}}{2}v \sin\left(\frac{1}{3}x\right), \quad u_t(x, 0) = -\frac{\sqrt{6}}{6}v^2 \cos\left(\frac{1}{3}x\right).
\]

\[
  n = m = 3, \quad u(x, 0) = \frac{\sqrt{6}}{2}v \cos\left(\frac{1}{3}x\right), \quad u_t(x, 0) = \frac{\sqrt{6}}{6}v^2 \sin\left(\frac{1}{3}x\right).
\]

Recently, Chen et al.\cite{9} investigated the \((2+1)\)-dimensional Boussinesq equation\(u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxxx} - u_{yy} = 0\),

using the Riccati equation expansion method\cite{10} such that many types of solutions were obtained, which include solitary wave solutions, periodic wave solutions and the combined formal solitary wave solutions and periodic wave solutions. El-Sayed and Kaya\cite{11} used the decomposition method to investigate the numerical solitary wave solutions of (1.11) with the initial conditions

\[
  u(x, 0, t) = K_1 - 6\alpha^2 R^2 \tanh^2[R(\alpha - ct)],
\]

\[
  u_y(x, 0, t) = -12\alpha^2 \beta R^2 \text{sech}^2[R(\alpha - ct)] \tanh[R(\alpha - ct)],
\]

To our knowledge, the Adomian decomposition method\cite{1,2} was not extended to investigate approximate or closed-from doubly-periodic solution of \((2+1)\)-dimensional Boussinesq equation (1.11) with the initial condition

\[
  u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxxx} - u_{yy} = 0, \quad u(x, y, 0) = f(x, y),
\]

\[
  u_t(x, y, 0) = g(x, y),
\]

\[
  u_t(x, y, 0) = h(x, y),
\]
where \( f(x, y) \) and \( g(x, y) \) are doubly-periodic functions. In this paper, we would like to extend the decomposition method to seek numerical doubly-periodic solutions of \((2+1)\)-dimensional Boussinesq equation \((1.13)\).

The rest of this paper is organized as follows: In Sec. 2, In this paper, The improved scheme is presented to investigate the approximate solutions of \((2+1)\)-dimensional Boussinesq equation with initial conditions based on the decomposition method. The improve scheme makes all components \( u_i \) of \( u = \sum_{i=0}^{\infty} u_i \) be of the same degree in \( t \). In Sec.3, we choose the \((2+1)\)-dimensional Boussinesq equation with initial conditions of doubly-periodic functions to illustrate the scheme. As a result, the approximate and exact doubly-periodic solutions are obtained. For differential modulus \( m = 0.1, 0.9 \), we compare the approximate solution with the exact solution by their graphs. Moreover we analyze the absolute error and relative error, and give the contour and density plots of the approximate doubly-periodic solutions. Finally, some conclusions and discussions are given in Sec. 4.

2. Leading to a new recursive formulate

\((2+1)\)-dimensional Boussinesq equation in the operator form

\[
L_t u = u_{xx} + (u^2)_{xx} + u_{xxxx} + u_{yy},
\]

(2.1)

where \( L_t = \frac{\partial^2}{\partial t^2} \). It is assumed that \( L_t^{-1} \) is a twofold integral operator given by

\[
L_t^{-1}(\cdot) = \int_0^t \int_0^t (\cdot) dt dt.
\]

(2.2)

Operating with the integral operator \( L_t^{-1} \) on both sides of \((2.1)\) and using the given conditions we have

\[
u(x, y, t) = f(x, y) + tg(x, y) + L_t^{-1}[u_{xx} + (u^2)_{xx} + u_{xxxx} + u_{yy}].
\]

(2.3)

where \( f \) and \( g \) are the functions that arise from the given initial conditions that are assumed to be prescribed.

According to the decomposition method [1-8], we assume that a series solution of the unknown function \( u(x, t) \) given by

\[
u(x, y, t) = \sum_{i=0}^{\infty} u_i(x, y, t).
\]

(2.4)

The nonlinear term \( F(u) = (u^2)_{xx} \) can be decomposed into the infinite series of polynomials given by

\[
F(u) = (u^2)_{xx} = \sum_{i=0}^{\infty} A_i,
\]

(2.5)

where the components \( u_i(x, y, t) \) will be determined recurrently, and \( A_i \) is the so-called Adomian polynomials of \( u_0, u_1, ..., u_i \) defined by

\[
A_i = \frac{1}{i!} \frac{d^i}{d\lambda^i} \left[ \left( \sum_{j=0}^{\infty} \lambda^j u_j \right)^2 \right]_{\lambda=0} = \left( \sum_{j=0}^{i} u_j u_{i-j} \right)_{xx}, \quad i = 0, 1, ...
\]

(2.6)
Substituting (2.4)-(2.6) into (2.3) gives rise to
\[
\sum_{i=0}^{\infty} u_i(x, y, t) = f(x, y) + g(x, y)t + L_t^{-1}\left(\sum_{i=0}^{\infty} A_i + \sum_{i=0}^{\infty} u_{i,xx} + \sum_{i=0}^{\infty} u_{i,xxx} + \sum_{i=0}^{\infty} u_{i,yy}\right).
\] (2.7)

To determine the components \(u_i(x, y, t), \ i \geq 0\), we employ the new recursive relationship
\[
\begin{align*}
    u_0(x, y, t) &= f(x, y), \\
    u_1(x, y, t) &= g(x, y)t, \\
    u_{i+2}(x, y, t) &= L_t^{-1}(A_i + u_{i,xx} + u_{i,xxx} + u_{i,yy}), \quad i \geq 0.
\end{align*}
\] (2.8)

where \(A_i\) is the Adomian polynomials that represent the nonlinear \((u^2)_{xx}\) and can be derived by
\[
\begin{align*}
    A_0 &= (u_0^2)_{xx}, \\
    A_1 &= (2u_1u_0)_{xx}, \\
    A_2 &= (2u_2u_0 + u_1^2)_{xx}, \\
    A_3 &= (2u_3u_0 + 2u_2u_1)_{xx}, \\
    A_4 &= (2u_4u_0 + 2u_3u_1 + u_2^2)_{xx}, \\
    A_5 &= (2u_5u_0 + 2u_4u_1 + 2u_3u_2)_{xx}.
\end{align*}
\] (2.9)

**Proposition.** All components \(u_i\)'s defined by (2.8) are all the homogenous polynomials in \(t\) for all \(i = 0, 1, 2, ..., \) and \(D^i(u_i) = i\), where \(D^i(u_i)\) denotes the degree of \(i\) in the polynomial \(u_i\).

**Remark 1.** The recursive relationship (2.8) is naturally different from one in ref.[3]. It is easy to calculate the elements \(u_i\).

In view of (2.8) and (2.9), we know that all of the components \(u_i(x, y, t)\) are calculable, and \(u(x, y, t) = \sum_{i=0}^{\infty} u_i(x, y, t)\) using the initial conditions without any need to transformations and linearization or perturbation of the equations. If the series converges, the n-term partial sum \(\phi_n = \sum_{i=0}^{n-1} u_i\), will be the approximate solution since \(\lim_{n \to \infty} \sum_{i=0}^{\infty} u_i = u\).

### 3. Numerical doubly-periodic solutions

In the following we would like consider the numerical doubly-periodic solutions of \((2+1)\)-dimensional Boussinesq equation with the following initial conditions:
\[
\begin{align*}
    u_t - u_{xx} - (u^2)_{xx} - u_{xxxx} - u_{yy} &= 0, \\
    u(x, y, 0) &= -6k^2m^2sn^2[k(x + ly), m] - \frac{1}{2}l^2 + 2k^2(1 + m^2), \\
    u_t(x, y, 0) &= -12k^3\lambda m^2sn[k(x + ly), m]cn[k(x + ly), m]dn[k(x + ly), m],
\end{align*}
\] (3.1)

where \(k, l, \lambda\) are constants, and \(m\) is the modulus of Jacobi elliptic functions[12,13].

Applying the inverse operator \(L_t^{-1}\) to both sides of (3.1) and using the decomposition series (2.7) and (2.8) yields
\[
\sum_{i=0}^{\infty} u_i(x, y, t) = -6k^2m^2sn^2[k(x + ly), m] - \frac{1}{2}l^2 + 2k^2(1 + m^2) + \frac{1}{2}\lambda^2 - \frac{1}{2}.
\]
\[-12k^3\lambda m^2 \text{sn}[k(x + ly), m] \text{cn}[k(x + ly), m] \text{dn}[k(x + ly), m]t + L_t^{-1}\left(\sum_{i=0}^{\infty} A_i + \sum_{i=0}^{\infty} u_{i,xx} + \sum_{i=0}^{\infty} u_{i,xxxx} + \sum_{i=0}^{\infty} u_{i,yy}\right),\]  \hspace{1cm} (3.2)

Where $A_i$ is the Adomian polynomial that represent the nonlinear operator $(u^2)_{xx}$. We introduce the recursive relationship:

\[\begin{align*}
    u_0(x, y, t) &= -6k^2m^2\text{sn}^2(k\xi, m) - \frac{1}{4}t^2 + 2k^2(1 + m^2) + \frac{1}{2}\lambda^2 - \frac{1}{2}, \\
    u_1(x, y, t) &= -12k^3\lambda m^2 \text{sn}(k\xi, m) \text{cn}(k\xi, m) \text{dn}(k\xi, m)t, \\
    u_{i+2}(x, y, t) &= L_t^{-1}(A_i + u_{i,xx} + u_{i,xxxx} + u_{i,yy}), \quad i \geq 0.
\end{align*}\]  \hspace{1cm} (3.3)

where $\xi = x + ly$.

With the symbolic computation (Maple), we have from (3.3)

\[\begin{align*}
    u_0 &= -6k^2m^2\text{sn}^2(k\xi, m) - \frac{1}{2}t^2 + 2k^2(1 + m^2) + \frac{1}{2}\lambda^2 - \frac{1}{2}, \\
    u_1 &= -12k^3\lambda m^2 \text{sn}(k\xi, m) \text{cn}(k\xi, m) \text{dn}(k\xi, m)t, \\
    u_2 &= L_t^{-1}(A_0 + u_{0,xx} + u_{0,xxxx} + u_{0,yy}) \\
    &= \left[12k^4m^2\lambda^2 (\text{sn}(k\xi, m))^2 - 6k^4m^2\lambda^2 - 18k^4m^4\lambda^2 (\text{sn}(k\xi, m))^4\right. \\
    &\quad + 12k^4m^4\lambda^2 (\text{sn}(k\xi, m))^2 \big] t^2, \\
    u_3 &= L_t^{-1}(A_1 + u_{1,xx} + u_{1,xxxx} + u_{1,yy}) \\
    &= -24k^5m^4\lambda^3 (\text{sn}(k\xi, m))^3 \text{dn}(k\xi, m)\text{cn}(k\xi, m) + (8k^5m^4\lambda^3 + 8k^5m^2\lambda^5) \text{dn}(k\xi, m)\text{cn}(k\xi, m)\text{sn}(k\xi, m) \big] t^3, \\
    u_4 &= L_t^{-1}(A_2 + u_{2,xx} + u_{2,xxxx} + u_{1,yy}) \\
    &= -30\lambda^6k^6m^6 (\text{sn}(k\xi, m))^6 + (30\lambda^4k^6m^6 + 30\lambda^2k^6m^4) (\text{sn}(k\xi, m))^4 \\
    &\quad + \left(-4\lambda^4k^6m^2 - 4\lambda^4k^6m^2 - 26\lambda^4k^6m^4\right) (\text{sn}(k\xi, m))^2 \\
    &\quad + 2\lambda^4k^6m^2 + 2\lambda^4k^6m^2 \big] t^4, \\
    u_5 &= L_t^{-1}(A_3 + u_{3,xx} + u_{3,xxxx} + u_{3,yy}) \\
    &= -36k^7m^6\lambda^5 \text{cn}(k\xi, m) (\text{sn}(k\xi, m))^5 \text{dn}(k\xi, m) \\
    &\quad + (24k^7m^4\lambda^5 + 24k^7m^6\lambda^5) \text{dn}(k\xi, m)\text{cn}(k\xi, m) (\text{sn}(k\xi, m))^3 \\
    &\quad + (8/5k^7m^6\lambda^5 - 8/5k^7m^6\lambda^5 - 32k^7m^4\lambda^5) \text{dn}(k\xi, m)\text{cn}(k\xi, m)\text{sn}(k\xi, m) \big] t^5.
\]
\( u_6 = L_t^{-1}(A_1 + u_{4,xx} + u_{4,xxxx} + u_{4,yy}) \)
\[
= \left[ -42 k^8 m^8 \lambda^6 (\text{sn}(k\xi, m))^8 + (56 k^2 m^6 \lambda^8 + 56 k^8 m^8 \lambda^6) (\text{sn}(k\xi, m))^6 \\
+ \left( -\frac{84}{5} k^8 m^4 \lambda^6 - \frac{84}{5} k^8 m^8 \lambda^6 - \frac{336}{5} k^8 m^8 \lambda^6 \right) (\text{sn}(k\xi, m))^4 \\
+ \left( \frac{8}{15} k^8 m^2 \lambda^6 + \frac{8}{15} k^8 m^6 \lambda^6 + 16 k^8 m^6 \lambda^6 + 16 k^8 m^4 \lambda^6 \right) (\text{sn}(k\xi, m))^2 \\
- \frac{4}{15} k^8 m^2 \lambda^6 - \frac{26}{15} k^8 m^4 \lambda^6 - \frac{4}{15} k^8 m^6 \lambda^6 \right] t^6,
\]
\( u_7 = L_t^{-1}(A_5 + u_{5,xx} + u_{5,xxxx} + u_{5,yy}) \)
\[
= \left[ -48 k^9 m^8 (\text{sn}(k\xi, m))^7 \text{dn}(k\xi, m)\text{cn}(k\xi, m)\lambda^7 \\
+ (48 k^9 m^6 \lambda^7 + 48 k^9 m^8 \lambda^7) \text{dn}(k\xi, m)\text{cn}(k\xi, m) (\text{sn}(k\xi, m))^5 + \left( -\frac{48}{5} k^9 m^8 \lambda^7 \\
- \frac{48}{5} k^9 m^4 \lambda^7 - \frac{192}{5} k^9 m^6 \lambda^7 \right) \text{dn}(k\xi, m)\text{cn}(k\xi, m) (\text{sn}(k\xi, m))^3 + \left( \frac{16}{105} k^9 m^2 \lambda^7 \\
+ \frac{32}{105} k^9 m^4 \lambda^7 + \frac{32}{27} k^9 m^6 \lambda^7 + \frac{16}{105} k^9 m^8 \lambda^7 \right) \text{dn}(k\xi, m)\text{cn}(k\xi, m)\text{sn}(k\xi, m) \right] t^7,
\]

Therefore we have the approximate solution of (3.1)
\[
u(x, y, t) = u_0(x, y, t) + u_1(x, y, t) + u_2(x, y, t) + u_3(x, y, t) + u_4(x, y, t) \\
+ u_5(x, y, t) + u_6(x, y, t) + u_7(x, y, t) + \cdots, \tag{3.5}\]
where \( u_i(x, y, t), i = 0, 1, 2, \ldots, \) are given by (3.4).

Using the Taylor series, we have the exact doubly-periodic solutions of (3.1)
\[
u(x, y, t) = -6k^2 m^2 \text{sn}^2[k(x + ly + \lambda t), m] - \frac{1}{2} l^2 + 2k^2(1 + m^2) + \frac{1}{2} \lambda^2 - \frac{1}{2}, \tag{3.6}\]

When \( m \to 1, \) the solution (3.5) becomes the known bell-shaped solitary wave solutions
\[
u(x, y, t) = -6k^2 \tanh^2[k(x + ly + \lambda t)] - \frac{1}{2} l^2 + 4k^2 + \frac{1}{2} \lambda^2 - \frac{1}{2}, \tag{3.6'}\]

In particular, in the special case \( k = \alpha R, \ l = \frac{\beta}{\alpha}, \ \lambda = \sqrt{\alpha^2 + \beta + 4\alpha^2 R^2}/\alpha, \) the solution (3.6') becomes the known bell-shaped solitary wave solutions[9].

In order to proof numerically whether the new scheme obtained from the Adomian decomposition method for numerical doubly-periodic solutions of (2+1)-dimensional Boussinesq equation (3.1) leads to higher accuracy, we evaluate the approximate solution using the 8-term approximation,
\[
\phi_{\text{app}} = u_0(x, y, t) + u_1(x, y, t) + u_2(x, y, t) + u_3(x, y, t) + u_4(x, y, t) \\
+ u_5(x, y, t) + u_6(x, y, t) + u_7(x, y, t), \tag{3.7}\]

4. Comparison of the approximate solution (3.7) and exact solution (3.6)
In the following, we compare the approximate solution (3.7) with exact solution (3.6) using tables and graphs.

Table 1. The absolute error and relative error between exact solution and approximate solution when \(m = 0.1, k = 2, \lambda = 1, l = 5\).

| \(x_i\) | \(y_i\) | \(t_i\) | \(|u(x, y, t) - \phi_{\text{appr}}|\) | \(\frac{|u(x, y, t) - \phi_{\text{appr}}|}{|u(x, y, t)|}\) |
|---|---|---|---|---|
| 5 | 15 | 0.1 | 0.1e-8 | 0.2214846691e-9 |
| 5 | 15 | 0.3 | 0.3029e-5 | 0.6852760476e-6 |
| 5 | 15 | 0.5 | 0.270331e-3 | 0.607162252e-4 |
| 3 | 10 | 0.1 | 0.2e-8 | 0.4383106235e-9 |
| 3 | 10 | 0.3 | 0.7791e-5 | 0.1742300619e-5 |
| 3 | 10 | 0.5 | 0.422777e-3 | 0.9561548005e-4 |
| 1 | 5 | 0.1 | 0.0 | 0.0 |
| 1 | 5 | 0.3 | 0.9561548005e-4 | 0.9561548005e-4 |
| 1 | 5 | 0.5 | 0.517534e-3 | 0.1154263345e-3 |

Fig.1, exact solution

Fig.2 approximate solution

The graphs of the approximate solution for \(\phi_{\text{appr}}\) in (3.7) and exact solution for \(u(x, y, t)\) in (3.6) with \(m = 0.1, k = 2, l = 1, \lambda = 5, t = 0.5\).

Fig.3, The comparison of approximate solution (3.7) and exact solution (3.6) with \(m = 0.1, k = 2, l = 1, \lambda = 5, y = 5, t = 0.5\).
The contour plot and density plot of the approximate solution for $\phi_{\text{appr}}$ in (3.7) with $m = 0.1, k = 2, l = 1, \lambda = 5, t = 0.5$.

Table 2. The absolute error and relative error between exact solution and approximate solution when $m = 0.9, k = 2, l = 1, t = 5$

| spatial variable | temporal variable | absolute error $|u(x, y, t) - \phi_{\text{appr}}|$ | relative error $\frac{|u(x, y, t) - \phi_{\text{appr}}|}{|u(x, y, t)|}$ |
|------------------|-------------------|---------------------------------|----------------------------------|
| $x_i$ | $y_i$ | $t_i$ | $u(x, y, t) - \phi_{\text{appr}}$ | $\frac{|u(x, y, t) - \phi_{\text{appr}}|}{|u(x, y, t)|}$ |
| 5 | 15 | 0.01 | 0.41e-8 | 0.6624643312e-8 |
| 5 | 15 | 0.03 | 0.15e-7 | 0.1321247205e-7 |
| 5 | 15 | 0.05 | 0.8e-8 | 0.475714566e-8 |
| 3 | 10 | 0.01 | 0.1e-7 | 0.8886569313e-9 |
| 3 | 10 | 0.03 | 0.1e-7 | 0.8526622064e-9 |
| 3 | 10 | 0.05 | 0.1e-7 | 0.820999018e-9 |
| 1 | 5 | 0.01 | 0.1e-7 | 0.5972896058e-9 |
| 1 | 5 | 0.03 | 0.1e-7 | 0.5926747600e-9 |
| 1 | 5 | 0.05 | 0.1e-7 | 0.588615602e-9 |

The graphs of the approximate solution for $\phi_{\text{appr}}$ in (3.7) and exact solution for $u(x, y, t)$ in (3.6) with $m = 0.9, k = 2, l = 1, \lambda = 5, t = 0.1$. 
Remark 2. For other modulus \( m \), we also make the comparison of approximate solution (3.7) and exact solution (3.6). We omit them here.

4. Conclusions and Discussion

In brief, w presented an improved scheme to investigate the approximate solutions of (2+1)-dimensional Boussinesq equation with initial conditions based on the decomposition method. The improve scheme makes all components \( u_i \) of \( u = \sum_{i=0}^{\infty} \) be of the same degree in \( t \). We choose the (2+1)-dimensional Boussinesq equation with initial conditions of doubly-periodic functions to illustrate the scheme. As a result, the approximate and exact doubly-periodic solutions are obtained. For differential modulus \( m = 0.1, 0.9 \), we compare the approximate solution with the exact solution by their graphs. Moreover we analyze the absolute error and relative error, and give the contour and density plots of the approximate doubly-periodic solutions.

In addition, the improved scheme is also extended to higher-dimensional nonlinear evolution equations, for example, the (3+1)-dimensional KP equation

\[
\frac{u_{xt}}{2} + 6u_x^2 + 6uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0, \tag{4.1}
\]

Eq(4.1) can be rewritten in the operator form

\[
\frac{u_{xt}}{2} + 6u_x^2 + 6uu_{xx} - L_yu - L_zu = 0, \tag{4.2}
\]
where \( L_y = \partial^2 / \partial y^2, L_z = \partial^2 / \partial z^2 \). They are all easily invertible. Without loss of generality, we choose \( L_y^{-1} \) to operate it on both side of (4.2) to obtain

\[
u = f(x, z, t) + g(x, z, t)y + L_y^{-1}(u_{xt} + 3(u^2)_{xx} - u_{xxx} - u_{zz}), \tag{4.3}\]

where \( L_y^{-1} = \int_0^y \int_0^y (\cdot) dsdt \), and \( f(x, 0, z, t), g(x, 0, z, t) \) can be regarded as the given initial conditions.

Similarly, we employ the new recursive relationship

\[
u_0(x, y, z, t) = f(x, z, t), \quad 
u_1(x, y, z, t) = g(x, z, t)y, \quad \nu_{i+2}(x, y, z, t) = L_y^{-1}(3A_i - u_{i,zz} - u_{i,xxx} - u_{i,yy}), \quad i \geq 0. \tag{4.4}\]

where \( A_i \) is the Adomian polynomials that represent the nonlinear \((u^2)_{xx}\) and can be derived by (2.9).

**Remark 3.** The scheme is different from one in [11].

According to the scheme we consider the following initial valuable problem

\[
u_{xt} + 6u_x^2 + 6uu_{xx} - u_{xxx} - u_{yy} - u_{zz} = 0, \]

\[
u(x, 0, z, t) = -2k^2m^2\sin^2[k(x + cz + \lambda t), m] + \frac{1}{6}(\lambda - l^2 - c^2 + 4k^2 + 4k^2m^2),
\]

\[
u_y(x, 0, z, t) = -4k^2m^2\sin[k(x + cz + \lambda t), m]\csc[k(x + cz + \lambda t), m]\csc[k(x + cz + \lambda t), m], \tag{4.5}\]

The corresponding recursive relationship is

\[
u_0(x, y, z, t) = -2k^2m^2\sin^2[k(x + cz + \lambda t), m] + \frac{1}{6}(\lambda - l^2 - c^2 + 4k^2 + 4k^2m^2),
\]

\[
u_1(x, y, z, t) = -4k^2m^2\sin[k(x + cz + \lambda t), m]\csc[k(x + cz + \lambda t), m]\csc[k(x + cz + \lambda t), m]y,
\]

\[
u_{i+2}(x, y, z, t) = L_y^{-1}(3A_i - u_{i,zz} - u_{i,xxx} - u_{i,yy}), \quad i \geq 0. \tag{4.6}\]

from which we can determine these components \( u_i \). Therefore we have \( u = \sum_{i=0}^\infty u_i \). These results will be further considered.

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**References**