The Investigation of New Solutions to Two Coupled Nonlinear Wave Equations Via a Weierstrass Semi-Rational Expansion Method

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Abstract.
In this paper a new Weierstrass semi-rational expansion method is developed via the Weierstrass elliptic function \( \wp(\xi; g_2, g_3) \). With the aid of Maple, we choose the coupled water wave equation and the generalized Hirota-Satsuma coupled KdV equation to illustrate the method. As a consequence, it is shown that the method is powerful to obtain many types of new doubly periodic solutions in terms of the Weierstrass elliptic function. Moreover the corresponding new Jacobi elliptic function solutions are also given.

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1. Introduction

Recently, starting from the Konopelchenko-Dubrovsky (KD) equation

\[
U_t - U_{xxx} - 6\beta U U_x + \frac{3}{2}a^2 U^2 U_x - 3U_x - 1U_y + 3a U_x \partial_x^{-1} U_y = 0,
\]

(1.1)

by using the reduction method, Maccari presented the new integrable Davey-Stewartson-type equation

\[
i\Psi_x + L_1 \Psi + \Psi \Phi + \Psi \chi = 0,
\]

\[
L_2 \chi = L_3 |\Psi|^2,
\]

\[
\Phi_{\xi} = \chi_{\eta} + \mu (|\Psi|^2)_{\eta}, \quad \mu = \mp 1.
\]

(1.2)

where the linear differential operators are given by

\[
L_1 = \frac{b^2 - a^2}{4} \frac{\partial^2}{\partial \xi^2} - a \frac{\partial^2}{\partial \xi \partial \eta} - \frac{\partial^2}{\partial \eta^2},
\]

(1.3)

\[
L_2 = \frac{b^2 + a^2}{4} \frac{\partial^2}{\partial \xi^2} + a \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2},
\]

(1.4)

\[
L_3 = \pm \frac{1}{4} (b^2 + a^2 + \frac{8b^2(a-1)}{(a-2)^2-b^2}) \frac{\partial^2}{\partial \xi^2} \pm (a + \frac{2b^2}{(a-2)^2-b^2}) \frac{\partial^2}{\partial \xi \partial \eta} \pm \frac{\partial^2}{\partial \eta^2},
\]

(1.5)
\(a, b\) are real parameters, and \(\Psi = \Psi(\xi, \eta, \tau)\) is complex while \(\Phi = \Phi(\xi, \eta, \tau), \chi = \chi(\xi, \eta, \tau)\) are real. The Lax pair of (1.2) has been derived using the Lax pair of (1.1) and the reduction technique. More recently, we used the sinh-Gordon equation expansion method \([1]\) and the extended Jacobi elliptic function expansion method \([1]\) to obtain many types of doubly periodic solutions and solitary wave solutions.

To our knowledge, the doubly periodic solutions of (1.2) were not given in terms of the Weierstrass elliptic function \(\wp(\xi; g_2, g_3)\) satisfying \([1, 2]\]

\[\wp^2(\xi) = 4\wp^3(\xi) - g_2\wp(\xi) - g_3, \quad (1.6a)\]

or, another form

\[\wp''(\xi) = 6\wp^2(\xi) - \frac{1}{2}g_2, \quad (1.6b)\]

In this paper we will develop a new transformation to investigate new doubly periodic solutions of (1.2) in terms of the Weierstrass elliptic function \(\wp(\xi; g_2, g_3)\).

The rest of this paper is arranged as follows: In section 2, we introduce our method called the Weierstrass semi-rational expansion method. In sections 3 we apply the method to the new integrable Davey-Stewartson-type equation (1.2). As a result, some new doubly periodic solutions are obtained in terms of the Weierstrass elliptic function. Moreover, some special solitary wave solutions are also given as simple limits of doubly periodic solutions. Finally some conclusions and discussions are given.

2. The introduction of the Weierstrass semi-rational expansion method

In what follows we introduce the Weierstrass semi-rational expansion method and its algorithm:

**Step I:** Consider a given nonlinear evolution equation with a physical field \(u\) and two independent variables \(x, t\)

\[F(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \quad (2.1)\]

we make the travelling wave transformation \(u(x, t) = u(\xi), \xi = k(x - \lambda t),\) which reduces (2.1) to a nonlinear ordinary differential equation

\[G(u, u', u'', u''', \ldots) = 0. \quad (2.2)\]

where the prime denotes derivative with respect to \(\xi\).

**Step II:** We assume (2.2) is of the power series solution

\[u(\xi) = u(\wp(\xi; g_2, g_3)) = A_0 + \sum_{i=1}^{n} \left[ \frac{A_i\wp(\xi; g_2, g_3) + B_i\wp'(\xi; g_2, g_3) + C_i\wp''(\xi; g_2, g_3)}{R + P\wp(\xi; g_2, g_3) + Q\wp'(\xi; g_2, g_3)} \right]^i, \quad (2.3)\]

where the prime denotes the derivative with respect to \(\xi\), \(n, A_i, B_i, A_0, R, P, Q\) are parameters to be determined later, and \(\wp(\xi; g_2, g_3)\) is the Weierstrass elliptic function satisfying

\[\wp^2(\xi) = 4\wp^3(\xi) - g_2\wp(\xi) - g_3, \quad (2.4)\]
or, another form

\[ \varphi''(\xi) = 6\varphi^2(\xi) - \frac{1}{2}g_2, \]  

(2.5)

with \( g_2, g_3 \) being real parameters and called invariants[1,2].

According to the equation (2.3)-(2.5), to determined the parameter \( n \), we define a polynomial degree function as \( D(u(\varphi)) = n \), thus we have

\[ D \left[ u^p(\varphi) \left( \frac{d^s u(\varphi)}{d\xi^s} \right)^q \right] = np + q(n + s). \]  

(2.6)

Therefore we can determine \( n \) in (2.3) by balancing the highest degree linear term and nonlinear terms in (2.1) or (2.2).

**Step III:** With the aid of Maple, the substitution of (2.3) into (2.2) along with (2.4) and (2.5) leads to a polynomial of \( \varphi^i \varphi^j (i = 0, 1; j = 0, 1, 2, 3...). Setting their coefficients to zero yields a set of nonlinear algebraic equations with respect to the unknowns \( k, \lambda, A_i, B_i, P, R, Q, g_2, g_3 \).

**Step VI:** With the aid of Maple, we solve the set of algebraic equations obtained in Step 3 to determine these unknowns. Finally we derive the doubly periodic solutions of the given nonlinear wave equations from (2.3) in terms of Weierstrass elliptic function.

3. The new doubly periodic solutions

We introduce the transformations

\[ \Psi(\xi, \eta, \tau) = \Psi(X) \exp(iY), \quad \Phi(\xi, \eta, \tau) = \Phi(X), \quad \chi(\xi, \eta, \tau) = \chi(X), \]  

(3.1)

where \( k, l, \lambda, \alpha, \beta, \gamma \) are constants to be determined later.

The substitution of (3.1) into (1.1) yields

\[ k^2M_1 \frac{d^2\Psi(X)}{dX^2} + M_0\Psi(X) + \Phi(X)\Psi(X) + \Psi(X)\chi(X) = 0, \]

\[ M_2 \frac{d^2\chi(X)}{dX^2} = M_3 \frac{d^2\Psi^2(X)}{dX^2}, \]  

(3.2)

\[ \frac{d\Phi(X)}{dX} = i\frac{d\chi(X)}{dX} + \mu l \frac{d^2\Psi^2(X)}{dX^2}. \]

with the condition

\[ \lambda = -\frac{\alpha}{4}(b^2 - a^2) + a(\beta + \alpha l) + 2l/\beta, \]  

(3.3)

where

\[ M_0 = -\gamma - \frac{1}{4}\alpha^2(b^2 - a^2) + a\alpha\beta + \beta^2, \]

\[ M_1 = \frac{b^2 - a^2}{4} - al - l^2, \quad M_2 = \frac{b^2 + a^2}{4} + al + l^2, \]

\[ M_3 = \pm \frac{1}{4} \left[ b^2 + a^2 + \frac{8b^2(a - 1)}{(a - 2)^2 - b^2} \right] \pm \left[ a + \frac{2b^2}{(a - 2)^2 - b^2} \right] l \pm l^2. \]  

(3.4)
Integrating the second and third equations yields

\[ \chi(X) = \frac{M_3}{M_2} \Psi^2(X) + C_1, \quad (3.5) \]

\[ \Phi(X) = l_1 \chi(X) + \mu \Psi^2(X) + C_2 = \left[ \frac{M_3}{M_2} + \mu \right] \Psi^2(X) + lC_1 + C_2, \quad (3.6) \]

which make the first equation reduce to

\[ k^2 M_1 \frac{d^2 \Psi(X)}{dX^2} + (M_0 + C_1 + lC_1 + C_2) \Psi(X) + [(l + 1) \frac{M_3}{M_2} + \mu \lambda] \Psi^3(X) = 0, \quad (3.7) \]

Note that if we obtain one solution \( \Psi(X) \) from (3.7), then we can derive the corresponding solutions \( \chi(X) \) and \( \Phi(X) \) using (3.5) and (3.6).

According to Step 2, we assume (3.7) has the solution

\[ \Psi(X) = A_0 + \frac{A_1 \varphi(\xi; g_2, g_3) + B_1 \varphi'(\xi; g_2, g_3)}{R + P \varphi(\xi; g_2, g_3) + Q \varphi'(\xi; g_2, g_3)} \]

where \( \varphi(\xi; g_2, g_3) \) satisfies (2.4) and (2.5)

With the aid of Maple, by inserting (3.6) into (3.5) along with (3.7a,b) and equating the coefficients of these terms \( \varphi^i \varphi'(i = 0, 1; j = 0, 1, 2, \ldots) \), we get the set of algebraic equations with respect to unknowns \( k, \lambda, A_0, A_1, B_1, R, P, Q, C_1, C_2, C_3 \), which are so complicated. Thus we omit them here. Solving the set of algebraic equations yields:

Family 1.

\[ v_1(\hat{\xi}) = \sqrt[3]{\frac{C_2}{a}} + \sqrt{\frac{10bk^2C_2^3 - \frac{2}{3} \sqrt[3]{\frac{C_2^3}{a^2}} \varphi(\hat{\xi}; g_2, g_3)}{aC_2^2 - \frac{2}{3} abk^2 \sqrt[3]{\frac{C_2^3}{a^2}} \varphi(\hat{\xi}; g_2, g_3)}}^2, \quad (3.8) \]

where

\[ g_2 = \frac{18a^2}{25bk^4} \sqrt[3]{\frac{C_2^3}{a^2}}, \quad g_3 = \frac{27a^2C_2^2}{250b^3k^5} \]

Family 2.

\[ v_2(\hat{\xi}) = A_0 + \sqrt{\frac{-2}{a} P k \varphi'(\hat{\xi}; g_2, g_3)} \left( \frac{R + P \varphi(\hat{\xi}; g_2, g_3)}{R + P \varphi(\hat{\xi}; g_2, g_3)} \right)^2, \quad (3.9) \]

where

\[ g_2 = \frac{1}{16} \frac{-24 a A_0^2 P k^2 b R - a^2 A_0^3 P^2 + a C_2^2 P^2 + 48 k^4 b^2 A_0 R^2}{P^2 k^4 b^2 A_0}, \]

\[ g_3 = -\frac{1}{16} \frac{R (a^2 A_0^3 P^2 + 24 a A_0^2 P k^2 b R + 16 k^4 b^2 A_0 R^2 - a C_2^2 P^2)}{k^4 b^2 P^2 A_0} \]

Family 3.

\[ v_3 = A_0 + \frac{A_1 \varphi(\hat{\xi}; g_2, g_3)}{\frac{3a A_0 P A_1 + 3 a A_0^2 P^2 A_1 + a A_0^3 P^3 + a A_1^3}{40 k^2 (A_1 + P A_0)(2 A_1 + P A_0)}} + P \varphi(\hat{\xi}; g_2, g_3), \quad (3.10) \]
where

\[ g_2 = \frac{1/4}{P^2 v^2 k^4 (2 A_1 + 3 P A_0)^2 (A_1 + P A_0)^3} (3 a A_0 P A_1^2 + 3 a A_0^2 P^2 A_1 + a A_0^3 P^3 + \ldots) \]

\[ + a A_1^3 - 2 C_2 P^3 (-3 C_2^2 P^3 + a A_1^3 + 6 a A_0^2 P^2 A_1 + 3 a A_0^3 P^3 + 4 a A_0 P A_1^2) \]

\[ g_3 = \frac{1/16}{b^3 k^6 (2 A_1 + 3 P A_0)^3 (A_1 + P A_0)^3 P^2} (3 a A_0 P A_1^2 + 3 a A_0^2 P^2 A_1 + \ldots) \]

\[ + a A_0^3 P^3 + a A_1^3 - 2 C_2 P^3 (2 a A_0^3 P^2 + 3 a A_0^2 A_1 P + a A_0 A_1^2 - 2 C_2^2 P^2) \]

**Family 4.**

\[ v_4 = A_0 + \frac{1/k F f'(\hat{\xi}; g_2, g_3) + B_1 f'(\hat{\xi}; g_2, g_3)}{\frac{a}{a^{1/2}} F + Q f'(\hat{\xi}; g_2, g_3)}, \]

(3.11)

where

\[ F = \sqrt{-\frac{1}{b c} (9 Q^2 a A_0 B_1 + 6 Q a A_0 B_1^2 + 4 Q^3 a A_0^3 + a B_1^3 - 4 Q^3 C_2^3), \]

\[ g_2 = \frac{3}{64} \frac{a^2 B_1^2}{k^{1/2} b^{1/2} Q^2}, \]

\[ g_3 = \frac{1}{512} \frac{9 Q^3 a A_0^3 + 72 Q^2 a A_0 B_1 + 48 Q a A_0 B_1^2 + 9 a B_1^3 - 32 Q^3 C_2^3}{k^{1/2} b^{1/2} P^2}. \]

**Family 5.**

\[ v_5(\hat{\xi}) = -\frac{4 k^2 b}{a} f(\hat{\xi}; g_2, g_3) + A_0, \]

(3.12)

where

\[ g_3 = \frac{a (-C_2^2 a + a^2 A_0^3 - 4 A_0 b^2 k^4 g_2)}{16 b^3 k^6} \]

**Remark 1.** In particular, the Weierstrass elliptic function in (3.7) can be written as

\[ f(\hat{\xi}; g_2, g_3) = e_2 - (e_2 - e_3) c n^2(\sqrt{e_1 - e_3}; m), \]

(3.13)

in terms of the Jacobi elliptic cosine function, where \( m^2 = (e_2 - e_3)/(e_1 - e_3) \) is the modulus of the Jacobi elliptic function, \( e_i (i = 1, 2, 3; e_1 \geq e_2 \geq e_3) \) are three roots of the cubic equation \( 4y^3 - g_2 y - g_3 = 0 \).

Therefore we have from (3.8)-(3.10)

\[ v'(\hat{\xi}) = \sqrt{\frac{C_2}{a}} + \left[ \frac{10 b k^2 C_2 \sqrt{-\frac{6}{a} \sqrt{\frac{C_2}{a^2}}} [e_2 - (e_2 - e_3) c n^2(\sqrt{e_1 - e_3}; m)]}{a C_2^2 - \frac{5}{4} a b k^2 \sqrt{\frac{C_2}{a}} [e_2 - (e_2 - e_3) c n^2(\sqrt{e_1 - e_3}; m)]} \right]^2, \]

(3.14)

\[ v'(\hat{\xi}) = A_0 + \left[ \frac{\sqrt{-\frac{2}{a} P k M \sqrt{N} \text{sn}(\sqrt{N} \hat{\xi}; m) \text{cn}(\sqrt{N} \hat{\xi}; m) \text{dn}(\sqrt{N} \hat{\xi}; m)}}{R + P[e_2 - (e_2 - e_3) c n^2(\sqrt{e_1 - e_3}; m)]} \right]^2, \]

(3.15)
where $M = e_2 - e_3, N = e_1 - e_3$.

$$v_4' = A_0 + \frac{A_1[e_2 - (e_2 - e_3)\text{cn}^2(\sqrt{e_2 - e_3}\xi; m)]}{4aA_0 P A_1 + 3aA_0^2 P^2 A_1 + aA_0 A_1^3 - C_2^2 P^3} + \frac{P[e_2 - (e_2 - e_3)\text{cn}^2(\sqrt{e_2 - e_3}\xi; m)]}{4aA_0 P A_1 + 3aA_0^2 P^2 A_1 + aA_0 A_1^3 - C_2^2 P^3},$$

$$v_4' = A_0 + \frac{1/kF[e_2 - M \text{cn}^2(\sqrt{N}\xi; m)] + B_1 M \sqrt{N} \text{sn}(\sqrt{N}\xi; m)\text{cn}(\sqrt{N}\xi; m)\text{dn}(\sqrt{N}\xi; m)}{a\text{sech}^2[\hat{N}]} F + QM \sqrt{N} \text{sn}(\sqrt{N}\xi; m)\text{cn}(\sqrt{N}\xi; m)\text{dn}(\sqrt{N}\xi; m),}$$

$$v_5' = A_0 - \frac{4k^2 b}{a} [e_2 - (e_2 - e_3)\text{cn}^2(\sqrt{e_1 - e_3}\xi; m)] + A_0.$$

The solution (3.18) is similar to the only result in [12]. But the solutions (3.14)-(3.17) were not found.

When $m \to 1$, i.e., $e_1 \to e_2$, we have solitary wave solutions from (3.11)-(3.13)

$$v_1' = \sqrt[3]{C_2} \frac{\sqrt{\frac{10bk^2 C_2}{a} \sqrt{\frac{-3}{a} \frac{\sqrt{\frac{8C_2}{a}}}{a}}} [e_2 - (e_2 - e_3)\text{sech}^2(\sqrt{e_2 - e_3}\xi)]^2}{aC_2^2 - \frac{5}{3} abk^2 \sqrt{\frac{C_2}{a}}} [e_2 - (e_2 - e_3)\text{sech}^2(\sqrt{e_2 - e_3}\xi)],}$$

$$v_2' = A_0 + \left[ \frac{-\frac{b}{a} P k(e_2 - e_3)\sqrt{e_2 - e_3} \text{tanh}(\sqrt{e_2 - e_3}\xi)\text{sech}^2(\sqrt{e_2 - e_3}\xi)}{R + P[e_2 - (e_2 - e_3)\text{sech}^2(\sqrt{e_2 - e_3}\xi)]} \right]^2,$$

$$v_3' = A_0 + \frac{A_1[e_2 - (e_2 - e_3)\text{sech}^2(\sqrt{e_2 - e_3}\xi)]}{4aA_0 P A_1 + 3aA_0^2 P^2 A_1 + aA_0 A_1^3 - C_2^2 P^3} + \frac{P[e_2 - (e_2 - e_3)\text{sech}^2(\sqrt{e_2 - e_3}\xi)]}{4aA_0 P A_1 + 3aA_0^2 P^2 A_1 + aA_0 A_1^3 - C_2^2 P^3},$$

$$v_4' = A_0 + \frac{1/kF[e_2 - M \text{cn}^2(\sqrt{N}\xi; m)] + B_1 M \sqrt{N} \text{tanh}(\sqrt{N}\xi)\text{sech}^2(\sqrt{N}\xi)}{a\text{sech}^2[\hat{N}]} F + QM \sqrt{N} \text{tanh}(\sqrt{N}\xi)\text{sech}^2(\sqrt{N}\xi),}$$

$$v_5' = A_0 - \frac{4k^2 b}{a} [e_2 - (e_2 - e_3)\text{sech}^2(\sqrt{e_2 - e_3}\xi)].$$

4. The generalized Hirota-Satsuma coupled KdV system

Consider the travelling wave solution of (1.3a,b) in the form

$$u(x, t) = u(\xi), \quad v(x, t) = v(\xi), \quad w(x, t) = w(\xi), \quad \xi = k(x - \lambda t).$$

where $\lambda$ denotes the wave speed. Therefore (1.3a,b) reduce to

$$-\lambda u' = \frac{1}{4} k^2 u''' + 3uv' + 3(-v^2 + w)'$$

$$-\lambda v' = -\frac{1}{2} k^2 v''' - 3uv',$$
\[-\lambda w' = -\frac{1}{2} k^2 w''' - 3uw'. \tag{4.2c}\]

According to Step 2, we know that there exist two cases:

**Case I.** $Du(\wp) = 2$, $Dv(\wp) = 1$, $Dw(\wp) = 1$

**Case II.** $Du(\wp) = 2$, $Dv(\wp) = 2$, $Dw(\wp) = 2$.

In the following we discuss (4.2) in two cases:

**4.1 Case I.** $Du(\wp) = 2$, $Dv(\wp) = 1$, $Dw(\wp) = 1$

In this case, let

\[ u = \alpha v^2 + \beta v + \gamma, \quad w = Av + B, \tag{4.3} \]

where $\alpha, \beta, \gamma, A$ and $B$ are constants to be determined later. By substituting (4.3) into (4.2b) and (4.2c) and integrating once, we know that (4.2b) and (4.2c) give rise to the same differential equation

\[ k^2 v''' + 6(\alpha v^2 + \beta v + \gamma)v' - 2\lambda v' = 0. \tag{4.4} \]

Integrating (4.4) with respect to $\xi$ yields the two nonlinear differential equations

\[ k^2 v'' = -2\alpha v^3 - 3\beta v^2 + 2(\lambda - 3\gamma)v + c_1, \tag{4.5} \]

\[ k^2 (v')^2 = -\alpha v^4 - 2\beta v^3 + 2(\lambda - 3\gamma)v^2 + 2c_1v + c_2. \]

where $c_1, c_2$ are integration constants.

From (4.3)-(4.5) we get

\[ k^2 u'' = 2\alpha k^2 v'^2 + k^2(2\alpha v + \beta)v'' \]

\[ = 2\alpha[-\alpha v^4 - 2\beta v^3 + 2(\lambda - 3\gamma)v^2 + 2c_1v + c_2] \]

\[ + (2\alpha v + \beta)[-2\alpha v^3 - 3\beta v^2 + 2(\lambda - 3\gamma)v + c_1] \tag{4.6} \]

Integrating (4.2a) once we have

\[ \frac{1}{4} k^2 u'' + \frac{3}{2} u^2 + \lambda u + 3(-v^2 + w) + c_3 = 0, \tag{4.7} \]

where $c_3$ is an integration constant. Inserting (4.5) and (4.6) into (4.7) yields

\[
\begin{cases}
3\alpha \lambda - 3\alpha \gamma + \frac{3}{4} \beta^2 - 3 = 0, \\
\frac{1}{2}(\alpha c_1 + \beta \lambda + \gamma \beta) + A = 0, \\
\frac{1}{4}(2\alpha c_2 + \beta c_1) + \frac{3}{2} \gamma^2 + \lambda \gamma + 3B + c_3 = 0.
\end{cases} \tag{4.8} \]

We find from (4.8) that

\[ \alpha = \frac{\beta^2 - 4}{4(\gamma - \lambda)}, \quad A = -\frac{-c_1 \beta^2 + 4c_1 + 4\beta \lambda^2 - 4\gamma^2 \beta}{8(\lambda - \gamma)}, \]

\[ B = -\frac{-c_2 \beta^2 + 4c_2 + 2\beta c_1 \lambda - 2\beta c_1 \gamma + 4\gamma^2 \lambda - 12\gamma^3 + 8\lambda^2 \gamma + 8c_3 \lambda - 8c_3 \gamma}{24(\lambda - \gamma)}. \tag{4.9} \]
Therefore we know that if we obtain a solution \( v \) from (4.4), then we can determine another two functions \( u \) and \( w \) by using (4.3) with parameters defined by (4.9) and \( v \) in (4.4).

In the following we consider (4.4) using our method. According to Step 2, we assume that (4.4) has the solution

\[
v(\xi) = A_0 + \frac{A_1 \varphi(\xi; g_2, g_3) + B_1 \varphi'(\xi; g_2, g_3)}{R + P \varphi(\xi; g_2, g_3) + Q \varphi'(\xi; g_2, g_3)},
\]

where \( \varphi(\xi; g_2, g_3) \) satisfies (2.4) and (2.5).

With the aid of Maple, by inserting (4.10) into (4.4) along with (2.4) and (2.5), and equating the coefficients of these terms \( \varphi^i \varphi^j (i = 0, 1; j = 0, 1, 2, \ldots) \), we get the set of algebraic equations with respect to unknowns \( k, \lambda, A_0, A_1, B_1, R, P, Q, \beta, \gamma \), which is so complicated. Thus we omit it here. Solving the set of algebraic equations yields:

**Family 1.**

\[
\begin{align*}
u_1 &= \alpha \left[ -\frac{P^3 k^2 g_2 + 12 k^2 R^2 P - 4 A_1^2 \alpha R - 2 \beta A_1 P R}{4 R \alpha P A_1} + \frac{A_1 \varphi(\xi; g_2, g_3)}{R + P \varphi(\xi; g_2, g_3)} \right]^2, \\
\lambda &= \frac{3}{16 R \alpha A_1^2 R^2} \left[ 16 \gamma R^2 \alpha P A_1^2 - 32 A_1^2 \alpha R^3 k^2 + P^5 k^4 g_2^2 - 24 P^3 k^4 g_2 R^2 \\
&\quad + 144 k^4 R^4 - 4 \beta^2 A_1^2 P R^2 \right] \\
\end{align*}
\]

Particularly we rewrite (4.11a,b,c) as

\[
\begin{align*}
u_1' &= \alpha \left[ -\frac{P^3 k^2 g_2 + 12 k^2 R^2 P - 4 A_1^2 \alpha R - 2 \beta A_1 P R}{4 R \alpha P A_1} + \frac{A_1 [e_2 - M cn^2(\sqrt{N} \xi; m)]}{R + P [e_2 - M cn^2(\sqrt{N} \xi; m)]} \right]^2, \\
\lambda &= \frac{3}{16 R \alpha A_1^2 R^2} \left[ 16 \gamma R^2 \alpha P A_1^2 - 32 A_1^2 \alpha R^3 k^2 + P^5 k^4 g_2^2 - 24 P^3 k^4 g_2 R^2 \\
&\quad + 144 k^4 R^4 - 4 \beta^2 A_1^2 P R^2 \right] \\
\end{align*}
\]
\[ w_1 = A \left[ \frac{-P^2k^2g_2 + 12k^2R^2 - 4A_1^2\alpha R - 2\beta A_1PR}{4Ra PA_1} A_1[e_2 - M\text{cn}^2(\sqrt{N}\xi; m)] + \right] + B, \]

where \( M = e_2 - e_3, N = e_1 - e_3. \)

**Family 2.**

\[ u_2 = \alpha \left[ -\frac{2\alpha A_1 + \beta P}{2\alpha P} + \frac{A_1\varphi(\xi; g_2, g_3) + B_1\varphi'(\xi; g_2, g_3)}{R + P\varphi(\xi; g_2, g_3)} \right]^2 + \beta \left[ -\frac{2\alpha A_1 + \beta P}{2\alpha P} + \frac{A_1\varphi(\xi; g_2, g_3) + B_1\varphi'(\xi; g_2, g_3)}{R + P\varphi(\xi; g_2, g_3)} \right] + \gamma, \quad (4.12a) \]

\[ v_2 = -\frac{2\alpha A_1 + \beta P}{2\alpha P} + \frac{A_1\varphi(\xi; g_2, g_3) + B_1\varphi'(\xi; g_2, g_3)}{R + P\varphi(\xi; g_2, g_3)}, \quad (4.12b) \]

\[ w_2 = A \left[ -\frac{2\alpha A_1 + \beta P}{2\alpha P} + \frac{A_1\varphi(\xi; g_2, g_3) + B_1\varphi'(\xi; g_2, g_3)}{R + P\varphi(\xi; g_2, g_3)} \right] + B, \quad (4.12c) \]

where \( \alpha, A, B \) are given by (4.9) and

\[ k = \frac{2B_1\sqrt{-\alpha}}{P}, \quad g_2 = \frac{PR^2A_1^2 + B_1^2P^3g_3 + 4B_1^2R^3}{P^2B_1^2R}, \]

\[ \lambda = -\frac{3P^3\beta^2 - 4\gamma P^3\alpha + 16Ra^2B_1^2}{4P^3\alpha}. \]

Particularly we rewrite (4.12a,b,c) as

\[ u'_2 = \alpha \left[ A_1[e_2 - M\text{cn}^2(\sqrt{N}\xi; m)] + B_1M\sqrt{N}\text{sn}(\sqrt{N}\xi; m)\text{cn}(\sqrt{N}\xi; m)\text{dn}(\sqrt{N}\xi; m) \right] \]
\[ \left. \frac{R + P[e_2 - M\text{cn}^2(\sqrt{N}\xi; m)]} \right] + B, \]

\[ v'_2 = \frac{A_1[e_2 - M\text{cn}^2(\sqrt{N}\xi; m)] + B_1M\sqrt{N}\text{sn}(\sqrt{N}\xi; m)\text{cn}(\sqrt{N}\xi; m)\text{dn}(\sqrt{N}\xi; m) \right] + \gamma, \]

\[ w'_2 = A \left[ A_1[e_2 - M\text{cn}^2(\sqrt{N}\xi; m)] + B_1M\sqrt{N}\text{sn}(\sqrt{N}\xi; m)\text{cn}(\sqrt{N}\xi; m)\text{dn}(\sqrt{N}\xi; m) \right] \]
\[ \left. \frac{R + P[e_2 - M\text{cn}^2(\sqrt{N}\xi; m)]} \right] + B, \]
Family 3.

\[ u_3 = \alpha \left[ A_0 + \frac{A_1 \psi(\xi; g_2, g_3) + B_1 \psi'(\xi; g_2, g_3)}{R + P \psi(\xi; g_2, g_3) + Q \psi'(\xi; g_2, g_3)} \right]^2 + \beta \left[ A_0 + \frac{A_1 \psi(\xi; g_2, g_3) + B_1 \psi'(\xi; g_2, g_3)}{R + P \psi(\xi; g_2, g_3) + Q \psi'(\xi; g_2, g_3)} \right] + \gamma, \]  

(4.13a)

\[ v_3 = A_0 + \frac{A_1 \psi(\xi; g_2, g_3) + B_1 \psi'(\xi; g_2, g_3)}{R + P \psi(\xi; g_2, g_3) + Q \psi'(\xi; g_2, g_3)}, \]  

(4.13b)

\[ w_3 = A \left[ A_0 + \frac{A_1 \psi(\xi; g_2, g_3) + B_1 \psi'(\xi; g_2, g_3)}{R + P \psi(\xi; g_2, g_3) + Q \psi'(\xi; g_2, g_3)} \right] + B, \]  

(4.13c)

where \( \alpha, A, B \) are given by (4.9) and

\[
A_0 = -\frac{1}{4 \alpha Q (B_1 P - QA_1)^2} \left[ 4 \alpha B_1^3 P^2 + 2 \beta B_1^2 Q P Q^2 - k^2 A_1 P^3 Q + k^2 B_1 P^4 + 12 k^2 B_1 Q^2 R P - 4 \beta B_1 Q^2 A_1 P - 8 \alpha B_1^2 A_1 Q P + 4 \alpha A_1^2 B_1^2 - 8 k^2 A_1 Q^3 R + 2 \beta A_1^2 Q^4 \right],
\]

\[
\lambda = \frac{3}{16 \alpha Q^2 (B_1 P - QA_1)^3} \left[ k^4 B_1^2 P^8 - 4 \beta^2 Q^6 A_1^4 - 24 \beta^2 B_1^2 Q^2 P^2 A_1^2 + 16 k^4 A_1^2 P^3 Q^4 R + k^4 A_1^2 P^6 Q^2 + 16 \gamma Q^6 \alpha A_1 + 4 \alpha B_1^4 P^6 k^2 - 4 \beta^2 B_1^2 Q^6 P^4 + 64 k^4 A_1^2 Q^6 R^2 - 64 \gamma P^3 Q^3 \alpha A_1 B_1^3 + 96 \gamma P^2 Q^4 \alpha A_1^2 B_1^2 - 64 \gamma PQ^5 \alpha A_1^3 B_1 + 16 \gamma P^4 Q^2 \alpha A_1^4 - 16 \alpha B_1^3 P^3 k^2 A_1 Q + 16 \alpha B_1^4 P^3 k^2 Q^2 R - 48 \alpha B_1^3 P^3 k^2 A_1 Q^3 R + 16 \beta^2 B_1^3 Q^3 P^3 A_1 + 24 k^2 A_1^2 P^4 Q^2 \alpha A_1^2 - 40 k^3 A_1 P^4 Q^3 B_1 R - 2 k^4 A_1 B_1 P^7 Q B_1 - 16 k^2 A_1^3 P^3 Q^3 \alpha A_1 + 24 k^2 B_1^2 P^5 Q^2 R + 144 k^4 B_1^2 Q^4 R^2 P^2 + 48 k^2 B_1^2 Q^4 R P \alpha A_1^2 - 192 k^4 B_1^2 Q^3 R^2 P A_1 + 16 \beta^2 B_1 Q^5 A_1^3 P - 16 \alpha A_1^3 B_1 Q^5 k^2 R + 4 \alpha A_1^4 Q^4 k^2 P^2 \right],
\]

\[
g_2 = \frac{1}{16 Q^2 k^2 (B_1 P - QA_1)^2} \left[ 2 P^5 B_1 k^2 A_1 Q + 16 P^3 A_1 Q B_1^3 \alpha - P^4 k^2 A_1^2 Q^2 + 56 RP^2 B_1 k^2 A_1 Q^3 - 24 P^2 A_1^2 Q^2 B_1^2 \alpha + 48 R^2 B_1^2 k^2 Q^4 - P^6 B_1^2 k^2 - 24 RP^3 B_1^2 k^2 Q^2 - 4 B_1^4 P^4 \alpha - 32 R A_1^2 Q^4 k^2 P - 4 \alpha A_1^4 Q^4 + 16 P A_1^3 Q^5 B_1 \alpha \right],
\]

\[
g_3 = -\frac{1}{16 k^2 Q^4 (B_1 P - QA_1)^3} \left[ k^2 B_1^3 P^6 - 2 k^2 A_1 P^5 Q B_1^2 + 4 \alpha B_1^5 P^4 + k^2 A_1^2 P^4 Q^2 B_1 - 16 \alpha B_1^4 P^3 A_1 Q + 24 k^2 B_1^3 Q^2 R P^3 + 24 \alpha B_1^3 P^2 Q^2 A_1^2 - 56 k^2 B_1^2 Q^3 R P^2 A_1 - 16 \alpha B_1^2 A_1^3 Q^3 P + 48 k^2 B_1^4 R P A_1^2 + 16 R^2 Q^4 k^2 B_1^3 + 4 \alpha A_1^4 B_1 Q^4 - 16 k^2 A_1^3 Q^5 R \right].
\]
Particularly we rewrite (4.13a,b,c) as

\[
\begin{align*}
    u_3' &= a \left[ A_0 + \frac{A_1[e_2 - M\text{cn}^2(\sqrt{N}T; m)] + B_1 M\sqrt{N}\text{sn}(\sqrt{N}T; m)\text{cn}(\sqrt{N}T; m)\text{dn}(\sqrt{N}T; m)}{R + P[e_2 - M\text{cn}^2(\sqrt{N}T; m)] + QM\sqrt{N}\text{sn}(\sqrt{N}T; m)\text{cn}(\sqrt{N}T; m)\text{dn}(\sqrt{N}T; m)} \right]^2 \\
    v_3' &= A_0 + \frac{A_1[e_2 - M\text{cn}^2(\sqrt{N}T; m)] + B_1 M\sqrt{N}\text{sn}(\sqrt{N}T; m)\text{cn}(\sqrt{N}T; m)\text{dn}(\sqrt{N}T; m)}{R + P[e_2 - M\text{cn}^2(\sqrt{N}T; m)] + QM\sqrt{N}\text{sn}(\sqrt{N}T; m)\text{cn}(\sqrt{N}T; m)\text{dn}(\sqrt{N}T; m)} + \gamma, \\
    w_3' &= A_0 \left[ A_1[e_2 - M\text{cn}^2(\sqrt{N}T; m)] + B_1 M\sqrt{N}\text{sn}(\sqrt{N}T; m)\text{cn}(\sqrt{N}T; m)\text{dn}(\sqrt{N}T; m) \right] + A_0 A + B,
\end{align*}
\]

\[4.2\text{ Case II. } Du(\varphi) = 2, \quad Dv(\varphi) = 2, \quad Dw(\varphi) = 2\]

In this case, let

\[u = av + b, \quad w = \mu v + \nu,\]  

(4.14)

where \(a, b, \mu, \nu\) are constants to be determined later. By substituting (4.14) into (4.2b) and (4.2c) and integrating once, we know that (4.2b) and (4.2c) give rise to the same differential equation

\[k^2 v'' + 6(au + b)v' - 2\lambda v = 0.\]  

(4.15)

Integrating (4.15) with respect to \(\xi\) yields the nonlinear differential equation

\[k^2 v'' = -3av^2 - 6bv + 2\lambda v + c_1,\]  

(4.16)

where \(c_1\) is an integration constant.

Integrating (4.2a) with respect to \(\xi\) once we have

\[\frac{1}{4} k^2 u'' + \frac{3}{2} u^2 + \lambda u + 3(-v^2 + w) + c_2 = 0,\]  

(4.17)

where \(c_2\) is an integration constant. Inserting (4.14) and (4.16) into (4.17) yields

\[
\begin{align*}
    \frac{3}{2} b^2 + \lambda b + \frac{1}{3} ac_1 + 3\nu + c_2 &= 0, \\
    \frac{3}{4} \lambda a + 3\mu + \frac{3}{2} ab &= 0, \\
    -3 + \frac{3}{4} a^2 &= 0. 
\end{align*}
\]

(4.18)

which leads to

\[a = \pm 2, \quad \mu = \mp(\lambda + b), \quad \nu = -\frac{1}{2} b^2 - \frac{1}{3} \lambda b + \frac{1}{6} c_1 - \frac{1}{3} c_2,\]  

(4.19)
Therefore we know that if we obtain a solution \( v \) from (4.15), then we can determine another two functions \( u \) and \( w \) by using (4.14) with parameters defined by (4.19) and \( v \) in (4.15).

In the following we consider (4.15) using our method. According to Step 2, we assume that (4.15) has the solution

\[
v(\xi) = A_0 + \frac{A_1 \varphi(\xi; g_2, g_3) + B_1 \varphi'(\xi; g_2, g_3)}{R + P \varphi(\xi; g_2, g_3) + Q \varphi'(\xi; g_2, g_3)} + \left[ \frac{A_2 \varphi(\xi; g_2, g_3) + B_2 \varphi'(\xi; g_2, g_3)}{R + P \varphi(\xi; g_2, g_3) + Q \varphi'(\xi; g_2, g_3)} \right]^2,
\]

where \( \varphi(\xi; g_2, g_3) \) satisfies (2.4) and (2.5).

With the aid of Maple, by inserting (4.20) into (4.15) along with (2.4) and (2.5), and equating the coefficients of these terms \( \varphi^j \varphi^j \) (\( i = 0, 1; j = 0, 1, 2, \ldots \)), we get the set of algebraic equations with respect to unknowns \( k, \lambda, A_0, A_1, B_1, A_2, B_2, R, P, Q, b \), which is so complicated. Thus we omit it here. Solving the set of algebraic equations yields:

**Family 4.**

\[
\begin{align*}
u_4 &= a \left[ A_0 + \frac{3k^2 P^3}{2aQ^2} \varphi(\xi; g_2, g_3) + \frac{k^2 P^2}{aQ^3} \varphi'(\xi; g_2, g_3) \right] + b, & (4.21a) \\
v_4 &= A_0 + \frac{3k^2 P^3}{2aQ^2} \varphi(\xi; g_2, g_3) + \frac{k^2 P^2}{aQ^3} \varphi'(\xi; g_2, g_3), & (4.21n) \\
w_4 &= \mu \left[ A_0 + \frac{3k^2 P^3}{2aQ^2} \varphi(\xi; g_2, g_3) + \frac{k^2 P^2}{aQ^3} \varphi'(\xi; g_2, g_3) \right] + \nu, & (4.21c)
\end{align*}
\]

where \( a, \mu, \nu \) are given by (4.19) and

\[
\lambda = -3/4 - 5k^2 P^3 + 8 Q^2 k^2 R - 4 b P Q^2 - 4 A_0 P a Q^2, \\
g_2 = 1/16 - P^6 + 192 Q^4 R^2 - 48 P^3 Q^2 R \frac{P^2 Q^4}{P^2 Q^4}, \quad g_3 = 1/8 R \left( P^6 + 64 Q^4 R^2 - 24 P^3 Q^2 R \right) \frac{P^2 Q^4}{P^3 Q^4}.
\]

**Family 5.**

\[
\begin{align*}
u_5 &= a \left[ A_0 + \frac{A_1 \varphi(\xi; g_2, g_3)}{1/8 - \frac{k^2 P^3 + 2a A_1 Q^2}{Q^2 k^2} + P \varphi(\xi; g_2, g_3) + Q \varphi'(\xi; g_2, g_3)} \right] + b, & (4.22a) \\
v_5 &= A_0 + \frac{A_1 \varphi(\xi; g_2, g_3)}{1/8 - \frac{k^2 P^3 + 2a A_1 Q^2}{Q^2 k^2} + P \varphi(\xi; g_2, g_3) + Q \varphi'(\xi; g_2, g_3)}, & (4.22b) \\
w_5 &= \mu \left[ A_0 + \frac{A_1 \varphi(\xi; g_2, g_3)}{1/8 - \frac{k^2 P^3 + 2a A_1 Q^2}{Q^2 k^2} + P \varphi(\xi; g_2, g_3) + Q \varphi'(\xi; g_2, g_3)} \right] + \nu, & (4.22c)
\end{align*}
\]

where

\[
\lambda = 3b + 3a A_1 + \frac{3k^2 P^2}{4Q^2}, \quad g_2 = -1/16 P \left( 8 a A_1 Q^2 - 3 k^2 P^3 \right) \frac{Q^4 k^2}{Q^4 k^2},
\]
\[ g_3 = -\frac{1}{64} \left( -k^2P^3 + 2aA_1Q^2 \right)^2 \]

Family 6.

\[ u_6 = a \left[ A_0 + \frac{4k^2R/a\varphi(\xi;g_2,g_3) + B_1\varphi'(\xi;g_2,g_3)}{R + Q\varphi'(\xi;g_2,g_3)} \right] + b, \quad (4.23a) \]

\[ v_6 = A_0 + \frac{4k^2R/a\varphi(\xi;g_2,g_3) + B_1\varphi'(\xi;g_2,g_3)}{R + Q\varphi'(\xi;g_2,g_3)}, \quad (4.23b) \]

\[ w_6 = \mu \left[ A_0 + \frac{4k^2R/a\varphi(\xi;g_2,g_3) + B_1\varphi'(\xi;g_2,g_3)}{R + Q\varphi'(\xi;g_2,g_3)} \right] + \nu, \quad (4.23c) \]

where

\[ \lambda = 3b + 3aA_0 + \frac{9aB_1}{4Q}, \quad g_2 = 3/16 \frac{B_1^2a^2}{Q^2k^4}, \quad g_3 = -\frac{1}{64} - \frac{B_1^3a^3 + 64Qk^6R^2}{k^6Q^3}. \]

Family 7.

\[ u_7 = a \left[ A_0 + \frac{1}{2} \frac{k^2(8R^3 - 9aP)}{aR^2} \varphi(\xi;g_2,g_3) \right] + b, \quad (4.24a) \]

\[ v_7 = A_0 + \frac{1}{2} \frac{k^2(8R^3 - 9aP)}{aR^2} \varphi(\xi;g_2,g_3), \quad (4.24b) \]

\[ w_7 = \mu \left[ A_0 + \frac{1}{2} \frac{k^2(8R^3 - 9aP)}{aR^2} \varphi(\xi;g_2,g_3) \right] + \nu, \quad (4.24c) \]

where

\[ \lambda = \frac{1}{2R^2P} (6R^2P + 6aA_0R^2P + 12k^2R^3 - 3k^2P^3g_3), \quad g_2 = \frac{4R^3 + g_3P^3}{P^2R}. \]

Family 8.

\[ u_8 = a \left[ A_0 - \frac{2k^2}{a} \varphi(\xi;g_2,g_3) \right] + b, \quad (4.25a) \]

\[ v_8 = A_0 - \frac{2k^2}{a} \varphi(\xi;g_2,g_3), \quad (4.25b) \]

\[ w_8 = \mu \left[ A_0 - \frac{2k^2}{a} \varphi(\xi;g_2,g_3) \right] + \nu, \quad (4.25c) \]

where \( \lambda = 3b + 3aA_0 \).

Family 9. When \( a = 2 \)

\[ u_9 = a \left\{ A_0 + \left[ \frac{18B_1^3\sqrt{\frac{7}{13}}}{7Q} \sqrt{\frac{7a}{13}} \varphi(\xi;g_2,g_3) + B_2\varphi'(\xi;g_2,g_3) \right] \right\}^2 + b, \quad (4.26a) \]
\[ v_9 = A_0 + \left[ \frac{18B_2^2}{7kQ^2} \sqrt{\frac{7a}{13}} \varphi(\xi; g_2, g_3) + B_2 \varphi'(\xi; g_2, g_3) \right]^2, \quad (4.26b) \]

\[ w_9 = \mu \left\{ A_0 + \left[ \frac{18B_2^2}{7kQ^2} \sqrt{\frac{7a}{13}} \varphi(\xi; g_2, g_3) + B_2 \varphi'(\xi; g_2, g_3) \right]^2 \right\} + \nu, \quad (4.26c) \]

where

\[ \lambda = 3/13 \frac{6aB_2^2 + 13aA_0Q^2 + 13bQ^2}{Q^2}, \quad g_2 = \frac{135}{1183} \frac{a^2B_2^4}{k^4Q^4}, \quad g_3 = \frac{108}{15379} \frac{a^3B_2^6}{k^6Q^6} \]

**Family 10.** When \( a = -2 \),

\[ u_{10} = a \left\{ A_0 + \left[ \frac{B_2 \varphi'(\xi; g_2, g_3)}{R + \frac{2B_2}{k} \varphi(\xi; g_2, g_3)} \right]^2 \right\} + b, \quad (4.27a) \]

\[ v_{10} = A_0 + \left[ \frac{B_2 \varphi'(\xi; g_2, g_3)}{R + \frac{2B_2}{k} \varphi(\xi; g_2, g_3)} \right]^2, \quad (4.27b) \]

\[ w_{10} = \mu \left\{ A_0 + \left[ \frac{B_2 \varphi'(\xi; g_2, g_3)}{R + \frac{2B_2}{k} \varphi(\xi; g_2, g_3)} \right]^2 \right\} + \nu, \quad (4.27c) \]

where

\[ \lambda = 3b + 3aA_0 + \frac{6k^3R}{B_2}, \quad g_3 = \frac{R \left( 2k^2R^2 + aB_2^2g_2 \right)}{2aB_2^2}. \]

**Family 11.**

\[ u_{11} = a \left\{ A_0 + \left[ \frac{A_2 \varphi(\xi; g_2, g_3)}{R + \frac{aA^2}{5k^2R} \varphi(\xi; g_2, g_3)} \right]^2 \right\} + b, \quad (4.28a) \]

\[ v_{11} = A_0 + \left[ \frac{A_2 \varphi(\xi; g_2, g_3)}{R + \frac{aA^2}{5k^2R} \varphi(\xi; g_2, g_3)} \right]^2, \quad (4.28b) \]

\[ w_{11} = \mu \left\{ A_0 + \left[ \frac{A_2 \varphi(\xi; g_2, g_3)}{R + \frac{aA^2}{5k^2R} \varphi(\xi; g_2, g_3)} \right]^2 \right\} + \nu, \quad (4.28c) \]

where

\[ \lambda = 3b + 3aA_0 + \frac{30k^4R^2}{aA^2}, \quad g_2 = 50 \frac{k^4R^4}{a^2A^4}, \quad g_3 = \frac{125}{2} \frac{R^6k^6}{a^3A^6}. \]

**Remark 2.** The Weierstrass elliptic function \( \varphi(\xi; g_2, g_3) \) can be also written as another form

\[ \varphi(\xi; g_2, g_3) = e_3 + (e_1 - e_3)ns^2(\sqrt{e_1 - e_3}; m), \quad (4.29) \]
in terms of the Jacobi elliptic function $\text{ns}(\xi; m) = \frac{1}{\sin(\xi; m)}$, where $m^2 = \frac{(e_2 - e_3)}{(e_1 - e_3)}$ is the modulus of the Jacobi elliptic function, $e_i (i = 1, 2, 3; e_1 \geq e_2 \geq e_3)$ are three roots of the cubic equation $4y^3 - g_2y - g_3 = 0$. According to (4.14) we also write the mentioned-above solutions as other forms.

5. Conclusions and discussions

In brief, we have developed a new Weierstrass semi-rational expansion method in terms of the Weierstrass elliptic function. The method is applied to the coupled water wave equation and the generalized Hirota-Satsuma coupled KdV equation such that many types of new doubly periodic solutions are obtained. Moreover solitary wave solutions are also given as simple limits of doubly periodic solutions. The method can be also extend to other types of nonlinear wave equations to seek many doubly periodic solutions. If we assume that $\xi \rightarrow \psi(x, t)$ and the parameters $A_0, A_i, B_i, R, P, Q$ are functions of $x, t$, then we may study non-travelling elliptic function solutions of nonlinear wave equations. We will further study this question.

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References