Structured Low Rank Approximation of a Bezout Matrix

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Abstract. The task of determining the approximate greatest common divisor (GCD) of more than two univariate polynomials with inexact coefficients can be formulated as computing for a given Bezout matrix a new Bezout matrix of lower rank whose entries are near the corresponding entries of that input matrix. We present an algorithm based on a version of structured nonlinear total least squares (SNTLS) method for computing approximate GCD and demonstrate the practical performance of our algorithm on a diverse set of univariate polynomials.

1. Introduction

The computation of approximate GCDs of univariate polynomials has been extensively studied in [21, 35, 29, 7, 11, 24, 18, 13, 4, 33, 40, 30, 34, 39, 8, 37, 38, 9, 10]. It can be formulated as an optimization problem:

Problem 1.1 *Given univariate polynomials $f_1, \dots, f_l \in \mathbb{R}[x] \setminus \{0\}$ with $\deg(f_1) = d_1, \dots, \deg(f_l) = d_l$, assume $d_1 = \max(d_1, \dots, d_l)$. For a positive integer k , $k \leq \min(d_1, \dots, d_l)$, we wish to compute $\Delta f_1, \dots, \Delta f_l \in \mathbb{R}[x]$ such that $\deg(\Delta f_1) \leq d_1, \dots, \deg(\Delta f_l) \leq d_l$, $\deg(\text{GCD}(f_1 + \Delta f_1, \dots, f_l + \Delta f_l)) \geq k$ and $\|\Delta f_1\|_2^2 + \dots + \|\Delta f_l\|_2^2$ is minimized.*

In [21, 22, 23], the authors transformed the above problem into computing for a generalized Sylvester matrix the nearest singular matrix with the generalized Sylvester structure. They presented iterative algorithms based on structured total least norm algorithms in [32, 31, 25, 26] to solve the optimization problem. It is well known that Bezout matrix can also be used to compute GCDs, see recent paper [10] and references there. Compared with the generalized Sylvester matrix, Bezout matrix has smaller size. However, entries of the Bezout matrix are bilinear in coefficients of the polynomials. Hence, we propose to apply the structured nonlinear total least squares (SNTLS) algorithm [19, 26] to compute the nearest singular matrix with Bezout structure. We show how to solve PROBLEM 1.1, at least for a local minimum, by applying SNTLS with L_2 norm to a submatrix of the Bezout matrix.

We organize the paper as follows. In Section 2, we introduce some notations and discuss the equivalence between the GCD problem and the low rank approximation of a matrix with Bezout structure. In Section 3, we consider solving an overdetermined system with Bezout structure based on SNTLS. In Section 4, we describe our algorithm for two examples

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and compare our results with previous work in [21, 22, 23]. We conclude in Section 5 with remarks on the complexity and the rate of convergence of our algorithm.

2. preliminaries

The GCD problems are closely related with structured matrices, such as Sylvester matrix, Bezout matrix, Hankel matrix and so forth. In the following subsections we introduce the structured matrices.

2.1. Bezout Matrix

Suppose we are given two univariate polynomials $f_1, f_2 \in \mathbb{R}[x] \setminus \{0\}$ with $\deg(f_1) = m$ and $\deg(f_2) = n$, we assume $m \geq n$,

$$\begin{aligned} f_1 &= u_m x^m + u_{m-1} x^{m-1} + \cdots + u_1 x + u_0, & u_m &\neq 0, \\ f_2 &= v_n x^n + v_{n-1} x^{n-1} + \cdots + v_1 x + v_0, & v_n &\neq 0. \end{aligned} \quad (1)$$

The Bezout matrix $\hat{B}(f_1, f_2) = (\hat{b}_{ij})$ is defined by

$$\hat{b}_{ij} = |u_0 v_{i+j-1}| + |u_1 v_{i+j-2}| + \cdots + |u_k v_{i+j-k-1}|,$$

where $|u_r v_s| = u_r v_s - u_s v_r$, $k = \min(i-1, j-1)$ and $v_r = 0$ if $r > n$ [3, 17]. It satisfies that

$$\frac{f_1(x)f_2(y) - f_1(y)f_2(x)}{x-y} = [1, x, x^2, \dots, x^{m-1}] \hat{B}(f_1, f_2) [1, y, y^2, \dots, y^{m-1}]^T. \quad (2)$$

Notice that the Bezout matrix $B(f_1, f_2)$ defined in Maple is as follows:

$$B(f_1, f_2) = -J \hat{B}(f_1, f_2) J, \quad (3)$$

where J is an anti-diagonal matrix with 1 as its nonzero entries.

The Bezout matrix can be generalized for nonzero univariate polynomials $f_1, \dots, f_l \in \mathbb{R}[x] \setminus \{0\}$ with $\deg(f_1) = d_1, \dots, \deg(f_l) = d_l$. Suppose $d_1 = \max(d_1, \dots, d_l)$, $B(f_1, \dots, f_l) \in \mathbb{R}^{(l-1)d_1 \times d_1}$ is defined by

$$B(f_1, \dots, f_l) = \begin{bmatrix} B(f_1, f_2) \\ B(f_1, f_3) \\ \vdots \\ B(f_1, f_l) \end{bmatrix}. \quad (4)$$

Suppose $l = 2$ and $m = n = 4$, the Sylvester matrix $S(f_1, f_2)$ and Bezout matrix $B(f_1, f_2)$ defined in Maple are:

$$S(f_1, f_2) = \begin{bmatrix} u_4 & u_3 & u_2 & u_1 & u_0 & 0 & 0 & 0 \\ 0 & u_4 & u_3 & u_2 & u_1 & u_0 & 0 & 0 \\ 0 & 0 & u_4 & u_3 & u_2 & u_1 & u_0 & 0 \\ 0 & 0 & 0 & u_4 & u_3 & u_2 & u_1 & u_0 \\ v_4 & v_3 & v_2 & v_1 & v_0 & 0 & 0 & 0 \\ 0 & v_4 & v_3 & v_2 & v_1 & v_0 & 0 & 0 \\ 0 & 0 & v_4 & v_3 & v_2 & v_1 & v_0 & 0 \\ 0 & 0 & 0 & v_4 & v_3 & v_2 & v_1 & v_0 \end{bmatrix},$$

$$B(f_1, f_2) =$$

$$\begin{bmatrix} u_3v_4 - u_4v_3 & u_2v_4 - u_4v_2 & u_1v_4 - u_4v_1 & u_0v_4 - u_4v_0 \\ u_2v_4 - u_4v_2 & u_2v_3 - u_3v_2 + u_1v_4 - u_4v_1 & u_1v_3 - u_3v_1 + u_0v_4 - u_4v_0 & u_0v_3 - u_3v_0 \\ u_1v_4 - u_4v_1 & u_1v_3 - u_3v_1 + u_0v_4 - u_4v_0 & u_1v_2 - u_2v_1 + u_0v_3 - u_3v_0 & u_0v_2 - u_2v_0 \\ u_0v_4 - u_4v_0 & u_0v_3 - u_3v_0 & u_0v_2 - u_2v_0 & u_0v_1 - u_1v_0 \end{bmatrix}.$$

The size of the Sylvester matrix is larger, while the entries of the Bezout matrix are bilinear in coefficients of f_1 and f_2 . When the degrees of the polynomials are high, the advantage of the Bezout matrix is obvious.

2.2. Hankel matrix

A Hankel matrix $H \in \mathbb{R}^{m \times m}$ is formed from $2m - 1$ real numbers $(h_i)_{i=0}^{2m-2}$:

$$H(f_1, f_2) = \begin{bmatrix} h_0 & h_1 & \dots & h_{m-1} \\ h_1 & h_2 & \dots & h_m \\ \vdots & \vdots & \ddots & \vdots \\ h_{m-1} & h_m & \dots & h_{2m-2} \end{bmatrix}. \quad (5)$$

Given two univariate polynomials $f_1, f_2 \in \mathbb{R}[x] \setminus \{0\}$ with $\deg(f_1) = m$ and $\deg(f_2) = n$, where $m > n$, assume $R(x) = \frac{f_2(x)}{f_1(x)} = \sum_{i=0}^{+\infty} h_i x^{-i-1}$. The vector $(h_i)_{i=0, \dots, +\infty}$ forms a Hankel matrix H_∞ . The m -th leading principal submatrix $H(f_1, f_2)$ is defined as the Hankel matrix of f_1 and f_2 . Given the coefficients of f_1 and f_2 , the entries of the Hankel matrix $H(f_1, f_2)$ can be computed [6].

Proposition 2.1 [15, 14, 6] *Given two univariate polynomials $f_1, f_2 \in \mathbb{R}[x] \setminus \{0\}$ with $\deg(f_1) = m, \deg(f_2) = n (m > n)$, then we have:*

$$B(f_1, f_2) = -B(f_1, 1) J H(f_1, f_2) J B(f_1, 1) \quad (6)$$

Proposition 2.2 [6] *Given two univariate monic polynomials $f_1, f_2 \in \mathbb{R}[x] \setminus \{0\}$ with $\deg(f_1) = m$ and $\deg(f_2) = n (m > n)$, let $c(x) = \text{GCD}(f_1(x), f_2(x))$ be a monic polynomial of degree $m - k$. Then we have:*

- $\text{rank}(H(f_1, f_2)) = k$, $\det(H_k) \neq 0$, where H_i is the $i \times i$ leading principal submatrix of $H(f_1, f_2)$, $\det(H_i) = 0$ for $i > k$;
- If $H_{k+1} \mathbf{w} = 0$, $\mathbf{w} = (w_i)_{i=0, \dots, k}$, that is, if

$$H_k \begin{pmatrix} w_0 \\ \vdots \\ w_{k-1} \end{pmatrix} = -w_k \begin{pmatrix} h_k \\ \vdots \\ h_{2k-1} \end{pmatrix} \quad (7)$$

then

$$f_1(x) = c(x) \sum_{i=0}^k w_i x^i \quad (8)$$

Remark 2.3 [6] Since $B(f_1, 1)J$ and $JB(f_1, 1)$ are lower triangular and upper triangular matrices respectively, we deduce from the relation $B(f_1, f_2) = -B(f_1, 1)JH(f_1, f_2)JB(f_1, 1)$, that $B(f_1, f_2)\mathbf{y} = 0$, $y_k = 1$, $y_i = 0$, $i > k$, if and only if $\mathbf{w} = JB(f_1, 1)y$, where $\mathbf{w}$ is the vector of Proposition 2.2. Moreover, $\det(B(f_1, f_2))_k \neq 0$, and $\det(B(f_1, f_2))_i = 0$, for $i > k$, where $(B(f_1, f_2))_i$ is the $i \times i$ leading principal submatrix of $B(f_1, f_2)$.

Remark 2.4 Notice that the Propositions 2.1, 2.2 are also correct for $m = n$. Let $\bar{f}_2(x) = f_2(x) - \frac{v_n}{u_m}f_1(x)$, and $R(x) = \frac{f_2(x)}{f_1(x)} = \frac{v_n}{u_m} + \frac{\bar{f}_2(x)}{f_1(x)} = h_0 + \sum_{i=1}^{+\infty} h_i x^{-i}$ where $h_0 = \frac{v_n}{u_m}$. The matrix $H(f_1, f_2)$ is defined as:

$$H(f_1, f_2) = \begin{bmatrix} h_1 & h_2 & \dots & h_m \\ h_2 & h_3 & \dots & h_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_m & h_{m+1} & \dots & h_{2m-1} \end{bmatrix}. \quad (9)$$

Hence $B(f_1, f_2) = B(f_1, \bar{f}_2)$ and $H(f_1, f_2) = H(f_1, \bar{f}_2)$.

Corollary 2.5 Given univariate polynomials $f_1, \dots, f_l \in \mathbb{R}[x] \setminus \{0\}$ with $\deg(f_1) = d_1, \dots$, $\deg(f_l) = d_l$, $d_1 = \max(d_1, d_2, \dots, d_l)$, then we have:

$$B(f_1, \dots, f_l) = \begin{bmatrix} B(f_1, f_2) \\ B(f_1, f_3) \\ \vdots \\ B(f_1, f_l) \end{bmatrix}_{(l-1)d_1 \times d_1} = -\Gamma H(f_1, \dots, f_l) JB(f_1, 1), \quad (10)$$

where

$$\Gamma = \begin{bmatrix} B(f_1, 1)J & & \\ & \ddots & \\ & & B(f_1, 1)J \end{bmatrix}_{(l-1)d_1 \times ((l-1)d_1)} \quad (11)$$

and

$$H(f_1, \dots, f_l) = \begin{bmatrix} H(f_1, f_2) \\ H(f_1, f_3) \\ \vdots \\ H(f_1, f_l) \end{bmatrix}_{(l-1)d_1 \times d_1}. \quad (12)$$

Proof: The proof is derived from Proposition 2.1. $\square$

Lemma 2.6 [12] Let H_∞ be an infinity Hankel matrix of rank r , then we have:

$$\det(H(f_1, f_2)_r) \neq 0. \quad (13)$$

Lemma 2.7 [12] Let $H_\infty(f_1, f_2)$ be an infinity Hankel matrix which is formed from two polynomials f_1, f_2 with degrees m, n ($m \geq n$) respectively, then

$$\text{rank}(H_\infty(f_1, f_2)) \leq m. \quad (14)$$

2.3. Main Theorems

The following three theorems summarize the relationship between the greatest common divisor (GCD) of $f_1, \dots, f_l$ and the Bezout matrix $B(f_1, \dots, f_l)$.

Theorem 2.8 *Given univariate polynomials $f_1, \dots, f_l \in \mathbb{R}[x] \setminus \{0\}$ with $\deg(f_1) = d_1, \dots, \deg(f_l) = d_l$, $d_1 = \max(d_1, \dots, d_l)$, then we have $\dim(\text{Ker} B(f_1, \dots, f_l))$ being equal to the degree of the GCD of $f_1, \dots, f_l$.*

Proof: A detailed proof is given in [9]. $\square$

Theorem 2.9 *Given univariate polynomials $f_1, \dots, f_l \in \mathbb{R}[x] \setminus \{0\}$ with $\deg(f_1) = d_1, \dots, \deg(f_l) = d_l$, $d_1 = \max(d_1, \dots, d_l)$, then the degree of the GCD of $f_1, \dots, f_l$ is at least k for $k \leq \min(d_1, \dots, d_l)$ if and only if the first $d_1 - k + 1$ columns of $B(f_1, \dots, f_l)$ are linearly dependent.*

Proof: For $l = 2$, since $B(f_1, 1)J$ and $JB(f_1, 1)$ are lower triangular and upper triangular matrices respectively, by Proposition 2.1, we have $\text{rank}(B(f_1, f_2)_i) = \text{rank}(H(f_1, f_2)_i)$. If the first $d_1 - k + 1$ columns of $B(f_1, f_2)$ are linearly dependent, then the first $d_1 - k + i$ ($i \geq 1$) columns of $H(f_1, f_2)$ are linearly dependent. By Lemma 2.6 and Lemma 2.7, we have $\text{rank}(B(f_1, f_2)) \leq d_1 - k$. The proof can be derived by Theorem 2.8 and generalized to the cases $l > 2$ by Corollary 2.5. $\square$

Theorem 2.10 *Given univariate polynomials $f_1, \dots, f_l \in \mathbb{R}[x] \setminus \{0\}$ with $\deg(f_1) = d_1, \dots, \deg(f_l) = d_l$, $d_1 = \max(d_1, \dots, d_l)$, let $c(x) = \text{GCD}(f_1(x), \dots, f_l(x))$ be a polynomial of degree k , then we have $\text{rank}(B(f_1, \dots, f_l)) = d_1 - k$. Suppose $\mathbf{y} = (y_0, y_1, \dots, y_{d_1-k-1})^T$ satisfies $C\mathbf{y} = \mathbf{b}$, where C consists of the first $d_1 - k$ columns of $B(f_1, \dots, f_l)$, and $\mathbf{b}$ is a vector formed from the $d_1 - k + 1$ -th column of $B(f_1, \dots, f_l)$. Let $\mathbf{w} = [w_0, \dots, w_{d_1-k}]^T = (JB(f_1, 1))_{d_1-k+1} [y_0, \dots, y_{d_1-k}]^T$, where $y_{d_1-k} = -1$, $(JB(f_1, 1))_{d_1-k+1}$ is the leading principal $d_1 - k + 1$ -th submatrix, then $f_1(x) = c(x)w(x)$, with $w(x) = \sum_{i=0}^{d_1-k} w_i x^i$.*

See [1, 2, 3, 15, 6, 27, 5, 16, 9, 10] for detailed proofs.

3. SNTLS for Overdetermined System with Bezout Structure

The Bezout matrix $B(f_1, \dots, f_l)$ can be parameterized by a vector ζ which contains the coefficients of $f_1, \dots, f_l$. By applying Theorem 2.8, we can transfer the PROBLEM 1.1 into solving the following minimization problem:

$$\min_{\Delta \mathbf{s} \in \mathbb{R}^{d+l}} \|\Delta \mathbf{s}\|_2 \quad \text{with} \quad \dim(\text{Ker} B(\mathbf{s} + \Delta \mathbf{s})) \geq k, \quad (15)$$

where $d = \sum_{i=1}^l d_i$ and $\mathbf{s}$ is fixed to the initial coefficient vector. Let $B_k(\zeta) = [D_1(\zeta), \mathbf{b}(\zeta), D_2(\zeta)]$ be the first $d_1 - k + 1$ columns of the Bezout matrix $B(\zeta)$ and let $A(\zeta) = [D_1(\zeta), D_2(\zeta)] \in \mathbb{R}^{d_1(l-1) \times (d_1-k)}$. According to Theorem 2.9, the minimization problem (15) can be transferred into the following structured nonlinear total least squares problem:

$$\min_{\Delta \mathbf{s} \in \mathbb{R}^{d+l}} \|\Delta \mathbf{s}\|_2 \quad \text{with} \quad A(\mathbf{s} + \Delta \mathbf{s}) \mathbf{x} = \mathbf{b}(\mathbf{s} + \Delta \mathbf{s}), \quad \text{for some vector } \mathbf{x}. \quad (16)$$

Similar to [21, 22, 23], we choose that column as $\mathbf{b}(\zeta) \in \mathbb{R}^{d_1(l-1) \times 1}$ for which the corresponding component in the first right singular vector of B_k was maximum in absolute value. In the following, we illustrate how to find the minimum solution of (16) using the structured nonlinear total least squares (SNTLS) method.

We can initialize $\mathbf{x}$ as the unstructured least squares solution $A(\mathbf{s})\mathbf{x} = \mathbf{b}(\mathbf{s})$ for the input parameters $\mathbf{s}$ and $\Delta\mathbf{s} = \mathbf{0}$. However, as pointed by [25, Section 4.5.3] and [22, 23], another way is to initialize $\Delta\mathbf{s}$ and $\mathbf{x}$ such that they satisfy the nonlinear constrains approximately, $A(\mathbf{s} + \Delta\mathbf{s})\mathbf{x} \approx \mathbf{b}(\mathbf{s} + \Delta\mathbf{s})$. We compute $\Delta\mathbf{s}$ as follows:

$$\Delta\mathbf{s} = -Y^T(\mathbf{s}, \mathbf{v})(Y(\mathbf{s}, \mathbf{v})Y^T(\mathbf{s}, \mathbf{v}))^{-1}B_k(\mathbf{s})\mathbf{v}, \quad (17)$$

where $\mathbf{v}$ is the right singular vector corresponding to the smallest singular value of $B_k(\mathbf{s})$ and matrix Y is the Jacobian of $B_k(\zeta)\mathbf{v}$ with respect to ζ , we have

$$B_k(\mathbf{s} + \Delta\mathbf{s})\mathbf{v} = B_k(\mathbf{s})\mathbf{v} + Y(\mathbf{s}, \mathbf{v})\Delta\mathbf{s} + O(\|\Delta\mathbf{s}\|_2^2) = O(\|\Delta\mathbf{s}\|_2^2).$$

Suppose $b(\mathbf{s})$ is the t -th column corresponding to the absolutely largest component in $\mathbf{v}$; We initialize the vector $\mathbf{x}$ by normalizing the vector $\mathbf{v}$ to make $\mathbf{v}[t] = -1$, i.e.,

$$\mathbf{x} = \left[-\frac{\mathbf{v}[1]}{\mathbf{v}[t]}, \dots, -\frac{\mathbf{v}[t-1]}{\mathbf{v}[t]}, -\frac{\mathbf{v}[t+1]}{\mathbf{v}[t]}, \dots \right]^T. \quad (18)$$

We have $A(\mathbf{s} + \Delta\mathbf{s})\mathbf{x} - \mathbf{b}(\mathbf{s} + \Delta\mathbf{s}) = O(\|\Delta\mathbf{s}\|_2^2)$. Since the initial values of $\Delta\mathbf{s}$ and $\mathbf{x}$ only satisfy the first order of the nonlinear constrains, for the second initialization method to be successful, we usually require that the initial perturbation (17) $\|\Delta\mathbf{s}\|_2 \ll 1$.

By introducing the Lagrangian multipliers, and neglecting the second-order terms in $\Delta\mathbf{s}$, the constrained minimization problem can be transformed into an unconstrained optimization problem [26, 19],

$$L(\Delta\mathbf{s}, \mathbf{x}, \lambda) = \frac{1}{2}\Delta\mathbf{s}^T\Delta\mathbf{s} - \lambda^T(\mathbf{b} - A\mathbf{x} - X\Delta\mathbf{s}), \quad (19)$$

where $X(\zeta, \mathbf{x})$ is the Jacobian of $\mathbf{r}(\zeta, \mathbf{x}) = A(\zeta)\mathbf{x} - \mathbf{b}$ with respect to ζ :

$$X(\zeta, \mathbf{x}) = \nabla_\zeta(A(\zeta)\mathbf{x}) - \nabla_\zeta(\mathbf{b}(\zeta)) = \sum_{j=1}^{d_1-k} x_j \nabla_\zeta a_j(\zeta) - \nabla_\zeta(\mathbf{b}(\zeta)), \quad (20)$$

where $a_j(\zeta)$ represents the j -th column of $A(\zeta)$. Applying the Newton method on the Lagrangian L yields:

$$\begin{bmatrix} I_{d+l} & \mathbf{0}_{(d+l) \times (d_1-k)} & X(\mathbf{s} + \Delta\mathbf{s}, \mathbf{x})^T \\ \mathbf{0}_{(d_1-k) \times (d+l)} & \mathbf{0}_{(d_1-k) \times (d_1-k)} & A(\mathbf{s} + \Delta\mathbf{s})^T \\ X(\mathbf{s} + \Delta\mathbf{s}, \mathbf{x}) & A(\mathbf{s} + \Delta\mathbf{s}) & \mathbf{0}_{(l-1)d_1 \times (l-1)d_1} \end{bmatrix} \begin{bmatrix} \Delta\tilde{\mathbf{s}} \\ \Delta\tilde{\mathbf{x}} \\ \Delta\tilde{\lambda} \end{bmatrix} = - \begin{bmatrix} \Delta\mathbf{s} + X(\mathbf{s} + \Delta\mathbf{s}, \mathbf{x})^T \lambda \\ A(\mathbf{s} + \Delta\mathbf{s})^T \lambda \\ A(\mathbf{s} + \Delta\mathbf{s})\mathbf{x} - \mathbf{b}(\mathbf{s} + \Delta\mathbf{s}) \end{bmatrix}, \quad (21)$$

where $d = \sum_{i=1}^l d_i$. The iterative update $\mathbf{x} = \mathbf{x} + \Delta\tilde{\mathbf{x}}$, $\lambda = \lambda + \Delta\tilde{\lambda}$, $\mathbf{s} = \mathbf{s} + \Delta\mathbf{s} + \Delta\tilde{\mathbf{s}}$ is stopped when $\|\Delta\tilde{\mathbf{x}}\|_2$ and/or $\|\Delta\tilde{\mathbf{s}}\|_2$ and/or $\|\Delta\tilde{\lambda}\|_2$ becomes smaller than a given tolerance.

Suppose $\tilde{B}_k$ is the nearest singular matrix with Bezout structure computed successfully by SNTLS algorithm. Let the perturbed polynomials be $\tilde{f}_1, \dots, \tilde{f}_l$. Suppose $k = \deg(\text{GCD}(\tilde{f}_1, \dots, \tilde{f}_l))$, the polynomial $c(x) = \text{GCD}(\tilde{f}_1, \dots, \tilde{f}_l)$ can be computed according to Theorem 2.10. However, we can also use the vector $\mathbf{x}$ returned from the iterations (21) to compute the GCD directly. Let $\mathbf{y} = [x_1, \dots, x_{t-1}, -1, x_t, \dots, x_{d_1-k}]^T$, compute the vector $\mathbf{w} = [w_0, \dots, w_{d_1-k}]^T = (JB(\tilde{f}_1, 1))_{d_1-k+1} \mathbf{y}^T$, where $(JB(\tilde{f}_1, 1))_{d_1-k+1}$ is the leading principal $d_1 - k + 1$ -th submatrix. The polynomial $c(x)$ is computed by dividing the polynomial $\tilde{f}_1(x)$ by the polynomial $w(x) = \sum_{i=0}^{d_1-k} w_i x^i$, and is returned as the approximate GCD of $f_1, \dots, f_l$.

Remark 3.1 When k is smaller than $\deg(\text{GCD}(\tilde{f}_1, \dots, \tilde{f}_l))$, as suggested in [23], we may increase k by $k+1$ and run our SNTLS algorithm on $\tilde{f}_1, \dots, \tilde{f}_l$ again until we find the correct k .

4. Algorithms and Experiments

We have implemented the algorithms in Maple 10 for computing the approximate GCDs of several univariate polynomials with real coefficients by structured low rank approximation of a Bezout matrix. We first present two examples and compare the results with other methods in [10, 22].

Example 4.1 [10] Consider the polynomials

$$\begin{aligned} f_1 &= (x^5 - 1)(x^4 - x + 1), \\ f_2 &= (x^5 - 0.9999)(x + 4.0001), \\ f_3 &= (x^5 - 0.9999)(x^4 - 3.0003x - 2.9999), \\ f_4 &= (x^5 - 1.0001)(x^4 - 3.0001x - 0.9999). \end{aligned}$$

The matrix $B_5(f_1, f_2, f_3, f_4)$ is of size 27×5 , whereas the generalized Sylvester matrix used in [22] is of size 39×17 . By our algorithms, for $k = 5$, after two iterations, we stop the algorithm at $\Delta\mathbf{x} = .28218 \times 10^{-5}$. The deformed polynomials are

$$\begin{aligned} \tilde{f}_1 &= x^9 + .16753 \times 10^{-5} x^8 + .26750 \times 10^{-5} x^7 - x^6 + x^5 - .99997 x^4 \\ &\quad + .19010 \times 10^{-5} x^3 + .23249 \times 10^{-5} x^2 + .99998 x - .99998, \\ \tilde{f}_2 &= .99998 x^6 + 4x^5 + .16014 \times 10^{-4} x^4 + .21290 \times 10^{-4} x^3 + .17999 \times 10^{-4} x^2 \\ &\quad - .99992 x - 3.9998, \\ \tilde{f}_3 &= .99996 x^9 - .10714 \times 10^{-5} x^8 - .80617 \times 10^{-5} x^7 - 3.0002 x^6 - 2.9998 x^5 \\ &\quad - .99994 x^4 - .10723 \times 10^{-5} x^3 - .80515 \times 10^{-5} x^2 + 3.0001 x + 2.9997, \\ \tilde{f}_4 &= 1.0001 x^9 - .63203 \times 10^{-5} x^8 - .43391 \times 10^{-5} x^7 - 3.0003 x^6 - .99998 x^5 \\ &\quad - x^4 - .63321 \times 10^{-5} x^3 - .43314 \times 10^{-5} x^2 + 3.0002 x + 0.99992. \end{aligned}$$

The backward error

$$\mathcal{N} = \sqrt{\|\tilde{f}_1 - f_1\|_2^2 + \|\tilde{f}_2 - f_2\|_2^2 + \|\tilde{f}_3 - f_3\|_2^2 + \|\tilde{f}_4 - f_4\|_2^2} = .41295 \times 10^{-3}.$$

The backward error computed by the STLS algorithm in [22] is $.41292 \times 10^{-3}$. However, our algorithm only takes 2.937 seconds while the STLS algorithm takes 213.596 seconds. The backward error given in [10] is larger than $.47610 \times 10^{-3}$. The approximate GCD computed by our algorithm is

$$c(x) = x^5 + .29055 \times 10^{-5}x^4 + .43923 \times 10^{-5}x^3 + .37214 \times 10^{-5}x^2 \\ + .31134 \times 10^{-5}x - .99995.$$

Example 4.2 *Consider the polynomials [22]*

$$f_1 = 1000x^{10} + x^3 - 1, \\ f_2 = x^2 - 0.01.$$

Case 1 *If we initialize $\Delta \mathbf{s} = 0$ and $\mathbf{x}$ being the unstructured least squares solution of $A(\mathbf{s})\mathbf{x} = \mathbf{b}(\mathbf{s})$. After 10 iterations, we obtain the deformed polynomials*

$$\tilde{f}_1 = 1000x^{10} - 0.00011x^9 - .00014x^8 - .00009x^7 + .00008x^6 - .00026x^5 \\ + .00049x^4 + .99901x^3 + .00195x^2 - .00386x - .99238, \\ \tilde{f}_2 = .95204x^2 + .09462x - .19666,$$

which have a common divisor $x + 0.50690$, and the backward error is

$$\mathcal{N} = \|\tilde{f}_1 - f_1\|_2^2 + \|\tilde{f}_2 - f_2\|_2^2 = 0.04617.$$

As discussed in [22], this is only one of the local minimum.

Case 2 *We initialize $\Delta \mathbf{s}$ by formula (17) and choose $\mathbf{v}$ being the right singular vector corresponding to the smallest singular value of $B_k(\mathbf{s})$ and normalized with respect to the largest entry. After 8 iterations, the algorithm returns*

$$\tilde{f}_1 = 1000x^{10} + .00012x^9 - .00013x^8 + .00006x^6 + .00023x^5 + .00049x^4 \\ + 1.0010x^3 + .00205x^2 + .00415x - .99156, \\ \tilde{f}_2 = .95614x^2 - .08876x - .18962,$$

which have a common divisor $x - 0.49415$, the backward error is

$$\mathcal{N} = \|\tilde{f}_1 - f_1\|_2^2 + \|\tilde{f}_2 - f_2\|_2^2 = .04216.$$

It is the non-monic global minimum similar to the one derived in [22].

In Table 1, we show the performance of our algorithm for computing approximate GCD of univariate polynomials on Pentium 4 at 2.0 Ghz for *Digits* = 14 in Maple 10 under

<i>Ex.</i>	d_i	k	<i>it.</i> (STLS)	<i>it.</i> (SNTLS)	<i>error</i> (STLS)	<i>error</i> (SNTLS)	<i>time(s)</i> (STLS)	<i>time(s)</i> (SNTLS)
1	2, 2	1	2.18	1.90	$6.96e-6$	$6.96e-6$	.25	.12
2	3, 3	2	2.17	1.93	$1.05e-5$	$1.07e-5$	.31	.14
3	5, 4	3	2.06	1.91	$1.56e-5$	$1.56e-5$	.44	.18
4	5, 5	3	2.27	2.00	$2.04e-5$	$2.75e-5$	.53	.20
5	6, 6	4	2.10	2.00	$2.18e-5$	$2.18e-5$	.58	.21
6	8, 7	4	2.10	1.90	$1.70e-5$	$1.70e-5$	.95	.31
7	10, 10	5	2.60	2.10	$3.43e-4$	$3.44e-5$	1.40	.43
8	14, 13	7	2.60	1.90	$5.73e-5$	$6.47e-5$	2.31	.80
9	28, 28	14	2.00	2.00	$2.60e-5$	$2.60e-5$	10.65.	11.97
10	10, 9, 8	5	4.00	3.002	$7.96e-5$	$9.86e-5$	4.17	1.99
11	8, 7, 8, 6	4	4.40	3.20	$3.24e-5$	$3.28e-5$	5.83	1.41

Table 1. Algorithm performance on benchmarks (univariate case)

Windows. For every example, we use 50 random cases for each $(d_1, \dots, d_l)$, and report the average over all results. For each example, the prime parts and GCD of polynomials are constructed by choosing polynomials with random integer coefficients in the range $-10 \leq c \leq 10$, and then adding a perturbation. For noise we choose a relative tolerance 10^{-e} , then randomly choose a polynomial that has the same degree as the product, and coefficients in $[-10^e, 10^e]$. Finally, we scale the perturbation so that the relative error is 10^{-e} . Here d_i denotes the degree of the polynomial f_i ; k is the degree of the approximate GCD of $f_1, \dots, f_l$; *it. (STLS)* is the number of the iterations needed by method in [23]; whereas *it. (SNTLS)* denotes the number of iterations by our algorithm; *error (STLS)* denotes the perturbation $\|\tilde{f}_1 - f_1\|_2^2 + \dots + \|\tilde{f}_l - f_l\|_2^2$ computed by algorithm in [23]; whereas *error (SNTLS)* is the minimal perturbation computed by our algorithm; the last two columns denote the time in seconds costed by two algorithms respectively.

5. Concluding Remarks

In this paper we present a new way based on SNTLS to compute the approximate GCD of several univariate polynomials. The overall computational complexity of the algorithm depends on the number of iterations needed for the first order update. If the starting values are good, then the iteration will converge quickly. This can be seen from the above table. Since the matrices involved in the minimization problems are all structured matrix, they have low displacement rank [20]. It would be possible to apply the fast algorithm to solve these minimization problems as in [28]. This would reduce the complexity of our algorithm to be only quadratic with respect to the degrees of the given polynomials.

Our methods can be generalized to several polynomials with arbitrary linear or nonlinear equational constraints imposed on the coefficients of the input and perturbed polynomials. However, at present, our algorithm can't deal with the polynomials with complex coefficients or the global minimal perturbations being complex. Notice that our algorithm also can not

deal with the case $k = d_1$, because in that case the Bezout matrix B_k is not defined.

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