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Linear Ordinary Differential Equations Satisfied by Modular Forms

Xiaolong Ji
Department of Foundation Courses
Jiyuan Vocational and Technical College
Henan 454650, P.R. China
E-mail: jixiaolong208@sohu.com
Yujie Ma
Key Laboratory of Mathematics Mechanization
Chinese Academy of Sciences
Beijing 100080, P.R. China
E-mail: yjma@mmrc.iss.ac.cn

Abstract. We present an efficient algorithm to calculate the linear homogeneous ordinary differential equations satisfied by modular forms.

1. Introduction

Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$ commensurable with $SL_2(\mathbb{Z})$. In the theory of modular forms, P. F. Stiller's result asserts that if $t(\tau)$ is a non-constant meromorphic modular function of weight zero and $F(\tau)$ a meromorphic modular form of weight k with respect to a multiplier system of Γ, then F, τF, $\cdots$, $\tau^k F$, as functions of t, are linearly independent solutions of a $(k+1)$-st order linear homogeneous ordinary differential equation with algebraic functions of t as coefficients. In [5], the author presents a new proof of this result by using the theory of modular functions and presents the following theorems.

Theorem 1.1 *Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$ commensurable with $SL_2(\mathbb{Z})$. Suppose that $t = t(q)$ is a non-constant (meromorphic) modular function invariant under Γ, and $F(t) = F(t(q))$ is a (meromorphic) modular form of weight k on the group Γ with respect to a multiplier system χ. Then the functions $F(t)$, $\tau F(t)$, $\cdots$, $\tau^k F(t)$ are linearly independent solutions of a $(k+1)-st$ order linear differential equation of the form*

$$D_t^{k+1}F + r_k(t)D_t^kF + r_{k-1}(t)D_t^{k-1}F + \cdots + r_0(t)F = 0, \tag{1}$$

where $D_t = t\frac{d}{dt}$ and $r_m(t)$ are algebraic functions of t.

Theorem 1.2 *Let t and F be given in Theorem 1.1, and suppose that F satisfies the $(k+1)$-st order linear differential equation*

$$D_t^{k+1}F + r_k(t)D_t^kF + r_{k-1}(t)D_t^{k-1}F + \cdots + r_0(t)F = 0, \tag{1}$$

where $r_m(t)$ are algebraic functions of t. Then the function $Q(t)$ in the Schwarzian differential equation

$$2Q(t)\left(\frac{dt}{d\tau}\right)^2 + \{t,\tau\} = 0$$

satisfied by t is equal to

$$Q(t) = \frac{1 + c_1 r_{k-1} + c_2 t r_k' + c_3 r_k^3}{4t^2},$$

where c_m are absolute constants depending only on k and $\{t,\tau\}$ denote the Schwarzian derivative of t w.r.t. τ.

In this paper, we present an efficient algorithm to calculate the linear homogeneous ordinary differential equations satisfied by a given modular forms.

This paper is organized as follows. In Section 2, we introduce the modular form and Schwarzian differential equation. In Section 3, we present the explicit formula of $r_k(t)$ and $r_{k-1}(t)$ by $p_1(t)$ and $p_2(t)$, and hence we get the explicit formula of $Q(t)$ in terms of $r_k(t)$, $r_{k-1}(t)$ and their derivatives. In Section 4, we present and efficient algorithm to calculate the linear homogeneous ordinary differential equations satisfied by a given modular forms.

2. Preliminary

Let F_1 and F_2 be two linearly independent solutions of the following linear differential equation

$$F'' + p_1(t)F'(t) + p_2(t)F = 0.$$

Define a variable $\tau = F_2(t)/F_1(t)$. It is well known that

$$F_1^2 = W\frac{dt}{d\tau},$$

where $W = F_1F_2' - F_1'F_2$ is the Wronskian, and Abel's identity states that

$$W = c\exp\left\{\int^t p_1(u)du\right\}$$

for some constant c depending only on the choices of F_1 and F_2. From the above two equalities, we can deduce that

$$F_1' = \frac{F_1}{2}(\frac{\ddot{t}}{\dot{t}^2} - p_1),$$

$$F_1'' = \frac{F_1}{4}\left\{p_1^2 - 2p_1\frac{\ddot{t}}{\dot{t}^2} - \frac{3\ddot{t}}{\dot{t}^4} - 2p_1' + \frac{2\dddot{t}}{\dot{t}^3}\right\},$$

where f' and $\dot{f}$ denote the derivatives of f with respect to t and τ, respectively. Substituting these two identities to the original differential equation one sees that the function $t(\tau)$ satisfies a non-linear differential equation

$$2Q(t)\left(\frac{dt}{d\tau}\right) + \{t,\tau\} = 0$$

with

$$\{t,\tau\} = \frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}^2}\right),$$

$$Q(t) = \frac{4p_2 - 2tp_1' - p_1^2}{4t^2}.$$

This differential equation is known as the Schwarzian differential equation in the literature, and $\{t,\tau\}$ is called the Schwarzian derivative of $t = t(\tau)$.

In [5], the author stated that give a $(k+1)$-st differential equation (1) satisfied by a modular form F of weight k it is easy to see from Theorem 1.1 that the algebraic functions p_1 and p_2 can be recovered from the first two coefficients r_k and r_{k-1} of the differential equation. On the other hand, the coefficients $r_{k-2}, \cdots, r_0$ are uniquely determined by p_1 and p_2. Thus, the functions r_m, $m = 0, \cdots, k-2$, can be expressed as polynomials of t, r_k, r_{k-1} and their derivatives. And the author presents such relations for the case $k = 2$ explicitly .

The author states that for general k it is not difficult to show that $r_k = k(k+1)p_1/2$ and $r_{k-1} = d_1p_2 + d_2tp_1' + d_3p_1^2$ for certain numerical constants. Moreover, the author asserts that the function $Q(t)$ in the associated Schwarzian differential equation can be written as the following form

$$Q(t) = \frac{1 + c_1r_{k-1} + c_2tr_k' + c_3r_k^2}{4t^2},$$

where c_m's are absolute constants depending only on k.

3. Differential Equations satisfied by modular forms

In this section, we will determined the coefficients in Theorem 1.1 and Theorem 1.2 explicitly.

Lemma 3.1

$$s_{n,n-1} = -p_1 \cdot \frac{n(n-1)}{2}\prod_{j=1}^{n-2}(1 - \frac{j}{k}).$$

Proof: By $D_t^nF = D_t(D_t^{n-1}F)$, we have the following recursive formulae on $s_{n-i,n-i-1}$ for $i = 0, \cdots, n-2$.

$$s_{n,n-1} = (1 - \frac{n-2}{k})s_{n-1,n-2} - p_1 \cdot (n-1)\prod_{j=1}^{n-2}(1 - \frac{j}{k})$$

$$s_{n-1,n-2} = (1 - \frac{n-3}{k})s_{n-2,n-3} - p_1 \cdot (n-2)\prod_{j=1}^{n-3}(1 - \frac{j}{k})$$

$$\cdots$$

Since $s_{2,1} = -p_1$, by elementary calculation and mathematical induction, we have

$$s_{n,n-1} = -p_1 \cdot \frac{n(n-1)}{2}\prod_{j=1}^{n-2}(1 - \frac{j}{k}).$$

Lemma 3.2

$$s_{n,n-2} = p_1^2 \cdot f_k(n) - tp_1' \cdot g_k(n) - p_2 \cdot h_k(n)$$

with

$$f_k(n) = \prod_{j=1}^{n-3}(1-\frac{j}{k})\sum_{m=1}^{n}\left[\frac{(m-1)(m-2)^2}{2}\right]$$

$$g_k(n) = \prod_{j=1}^{n-3}(1-\frac{j}{k})\sum_{m=1}^{n}\left[\frac{(m-1)(m-2)}{2}\right]$$

$$h_k(n) = \prod_{j=1}^{n-3}(1-\frac{j}{k})\sum_{m=1}^{n}\left[(m-1)(1-\frac{m-2}{k})\right]$$

Proof: By $D_t^n F = D_t(D_t^{n-1}F)$, we have the following recursive formulae on $s_{n-i,n-i-2}$ for $i = 0, \cdots, n-2$.

$$\begin{aligned} s_{n,n-2} = \quad & (1-\tfrac{n-3}{k})s_{n-1,n-3} - p_2 \cdot (n-1)\textstyle\prod_{j=1}^{n-2}(1-\frac{j}{k}) \\ & -(n-2)p_1 s_{n-1,n-2} + D_t(s_{n-1,n-2}) \end{aligned}$$

$$\begin{aligned} s_{n-1,n-3} = \quad & (1-\tfrac{n-4}{k})s_{n-2,n-4} - p_2 \cdot (n-2)\textstyle\prod_{j=1}^{n-3}(1-\frac{j}{k}) \\ & -(n-3)p_1 s_{n-2,n-3} + D_t(s_{n-2,n-3}) \end{aligned}$$

$$\cdots$$

Since $s_{2,0} = -p_2$, by elementary calculation and mathematical induction, we then prove the lemma.

Theorem 3.3 *Let F be a modular form satisfying the differential equation* (1)*. Then*

$$r_k = k(k+1)p_1/2$$

and

$$r_{k-1} = d_1 p_2 + d_2 tp_1' + d_3 p_1^2$$

with

$$d_1 = \frac{k^2}{6} + \frac{k}{2} + \frac{1}{3} = \frac{(k+1)(k+2)}{6},$$

$$d_2 = \frac{k^3-k}{6} = \frac{(k-1)k(k+1)}{6},$$

$$d_3 = \frac{k^4}{8} + \frac{k^3}{12} - \frac{k^2}{8} - \frac{k}{12} = \frac{(k-1)k(k+1)(3k+2)}{24}.$$

Corollary 3.4

$$p_1 = \frac{2r_k}{k(k+1)}$$

$$p_2 = \frac{6k(1+k)r_{k-1} + (2+k-3k^2)r_k^2 + 2kt(1-k^2)r_k'}{k(k+1)^2(k+2)}.$$

Proposition 3.5 *The function $Q(t)$ appeared in the associated Schwarzian differential equation can be write as the following form*

$$Q(t) = \frac{1 + c_1 r_{k-1} + c_2 t r'_k + c_3 r_k^2}{4t^2}$$

where

$$c_1 = 24 + 24k,$$
$$c_2 = -12k - 12k^2,$$
$$c_3 = -12k.$$

Note that the description of $Q(t)$ in [5] is problematic.

Proposition 3.6 *The function $Q(t)$ appeared in the associated Schwarzian differential equation can be write as the following form*

$$\begin{aligned} Q(t) &= -\frac{-k^4-4\,k^3+12\,k^2tr'_k-5\,k^2+12\,ktr'_k-2\,k-24\,r_{k-1}k+12\,r_k^2k-24\,r_{k-1}}{4t^2(k+1)^2(k+2)k} \\ &= \frac{k^4+4k^3-12k^2tr'_k+5k^2-12ktr'_k+2k+24r_{k-1}k-12r_k^2k+24r_{k-1}}{4t^2(k+1)^2(k+2)k} \\ &= \frac{(k^4+4k^3+5k^2+2k)-12kt(1+k)r'_k-12kr_k^2+24(1+k)r_{k-1}}{4t^2(k+1)^2(k+2)k} \\ &= \frac{1+c_1r_{k-1}+c_2tr'_k+c_3r_k^2}{4t^2} \\ &= \frac{k^3}{4(k+2)(k+1)^2t^2} + \frac{k^2}{(k+2)(k+1)^2t^2} + \frac{5k}{4(k+2)(k+1)^2t^2} \\ &\quad -\frac{3kr'_k}{t(k+1)^2(k+2)} - \frac{3r'_k}{t(k+1)^2(k+2)} + \frac{1}{2(k+2)(k+1)^2t^2} \\ &\quad -\frac{3r_k^2}{(k+2)(k+1)^2t^2} + \frac{6r_{k-1}}{(k+2)(k+1)^2t^2} + \frac{6r_{k-1}}{(k+2)(k+1)^2t^2k}. \end{aligned}$$

where

$$c_1 = \frac{24}{k(k+1)(k+2)},$$
$$c_2 = \frac{-12}{(k+1)(k+2)},$$
$$c_3 = \frac{-12}{(k+2)(k+1)^2}.$$

4. Algorithm

Following the proof of Theorem 1.1 of [5], we have the following theorem on the algorithm for the explicit formula of the linear homogeneous ordinary differential equations satisfied by a given modular form.

Theorem 4.1 *Let Γ be a discrete subgroup of $SL(2,\mathbb{R})$ commensurable with $SL(2,\mathbb{Z})$. Suppose that $t = t(q)$ is a non-constant (meromorphic) modular function invariant under Γ. Then the functions $F(t)$, $\tau F(t)$, $\cdots$, $\tau^k F(t)$ are linearly independent solutions of a $(k+1)-st$ order linear differential equation of the form*

$$D_t^{k+1}F + r_k(t)D_t^kF + r_{k-1}(t)D_t^{k-1}F + \cdots + r_0(t)F = 0, \tag{1}$$

where $r_m(t)$ are algebraic functions of t. The equation can be explicitly obtained by a polynomial time algorithm.

Proof: First define

$$D_t = t\frac{d}{dt}, \qquad D_q = q\frac{d}{dq}$$

with $q = \exp 2\pi\sqrt{-1}\tau$ for $Im\tau > 0$. Set

$$G_1 = \frac{D_q t}{t}, \qquad G_2 = \frac{D_q F}{F}.$$

We have

$$D_t F = t\frac{D_q F}{D_q t} = t\frac{FG_2}{tG_1} = F\frac{G_2}{G_1}$$

and

$$D_t\frac{G_2}{G_1} = \frac{t}{G_1}\frac{D_q G_2}{D_q t} - t\frac{G_2}{G_1^2}\frac{D_q G_1}{D_q t} = \frac{D_q G_2}{G_1^2} - \frac{G_2}{G_1^3}D_q G_1.$$

By Lemma 1 of [5], the functions $D_q G_1 - 2G_1G_2/k$ and $D_q G_2 - G_2^2/k$ are meromorphic modular forms of weight 4, and so is G_1^2. Therefore, the following functions

$$p_1(t) = \frac{D_q G_1 - 2G_1G_2/k}{G_1^2}$$

and

$$p_2(t) = \frac{D_q G_2 - G_2^2/k}{G_1^2}$$

are rational functions of t.

Thus, we have

$$D_t\frac{G_2}{G_1} = -\frac{G_2^2}{kG_1^2} - p_1\frac{G_2}{G_1} - p_2.$$

We then can compute the higher order derivatives of F inductively,

$$D_t^2 F = D_t(D_t F) = D_t(F\frac{G_2}{G_1}) = F\left\{(1-\frac{1}{k})\frac{G_2^2}{G_1^2} - p_1\frac{G_2}{G_1} - p_2\right\} \tag{7}$$

and

$$\frac{G_2^2}{G_1^2} = \frac{k}{k-1}\left\{\frac{D_t^2 F}{F} + p_1\frac{G_2}{G_1} + p_2\right\} = \frac{k}{k-1}\left\{\frac{D_t^2 F}{F} + p_1\frac{D_t F}{F} + p_2\right\}$$

$$\begin{aligned} D_t^3 F &= D_t(D_t^2 F) = F\tfrac{G_2}{G_1}\left\{(1-\tfrac{1}{k})\tfrac{G_2^2}{G_1^2} - p_1\tfrac{G_2}{G_1} - p_2\right\} \\ &\quad +F\left\{2(1-\tfrac{1}{k})\tfrac{G_2}{G_1}D_t\tfrac{G_2}{G_1} - tp_1'\tfrac{G_2}{G_1} - p_1 D_t\tfrac{G_2}{G_1} - tp_2'\right\} \\ &= F\left\{(1-\tfrac{1}{k})(1-\tfrac{2}{k})\tfrac{G_2^3}{G_1^3} + (\tfrac{3}{k}-3)p_1\tfrac{G_2^2}{G_1^2}\right. \\ &\quad \left. +[(2/k-3)p_2 - tp_1' + p_1^2]\tfrac{G_2}{G_1} + p_1p_2 - tp_2'\right\} \end{aligned}$$

and

$$\begin{aligned} \tfrac{G_2^3}{G_1^3} &= \tfrac{k^2}{(k-1)(k-2)}\left\{\tfrac{D_t^3 F}{F} - (\tfrac{3}{k}-3)p_1\tfrac{G_2^2}{G_1^2} - [(2/k-3)p_2 - tp_1' + p_1^2]\tfrac{G_2}{G_1} - p_1p_2 + tp_2'\right\} \\ &= \tfrac{k^2}{(k-1)(k-2)}\left\{\tfrac{D_t^3 F}{F} - (\tfrac{3}{k}-3)p_1\tfrac{k}{k-1}\left\{\tfrac{D_t^2 F}{F} + p_1\tfrac{D_t F}{F} + p_2\right\}\right. \\ &\quad \left. -[(2/k-3)p_2 - tp_1' + p_1^2]\tfrac{D_t F}{F} - p_1p_2 + tp_2'\right\} \\ &= \tfrac{k^2}{(k-1)(k-2)}\left\{\tfrac{D_t^3 F}{F} + 3p_1\tfrac{D_t^2 F}{F} - [(2/k-3)p_2 - tp_1' - 2p_1^2]\tfrac{D_t F}{F} + 2p_1p_2 + tp_2'\right\} \end{aligned}$$

In general, the n-th derivative of F takes the following form

$$D_t^n F = F\left\{\frac{G_2^n}{G_1^n}\prod_{j=1}^{n-1}(1-j/k) + s_{n,n-1}\frac{G_2^{n-1}}{G_1^{n-1}} + s_{n,n-2}\frac{G_2^{n-2}}{G_1^{n-2}} + \cdots + s_{n,0}\right\},$$

where $s_{n,j}$ are polynomials of t, p_1, p_2 and their derivatives which can be explicitly determined by chain rule of derivatives inductively.

When $n = k+1$, the term of $D_t^{k+1}F$ involving $\frac{G_2^{k+1}}{G_1^{k+1}}$ vanishes, and we have

$$D_t^{k+1} F = F\left\{s_{k+1,k}\frac{G_2^k}{G_1^k} + s_{k+1,k-1}\frac{G_2^{k-1}}{G_1^{k-1}} + \cdots + s_{k+1,0}\right\},$$

Therefore, for $1 \le n \le k$, the $\frac{G_2^n}{G_1^n}$'s are linear sum of lower order derivatives of F with coefficients as polynomials of t, p_1, p_2 and their derivatives, and henceforth, D_t^{k+1} is equal to a linear sum of lower order derivatives of F with rational functions of t as coefficients. ∎

Since the rational functions $p_1(t)$ and $p_2(t)$ can be calculated explicitly by Magma, we have the following algorithm.

Algorithm:

Input: $F(t) = F(t(q))$ is a (meromorphic) modular form of weight k on the group Γ with respect to a multiplier system χ.

Output: The coefficients of the $(k+1)$–order linear homogeneous ordinary differential equation satisfied by F.

1. Compute $G_1 = \frac{D_q t}{t}$ and $G_2 = \frac{D_q F}{F}$.
2. Compute $p_1(t) = \frac{D_q G_1 - 2G_1G_2/k}{G_1^2}$ and $p_2(t) = \frac{D_q G_2 - G_2^2/k}{G_1^2}$.
3. Compute $D_t^i F$ for i from 1 to $k+1$.
4. Compute $(\frac{G_2}{G_1})^i$ for i from 2 to k. Substitute the $(\frac{G_2}{G_1})^i$'s into the equation involved $D_t^{k+1}F$ to obtain the linear ordinary differential equations.
5. Transform the resulted equation by taking $D_t = t\frac{d}{dt}$.

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