Structured Condition Numbers of Sylvester Matrices 1)

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Abstract. We investigate the structured normwise and componentwise condition numbers for solving linear systems with Sylvester structure. Numerical examples show that the Sylvester structured condition numbers can be much smaller than the unstructured condition numbers.

Here and hereafter, we denote \( \| \cdot \| \) the spectral norm \( \| \cdot \|_2 \) and \( \| \cdot \|_\infty \) the infinity norm of its arguments. If \( A \) is a matrix, we write \( |A| = (|A_{ij}|) \), where \( A_{ij} \) is the \((i,j)\)th entry of \( A \). Let \( M_{n}^{Sylv}(\mathbb{R}) \) denote the set of \( n \times n \) real matrices with Sylvester structure.

1. Normwise structured perturbations

The structured normwise condition number for the linear system \( Ax = b \) with \( x \neq 0 \) is defined as

\[
\kappa_{A,Ax}^{\text{Struct}}(A, x) := \lim_{\varepsilon \to 0} \sup \left\{ \frac{\|\Delta x\|}{\varepsilon \|x\|} : (A + \Delta A)(x + \Delta x) = b + \Delta b, \Delta A \in M_n^{\text{Struct}}(\mathbb{R}), \right. \\
\left. \Delta b \in \mathbb{R}^n, \|\Delta A\| \leq \varepsilon \|A\|, \|\Delta b\| \leq \varepsilon \|b\| \}\right. .
\]

(1)

When \( M_n^{\text{Struct}}(\mathbb{R}) = M_n(\mathbb{R}) \), the unstructured condition number is denoted by \( \kappa_{A,Ax}(A, x) \); When \( M_n^{\text{Struct}}(\mathbb{R}) = M_n^{Sylv}(\mathbb{R}) \), we get the Sylvester structured condition number and

\[
\kappa_{A,Ax}^{\text{Sylv}}(A, x) \leq \kappa_{A,Ax}(A, x).
\]

It is well known [3] that

\[
\kappa_{A,Ax}(A, x) = \|A^{-1}\| \|A\| + \frac{\|A^{-1}\| \|b\|}{\|x\|}.
\]

(2)

Suppose \( A \) is the Sylvester matrix of two univariate polynomials \( f \) and \( g \) with degrees \( m \) and \( n \) respectively, then entries of \( A \) depend linearly on the \( m + n + 2 \) coefficients of \( f \) and \( g \).

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Lemma 1.1 Let \( A \in M_{m+n}(\mathbb{R}) \) be defined by the coefficient vector \( p \in \mathbb{R}^{m+n+2} \) of univariate polynomials \( f \) and \( g \). Then

\[
\frac{1}{\sqrt{k}} \|A\| \leq \|p\| \leq 2\|A\|, \quad \text{where} \quad k = \max(m,n).
\]

As shown in [4, 5], given a nonzero vector \( x \in \mathbb{R}^{m+n} \), and a Sylvester structured perturbation \( \Delta A \) defined by the vector \( \Delta p = [z_1, z_2, \ldots, z_{m+n+2}]^T \), the matrix \( \Psi_{x}^{\text{Sylv}} \):

\[
\Psi_{x}^{\text{Sylv}} = \begin{bmatrix}
    x_1 & x_{n+1} \\
    x_2 & \ddots & x_{n+2} & \ddots \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    x_n & \cdots & x_{m+n} & x_{n+1} \\
    \cdots & \cdots & \cdots & \cdots & \ddots \\
    x_n & \cdots & \cdots & \cdots & x_{m+n}
\end{bmatrix}_{m+n+1}
\]

(4)

satisfies

\[ \Delta A \cdot x = \Psi_{x}^{\text{Sylv}} \cdot \Delta p. \]

The analysis in [6] can be extended to obtain a computable bound for the Sylvester structured normwise condition number:

Theorem 1.2 Let nonsingular \( A \in M_{m+n}(\mathbb{R}) \) and \( 0 \neq x \in \mathbb{R}^{m+n} \) be given, we have

\[
\kappa_{A,x}^{\text{Sylv}}(A, x) = \frac{\gamma \|A^{-1}\Psi_{x}^{\text{Sylv}}\||A| + \|A^{-1}\|\|Ax\|}{\|x\|},
\]

where \( \frac{1}{\sqrt{2}} \leq c \leq 1 \), \( \frac{1}{\sqrt{k}} \leq \gamma \leq 2 \) and \( k = \max(m,n) \).

Corollary 1.3 Let nonsingular \( A \in M_{m+n}(\mathbb{R}) \) and \( 0 \neq x \in \mathbb{R}^{m+n} \) be given. Then

\[
\frac{\kappa_{A,Ax}^{\text{Sylv}}(A, x)}{\kappa_{A,Ax}(A, x)} \geq 2^{-1/2} \frac{1}{\sqrt{k}} \|A^{-1}\| \frac{\sigma_{\min}(\Psi_{x}^{\text{Sylv}})}{\|x\|} \|A\| + \|A^{-1}\| \frac{\|Ax\|}{\|x\|}. 
\]

(6)

According to Corollary 1.3, for a linear system whose \( \tau(x) := \sigma_{\min}(\Psi_{x}^{\text{Sylv}}) \) is not so small, the structured condition number can not be much smaller than the unstructured one. We test the minimum and median of \( \tau(x) \) for some \( 10^4 \) samples of \( x \) with entries with the standard normal distribution. As shown in the table 1., small \( \tau(x) \) are possible but seem to be rare.

The matrix \( \Psi_{x}^{\text{Sylv}} \) is singular when \( x_1 = x_{n+1} = 0 \). Even if \( x_1 x_{n+1} \neq 0 \), we can still construct a singular \( \Psi_{x}^{\text{Sylv}} \) by forming the Sylvester matrix for \( x^2 u, x^2 v \), where \( u \) and \( v \) have a non-trivial GCD,

\[
u = x_1 x^{n-1} + x_2 x^{n-2} + \ldots + x_{n}, \quad \nu = x_{n+1} x^{m-1} + x_{n+2} x^{m-2} + \ldots + x_{m+n},
\]

(7)
Table 1. Minimum value and median of $\tau(x) = \sigma_{\min}(\Psi_{\text{Sylv}}^x)/\|x\|

<table>
<thead>
<tr>
<th>k</th>
<th>minimum $\tau(x)$</th>
<th>median $\tau(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$5.32e-6$</td>
<td>$1.30e-1$</td>
</tr>
<tr>
<td>20</td>
<td>$1.55e-5$</td>
<td>$5.52e-2$</td>
</tr>
<tr>
<td>50</td>
<td>$6.36e-6$</td>
<td>$1.84e-2$</td>
</tr>
<tr>
<td>100</td>
<td>$3.85e-5$</td>
<td>$8.01e-3$</td>
</tr>
</tbody>
</table>

and deleting the last two rows of zero entries. However, we note that it is possible that the structured normwise condition number is near to the unstructured normwise condition number in the case $\tau(x) = 0$.

A method based on solving the linear system $Ax = (0, \ldots, 0, 1)^T$ with $A$ being the Sylvester matrix of two univariate polynomials $f$ and $g$ is given in [1] to test whether $f$ and $g$ are relatively prime. We construct 100 Sylvester matrices and compute the structured and unstructured normwise condition numbers of solving those linear systems. The random polynomials $f, g$ are generated as follows:

$$f = f_1 h, \quad g = g_1 h + \varepsilon,$$

where $0.05 \leq \varepsilon \leq 0.5$, $f_1, g_1$ and $h$ are randomly generated polynomials with random integer coefficients in $[-5, 5]$, $\deg h = 4$, $\deg f_1 + \deg g_1 = 100$. Among the 100 linear systems, we find 17 ones with $\kappa_{\text{Sylv}}^{\text{Sylv}}(A, Ax) < \frac{\kappa_{\text{A, Ax}}^{\text{A, Ax}}(A, x)}{10^4}$.

**Example 1.4**

$$f = -12x^{54} + 3x^{53} - 15x^{53} - 15x^{51} + \ldots - 27x^{10} - 25x^9 - 10x^8,$$

$$g = 12x^{50} - 3x^{49} + 15x^{47} + \ldots + 5x^7 + 2x^6 + 0.4.$$  

By the formulas (2), (5) we get $5.39e5 \leq \kappa_{\text{A, Ax}}^{\text{Sylv}}(A, x) \leq 7.65e5$, whereas $\kappa_{\text{A, Ax}}(A, x) = 2.71e10$.

2. Componentwise structured perturbations

The normwise structured condition number is not satisfactory all the time. We turn to so-called componentwise analysis and compute componentwise structured condition numbers. The componentwise structured condition number for the linear system $Ax = b$ with $x \neq 0$ is defined as

$$\text{cond}_{A, Ax}^{\text{struct}} := \lim_{\varepsilon \to 0} \sup \left\{ \frac{\|\Delta x\|_{\infty}}{\|x\|_{\infty}} : (A + \Delta A)(x + \Delta x) = b + \Delta b, \Delta A \in M_n^{\text{struct}}(\mathbb{R}), \Delta b \in \mathbb{R}^n, \|\Delta A\| \leq \varepsilon |A|, |\Delta b| \leq \varepsilon |b| \right\}.$$  

When $M_n^{\text{struct}}(\mathbb{R}) = M_n(\mathbb{R})$, the unstructured componentwise condition number, denoted by $\text{cond}_{A, Ax}(A, x)$ is defined and it was known in [3] that

$$\text{cond}_{A, Ax}(A, x) = \frac{\|A^{-1}\|A\|x\| + |A^{-1}|\|Ax\|_{\infty}}{\|x\|_{\infty}}.$$  

When $M_n^{\text{Struct}}(\mathbb{R}) = M_n^{\text{Sylv}}(\mathbb{R})$, we get the Sylvester structured condition number, $\text{cond}_{A,Ax}^{\text{Sylv}}(A, x)$. Extending Theorem 3.1 in [7] we get:

**Theorem 2.1** For nonsingular $A \in M_{m+n}^{\text{Sylv}}(\mathbb{R})$, $0 \neq x \in \mathbb{R}^{m+n}$, assume that $A$ is defined by the coefficient vector $p \in \mathbb{R}^{m+n+2}$ of univariate polynomials $f$ and $g$, we have

$$\text{cond}_{A,Ax}^{\text{Sylv}}(A, x) = \frac{\|A^{-1}\Psi_x^{\text{Sylv}}\| \|A^{-1}\| \|Ax\|_\infty}{\|x\|_\infty},$$

(11)

where $\Psi_x^{\text{Sylv}}$ is defined as in (4).

**Remark 2.2** The componentwise condition number defined here is also referred to as the mixed condition number (see [2, 8] and references therein). Applying a Kronecker product-based technique described in [8] we can derive a mixed Sylvester structured condition number expression equivalent to (11).

**Example 2.3** [1] Suppose we are given two univariate polynomials:

$$f(x) = x^m, \quad g(x) = \left(\frac{x - 1}{2}\right)^m.$$

Denote $A$ as the Sylvester matrix of $f, g$, we compute the normwise and componentwise condition numbers for solving the linear system $Ax = b_i$, where $b_1 = (0, \ldots, 0, 1)^T$ and $b_2 = (1, 0, \ldots, 0)^T$. We use the closed interval comprised by the lower and upper bounds derived from Theorem 1.2 to give an estimate of $\kappa_{A,Ax}^{\text{Sylv}}(A, x)$. As shown in the following tables, despite of the ill-conditioning in the normwise sense, the componentwise condition numbers are small.

<table>
<thead>
<tr>
<th>Table 2. Condition numbers for $Ax = b_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
</tr>
<tr>
<td>15</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>30</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3. Condition numbers for $Ax = b_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
</tr>
<tr>
<td>15</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>30</td>
</tr>
</tbody>
</table>

The componentwise structured condition number can be much smaller than the unstructured one. We test the componentwise structured and unstructured condition numbers for the same 100 linear systems as used in section 1, we find 20 ones with $\text{cond}_{A,Ax}^{\text{Sylv}}(A, x) < \frac{\text{cond}_{A,Ax}(A, x)}{10^6}$. 
Example 2.4 For the linear system described in Example 1.4, by the formulas (10) and (11) we get \( \text{cond}_{Sylv}^{A,A_2}(A, x) = 11.4 \), whereas \( \text{cond}_{A,A_2}(A, x) = 1.29 \times 10^9 \).

References


