A Criterion for the Similarity of Length-Two Elements in a Noncommutative PID ¹⁾

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Abstract. We present a criterion for the similarity of length-two elements in a noncommutative principal ideal domain. The criterion enables us to develop an algorithm for determining whether B_1A_1 and B_2A_2 are similar, where A_1, A_2, B_1, B_2 are first-order differential (difference) operators. The main step in the algorithm is to find a rational solution of a parametric differential (difference) Risch's equation, which has been well-studied in symbolic integration (summation).

1. Introduction

Similarity is an equivalence relation among ordinary differential (difference) operators. This notion was extensively used by O. Ore in [7] to describe the uniqueness of factorization of differential (difference) operators. Moreover, two differential (difference) operators are similar if and only if there exists a one-to-one correspondence, given by a differential (difference) operator, between their solution spaces. When the solution space of an operator is determined, so are the solution spaces of the similar ones (see [4]).

A general method for determining similarity is as follows. Let L_1 and L_2 be two differential operators of the same order over a differential field k . Then L_1 and L_2 are similar if and only if there exists a nonzero differential operator U over k such that L_1U is right divisible by L_2 , and that U and L_2 have a trivial greatest common right divisor. The operator U is called a solution of the mixed equation

$$L_1Z \equiv 0 \pmod{L_2}.$$

Some applications of differential mixed equations are given in [10]. An efficient method for solving differential mixed equations is developed in [4].

Suppose that L_1 and L_2 are given as products of irreducible factors. It is naturally expected that one can use their factors to determine whether L_1 and L_2 are similar. In this paper, we present a criterion for the similarity of length-two elements in a noncommutative principal ideal domain (abbreviated as PID). Note that a ring of differential (difference) operators is a special instance for PID's. As an application, we develop an algorithm for

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determining whether B_1A_1 and B_2A_2 are similar, where A_1, A_2, B_1, B_2 are first-order differential (difference) operators. Proposition 5.4, the main result in Section 5., may be viewed as a generalization of the criterion for the similarity of degree-one Ore polynomials in [6, page 15]. The input operators L_1 and L_2 are not in factored form in van Hoeij's algorithm. So one would have to expand these two operators to apply the algorithm directly.

The rest of this paper is organized as follows. In Section 2., we present some basic results about noncommutative PID's that will be used later. In Section 3., we study how to determine the similarity of completely reducible elements using their irreducible decompositions. The results in Sections 2. and 3. are well-known, but their proofs are scattered in [5, 6, 7]. So we prove some of the results for a later convenience of reference. Section 4. contains a criterion for the similarity of non-completely-reducible elements with length two (Proposition 4.2); while its applications to Ore polynomials are given in Section 5.. We present a few examples in Section 6., and discuss on future work in Section 7..

2. Preliminaries

A PID is a (noncommutative) ring with a multiplicative unit 1 and without zero divisors, in which all one-sided ideals are principal. Typical examples are the rings of ordinary differential and difference operators, and univariate Ore polynomial rings over a field.

Through this paper, R is a PID. For an element a of R , Ra stands for the left ideal generated by a . We shall work with left ideals Ra and the corresponding factor R/Ra , which is a cyclic left module over R with a generator $1 + Ra$. By symmetry, the results will apply equally well to right ideals. Two nonzero elements a and b in R are said to be *similar* if the left R -modules R/Ra and R/Rb are isomorphic. We denote by $a \sim b$ if a and b are similar. The similarity is an equivalence relation.

A remark is needed about homomorphisms between R/Ra and R/Rb . For an element c in R , the right-hand multiplication by c gives rise to a left-module homomorphism:

$$\begin{aligned} \phi_c : R/Ra &\rightarrow R/Rb \\ r + Ra &\mapsto rc + Rb. \end{aligned}$$

if and only if $R(ac) \subset Rc \cap Rb$. Now, let ψ be a homomorphism from R/Ra to R/Rb with $\psi(1 + Ra) = c + Rb$. Then $\psi = \phi_c$, because R/Ra is generated by $1 + Ra$ over R . In other words, every left module homomorphism from R/Ra to R/Rb is given by a right-hand multiplication. This observation allows us to say that a left module homomorphism ψ from R/Ra to R/Rb is given by $1 + Ra \mapsto c + Rb$. Assume further that a, b, c are nonzero elements of R , and that ϕ_c given above is well-defined. It is straightforward to verify that ϕ_c is surjective if and only if $Rb + Rc = R$; and that ϕ_c is injective if and only if $R(ac) = Rb \cap Rc$.

The next lemma is an ideal-theoretic characterization on similarity of elements, which is derived from [5, page 33].

Lemma 2.1 *Let R be a PID. For two nonzero elements $a, b \in R$, $a \sim b$ if and only if there exists a nonzero element $c \in R$ such that*

$$R(ac) = Rb \cap Rc \text{ and } Rb + Rc = R. \quad (1)$$

Moreover, the element c in the two equalities in (1) can be replaced by any element that is congruent to c modulo Rb .

Proof. By the above remark, $a \sim b$ if and only if (1) holds for some $c \in R$. The last assertion follows from the observation that

$$1 \mapsto c + Rb \quad \text{and} \quad 1 \mapsto \hat{c} + Rb$$

define the same homomorphism, provided that $c - \hat{c}$ is in Rb . $\square$

For nonzero elements $a_1, \dots, a_n$ in a PID R , there exist d and m in R such that

$$Ra_1 + \dots + Ra_n = Rd \quad \text{and} \quad Ra_1 \cap \dots \cap Ra_n = Rm.$$

Note that d and m are nonzero and unique up to a unit left factor, respectively (see [5, page 31]). We call d a *greatest common right-hand divisor* (abbreviated as *gcd*) of $a_1, \dots, a_n$, and m a *least common left-hand multiple* (abbreviated as *lcm*) of $a_1, \dots, a_n$.

A nonzero element a of R is said to be *irreducible* if it is not a unit and has no proper factors, or, equivalently, the left ideal Ra is maximal in R . The following lemma describes a fundamental property of PID's (see [5, Chapter 3, Theorem 5]).

Lemma 2.2 *Let R be a PID. Then any nonzero nonunit element $a \in R$ can be written as a product of irreducible elements. If $a = p_1 \cdots p_n = q_1 \cdots q_\ell$, where the p_i and q_j are irreducible, then $n = \ell$, and the p_i and the q_j may be paired into similar pairs.*

The number n of irreducible factors p_i in $a = p_1 \cdots p_n$ is called the *length* of a and denoted by $\text{length}(a)$. The length of a unit in R is set to be zero. Then

$$\text{length}(ab) = \text{length}(a) + \text{length}(b) = \text{length}(d) + \text{length}(m) \quad (2)$$

for all nonzero $a, b \in R$, where d and m are a *gcd* and an *lcm* of a and b , respectively (see [5, Chapter 3, Theorem 6] and [6, Chapter 1, Proposition 1.2.11]). The above irreducible decomposition of a induces a composition series

$$0 \subset R(p_2 \cdots p_n)/Ra \subset \dots \subset R(p_{n-1}p_n)/Ra \subset Rp_n/Ra \subset R/Ra.$$

The factors of this series are

$$R(p_2 \cdots p_n)/Ra, R(p_3 \cdots p_n)/R(p_2p_3 \cdots p_n), \dots, R(p_n)/R(p_{n-1}p_n), R/Rp_n.$$

These factors are isomorphic to

$$R/Rp_1, R/Rp_2, \dots, R/Rp_{n-1}, R/Rp_n,$$

respectively, because $Rc/R(bc)$ is isomorphic to R/Rb as left modules for all nonzero elements $b, c \in R$, which can be proved by observing that the left module homomorphism $\phi_c : R \rightarrow R/R(bc)$ given by $1 \mapsto c + R(bc)$ has kernel Rb , and maps R to $Rc/R(bc)$.

An easy consequence of the length formula (2) is

Lemma 2.3 *Let a, b and u be nonzero elements in a PID R . If $R(au) = Rb \cap Ru$ and $\text{length}(a) = \text{length}(b)$, then $Rb + Ru = R$.*

Proof. Assume that $Rb + Ru = Rd$ for some $d \in R$. It suffices to show $\text{length}(d) = 0$. By (2), $\text{length}(bu) = \text{length}(b) + \text{length}(u) = \text{length}(d) + \text{length}(au)$, which, together with $\text{length}(a) = \text{length}(b)$, implies that $\text{length}(d) = 0$. $\square$

3. Completely reducible elements

A nonzero element a in R is said to be *completely reducible* if it is a least common left multiple of irreducible elements in R , or, equivalently, the left module R/Ra is isomorphic to a direct sum of finitely many irreducible submodules. In particular, every irreducible element is completely reducible. The irreducible factors of a completely reducible element behave like those of a square-free element in a commutative PID.

Lemma 3.1 *Let a be a completely reducible elements in a PID R . If*

$$Ra = Rp_1 \cap \cdots \cap Rp_s = R(q_1 \cdots q_t),$$

where $p_1, \dots, p_s$ and $q_1, \dots, q_t$ are irreducible elements in R , and s is minimal, then $s = t$ and the p_i and q_j are similar in pairs.

Proof. The left ideals Rp_i and $Rp_1 \cap \cdots \cap Rp_{i-1} \cap Rp_{i+1} \cap \cdots \cap Rp_s$ are co-maximal for every i with $1 \leq i \leq s$, because s is minimal. It follows from a standard argument using the Chinese remainder theorem, that R/Ra is isomorphic to the direct sum $\oplus_{i=1}^s R/Rp_i$, which has a composition series:

$$0 \subset R/Rp_1 \subset R/Rp_1 \oplus R/Rp_2 \subset \cdots \subset \oplus_{i=1}^{s-1} R/Rp_i \subset \oplus_{i=1}^s R/Rp_i.$$

From the irreducible decomposition of a , we have a composition series:

$$0 \subset R(q_2 \cdots q_t)/Ra \subset \cdots \subset R(q_{t-1}q_t)/Ra \subset Rq_t/Ra \subset R/Ra.$$

Therefore, $s = t$. Note that the factors of the first composition series are

$$R/Rp_1, \dots, R/Rp_s,$$

while those of the second are isomorphic to

$$R/Rq_1, R/Rq_2, \dots, R/Rq_{s-1}, R/Rq_s,$$

respectively. The p_i and q_j are similar in pairs by the Jordan-Hölder Theorem. $\square$

The next proposition enables us to determine the similarity of two completely reducible elements by their irreducible decompositions.

Proposition 3.2 *Let a and b be nonzero elements of a PID R . Assume that $a = p_1 \cdots p_s$ and $b = q_1 \cdots q_t$, where the p_i and q_j are irreducible. If a and b are similar, then $s = t$ and the p_i and q_j are similar in pairs. The converse holds, provided that both a and b are completely reducible.*

Proof. If a and b are similar, then the two irreducible decompositions induce two equivalent composition series. Therefore, $s = t$ and the p_i and q_j are similar in pairs.

Assume that both a and b are completely reducible. If $s = t$ and the p_i and q_j are similar in pairs, then, by Lemma 3.1, there exist irreducible elements $p'_1, \dots, p'_s$, and $q'_1, \dots, q'_s$ such that

- R/Ra and R/Rb are isomorphic to $\oplus_{i=1}^s R/Rp'_i$ and $\oplus_{i=1}^s R/Rq'_i$, respectively;

- $p_i \sim p'_i$ and $q_i \sim q'_i$ for all i with $1 \leq i \leq s$.

Therefore, a and b are similar. $\square$

To study the similarity among length-two elements, we need to determine whether a product of two irreducible elements is completely reducible. The following proposition is a generalization of the mixed-equation-based criterion on whether a differential operator is a lcm of two differential operators whose gcd is trivial (see [4]).

Proposition 3.3 *Let a, b be two nonzero elements in a PID R . There exists a nonzero element c in R such that*

$$R(ba) = Ra \cap Rc \quad \text{and} \quad Ra + Rc = R \quad (3)$$

if and only if there exists a nonzero element f in R such that $af - 1$ is in Rb .

Proof. Assume that there exists a nonzero element c in R such that (3) holds. Then the map $\phi_a : R/Rb \rightarrow R/Rc$ given by $1 + Rb \mapsto a + Rc$ is a well-defined left module isomorphism by Lemma 2.1. Assume that ϕ_a^{-1} is given by $1 + Rc \mapsto f + Rb$. Then

$$1 + Rb = \phi_a^{-1} \circ \phi_a(1 + Rb) = \phi_a^{-1}(a + Rc) = af + Rb.$$

Thus, $af - 1$ is in Rb .

Conversely, assume that there exists a nonzero element f in R such that $af - 1$ is in Rb . Then the left module homomorphism $\psi : R \rightarrow R/Rb$ given by $1 \mapsto f + Rb$ is surjective, because $\psi(a) = 1 + Rb$. Its kernel $\ker(\psi)$ is not 0, since $ba \in \ker(\psi)$. Let c be a generator of $\ker(\psi)$. Then the induced homomorphism $\bar{\psi} : R/Rc \rightarrow R/Rb$ given by $1 + Rc \mapsto f + Rb$ is an isomorphism, that is, c is similar to b . Since

$$\bar{\psi}(a + Rc) = 1 + Rb,$$

the isomorphism $\bar{\psi}^{-1} : R/Rb \rightarrow R/Rc$ is given by $1 + Rb \mapsto a + Rc$. So (3) holds by the remark before Lemma 2.1. $\square$

4. Non-completely reducible elements

Recall that, for a nonzero element a in R , the *maximal completely reducible (right-hand) divisor* of a is defined to be the lcm m of all the right-hand irreducible divisors of a . In other words, Rm/Ra is the intersection of all maximal left submodules in R/Ra , that is, Rm/Ra is the Jacobson radical of R/Ra (see [1, page 173]).

The next proposition describes a relation between two similar reducible elements and their factors.

Proposition 4.1 *Let R be a PID, and p_1, p_2, q_1 and q_2 be nonzero and nonunit elements in R . Suppose that $\text{length}(p_1) = \text{length}(p_2)$ and $\text{length}(q_1) = \text{length}(q_2)$. Then*

- (i) *If there exist nonzero elements u and v in R such that*

$$R(p_1u) = Rp_2 \cap Ru, \quad p_1u = vp_2 \quad \text{and} \quad R(q_1v) = Rq_2 \cap Rv, \quad (4)$$

then q_1p_1 and q_2p_2 are similar.

(ii) If q_1p_1 and q_2p_2 are similar, and p_1, p_2 are maximal completely reducible divisors of q_1p_1, q_2p_2 , respectively. Then the three equalities in (4) hold. In addition, we have $p_1 \sim p_2$, and $q_1 \sim q_2$.

Proof. Assume that there exist nonzero u and v in R such that the three equalities in (4) hold. By Lemma 2.1, it suffices to show that

$$R(q_1p_1u) = R(q_2p_2) \cap Ru \quad \text{and} \quad R(q_2p_2) + Ru = R. \quad (5)$$

We compute

$$\begin{aligned} R(q_2p_2) \cap Ru &= R(q_2p_2) \cap Ru \cap Rp_2 \\ &= R(q_2p_2) \cap R(p_1u) \quad (\text{by the first equality in (4)}) \\ &= R(q_2p_2) \cap R(vp_2) \quad (\text{by the second equality in (4)}) \\ &= R(q_1vp_2) \quad (\text{by the third equality in (4)}) \\ &= R(q_1p_1u) \quad (\text{by the first equality in (4)}). \end{aligned}$$

The first equality in (5) holds. This equality and Lemma 2.3 yield the second in (5).

We now assume that p_1 and p_2 are maximal completely reducible divisors of q_1p_1 and q_2p_2 , respectively. Put $I_1 = R(q_1p_1)$ and $I_2 = R(q_2p_2)$. If $q_1p_1 \sim q_2p_2$, then there is an isomorphism ϕ from R/I_1 to R/I_2 . Assume that $\phi(1 + I_1) = u + I_2$ with $u \in R$. Then the two equalities in (5) hold by Lemma 2.1. Note that Rp_1/I_1 and Rp_2/I_2 are the Jacobson radicals of R/I_1 and R/I_2 , respectively. So $\phi(Rp_1/I_1) = Rp_2/I_2$ ([1, Proposition 4.3.6]), which induces an isomorphism from $(R/I_1)/(Rp_1/I_1)$ to $(R/I_2)/(Rp_2/I_2)$. As a direct consequence of the third isomorphism theorem, there is an isomorphism from R/Rp_1 to R/Rp_2 that maps $1 + Rp_1$ to $u + Rp_2$. Hence, $R(p_1u) = Rp_2 \cap Ru$ by Lemma 2.1. Let v be the element in R such that $p_1u = vp_2$. It remains to verify the third equality in (4). We compute

$$\begin{aligned} R(q_1vp_2) &= R(q_1p_1u) \quad (\text{by the second equality in (4)}) \\ &= R(q_2p_2) \cap Ru \quad (\text{by the first equality in (5)}) \\ &= R(q_2p_2) \cap Ru \cap Rp_2 \\ &= R(q_2p_2) \cap R(p_1u) \quad (\text{by the first equality in (4)}) \\ &= R(q_2p_2) \cap R(vp_2) \quad (\text{by the second equality in (4)}). \end{aligned}$$

Hence, $R(q_1vp_2) = R(q_2p_2) \cap R(vp_2)$. Removing p_2 yields the third equality in (4).

From the first and third equalities in (4) and Lemma 2.3 it follows that

$$Rp_2 + Ru = R = Rq_2 + Rv.$$

The last assertion holds by Lemma 2.1. □

In practice, it is not easy to compute the Jacobson radical of a left module. However, Proposition 4.1 can be applied to the following simple situation. Let p_1, p_2, q_1, q_2 be irreducible elements in a PID R . To determine whether q_1p_1 and q_2p_2 are similar, we make the following case distinction:

1. If one of the $q_i p_i$ is completely reducible, while the other is not, then they are not similar, because their corresponding modules are not isomorphic.

2. If both of them are completely reducible, then they are similar if and only if the factors are similar in pairs by Proposition 3.2.
3. If neither of them is completely reducible, then we use the next conclusion.

Proposition 4.2 *Let R be a PID, p_1, p_2, q_1 and $q_2 \in R$ be irreducible. Suppose that neither q_1p_1 nor q_2p_2 is completely reducible. Then $q_1p_1 \sim q_2p_2$ if and only if the three equalities in (4) hold.*

Proof. According to Proposition 4.1, it suffices to show that p_1 (resp. p_2) is the maximal completely reducible divisor of q_1p_1 (resp. q_2p_2). Assume that p'_1 is an irreducible right-hand divisor of q_1p_1 with $Rp'_1 \neq Rp_1$. Then the lclm s of p_1 and p'_1 has length two by Lemma 3.1, and s is a right-hand divisor of q_1p_1 by definition. Since $\text{length}(q_1p_1) = 2$,

$$R(q_1p_1) = Rs = Rp_1 \cap Rp'_1.$$

The element q_1p_1 is completely reducible, a contradiction. Hence, p_1 is the maximal completely reducible divisor of q_1p_1 . The same conclusion holds for p_2 . $\square$

Let q_1, p_1, q_2, p_2 be given in Proposition 4.2. To decide whether $q_1p_1 \sim q_2p_2$, we first decide whether $p_1 \sim p_2$ and $q_1 \sim q_2$. If either does not hold, $q_1p_1 \not\sim q_2p_2$. Otherwise, there exists a pair (u, v) with $uv \neq 0$ such that

$$R(p_1u) = Rp_2 \cap Ru \quad \text{and} \quad R(q_1v) = Rp_2 \cap Rv$$

by Lemma 2.1. There are many such pairs, some of which are described in the next corollary.

Corollary 4.3 *Let p_1, p_2, q_1 and q_2 be the same as those in Proposition 4.2. If the three equalities in (4) hold, then, for any $\hat{u} \in R$ such that $\hat{u} - u \in R(q_2p_2)$, there exists $\hat{v} \in R$ such that*

$$R(p_1\hat{u}) = Rp_2 \cap R\hat{u}, \quad p_1\hat{u} = \hat{v}p_2 \quad \text{and} \quad R(q_1\hat{v}) = Rq_2 \cap R\hat{v}. \quad (6)$$

Proof. By the first part in the proof of Proposition 4.1, we have that

$$R(q_1p_1u) = R(q_2p_2) \cap Ru \quad \text{and} \quad R(q_2p_2) + Ru = R.$$

Then, for every $\hat{u} \in R$ with $\hat{u} - u \in R(q_2p_2)$,

$$R(q_1p_1\hat{u}) = R(q_2p_2) \cap R\hat{u} \quad \text{and} \quad R(q_2p_2) + R\hat{u} = R$$

by Lemma 2.1. Hence, there exists $\hat{v} \in R$ such that the three equalities in (6) hold by the second part of the proof of Proposition 4.1. $\square$

5. Applications to univariate Ore polynomial rings

In this section, we apply the results obtained in Sections 3. and 4. to determine whether two reducible quadratic Ore polynomials are similar, assuming that their irreducible decompositions are given. Such a situation arises when one classifies second-order right-hand divisors of a differential (difference) operator with respect to similarity.

Let k be a commutative field, σ an automorphism of k , and δ a derivation on k with respect to σ , that is, δ is additive and

$$\delta(r_1 r_2) = \sigma(r_1) \delta(r_2) + \delta(r_1) r_2 \quad \text{for all } r_1, r_2 \in k.$$

A univariate Ore polynomial ring $k[\partial; \sigma, \delta]$ over k is the ring of polynomials in ∂ over k with the commutation rule $\partial r = \sigma(r) \partial + \delta(r)$ for any $r \in k$. For example, $\mathbb{Q}(x)[\partial; \mathbf{1}, \delta]$, where $\mathbf{1}$ maps each element to itself, and $\delta = \frac{d}{dx}$, is the ring of differential operators over $\mathbb{Q}(x)$; and $\mathbb{Q}(n)[\partial; \sigma, \mathbf{0}]$, where σ maps $f(n)$ to $f(n+1)$ for all $f(n) \in \mathbb{Q}(n)$, and $\mathbf{0}$ maps everything to zero, is the ring of shift operators over $\mathbb{Q}(n)$. The ring $k[\partial; \sigma, \delta]$ possesses the right and left division algorithms, hence it is a PID ([7, 6]).

The following lemma is immediate from [6, page 15]. Our proof is an easy application of Lemma 2.1.

Lemma 5.1 *Two degree-one Ore polynomials $\partial - a$ and $\partial - b$ in $k[\partial; \sigma, \delta]$ are similar if and only if there exists nonzero element $u \in k$ such that*

$$\delta(u) + b\sigma(u) - au = 0. \quad (7)$$

Proof. Put $A = \partial - a$, $B = \partial - b$ and $R = k[\partial; \sigma, \delta]$. By Lemma 2.1, $A \sim B$ if and only if there exists a nonzero element $u \in k$ such that $R(Au) = RB \cap Ru$, which equals RB . It is straightforward to verify that $R(Au) = RB$ if and only if u satisfies (7). $\square$

When σ is the identity map, (7) says that $\partial - a$ and $\partial - b$ are similar if and only if $a - b$ is a logarithmic derivative. When δ maps everything to zero, the two operators are similar if and only if either $a = b = 0$ or $a/b = \sigma(u)/u$. The former coincides with the similarity of exponential functions, while the latter with hypergeometric terms.

The next lemma tells us how to determine whether a product of two degree-one Ore polynomials is completely reducible.

Lemma 5.2 *Let $A = \partial - a$, $B = \partial - b$ be two operators in $k[\partial; \sigma, \delta]$. Then BA is completely reducible if and only if the equation*

$$\delta(z) + b\sigma(z) - az = 1 \quad (8)$$

has a solution in k . Moreover, $BA = \text{lcm}(A, A - \frac{1}{u})$, provided that $u \in k$ solves (8).

Proof. By Proposition 3.3, BA is completely reducible if and only if there exists r in $k[\partial; \sigma, \delta]$ such that $Ar \equiv 1 \pmod{B}$. Clearly, the right-hand remainder of r by B also satisfies the above congruence when r does. Hence, we may assume further that r belongs to k . Expanding Ar yields (8). The first part of the lemma follows. The last assertion follows from the fact that the right-hand remainder of BA by $A - \frac{1}{u}$ is zero if and only if $u \in k$ is a solution of (8). $\square$

Example 5.3 *If ∂^2 is considered as a differential operator in $\mathbb{Q}(x)[\partial; \mathbf{1}, \frac{d}{dx}]$, then (8) has a solution x . So ∂^2 is completely reducible and $\partial^2 = \text{lcm}(\partial, \partial - \frac{1}{x})$. If ∂^2 is regarded as an element in $k[\partial; \sigma, \mathbf{0}]$, where σ is an automorphism on k , then (8) does not hold. So ∂^2 is not completely reducible in $k[\partial; \sigma, \mathbf{0}]$.*

Lemma 5.2 yields the following algorithm.

Algorithm IsCompletelyReducible. Given two elements a, b in a differential field (k, δ) , decide whether $(\partial - b)(\partial - a)$ is completely reducible in $k[\partial; \mathbf{1}, \delta]$.

1. Determine whether the equation

$$\delta(z) + (b - a)z = 1$$

has a solution in k .

2. If there is a solution, then return true $[(\partial - b)(\partial - a)$ is completely reducible]. Otherwise, false is returned.

One can determine whether the product of two first-order difference operators is completely reducible in a similar manner.

The main result of this section is given in the next proposition.

Proposition 5.4 *Let $A_i = \partial - a_i$ and $B_i = \partial - b_i$ be in $k[\partial; \sigma, \delta]$ for $i = 1, 2$. If neither B_1A_1 nor B_2A_2 is completely reducible, then B_1A_1 is similar to B_2A_2 if and only if there exist u, v and w in k with $uv \neq 0$ such that*

$$\delta(u) + a_2\sigma(u) - a_1u = 0 \tag{9}$$

$$\delta(v) + b_2\sigma(v) - b_1v = 0 \tag{10}$$

$$\delta(w) + b_2\sigma(w) - a_1w + \sigma(u) - v = 0. \tag{11}$$

Proof. Put $R = k[\partial; \sigma, \delta]$. Assume that $B_1A_1 \sim B_2A_2$. By Proposition 4.2, there exist nonzero elements U and V in $k[\partial; \sigma, \delta]$ such that

$$R(A_1U) = RA_2 \cap RU, \quad A_1U = VA_2, \quad R(B_1V) = RB_2 \cap RV. \tag{12}$$

We may further assume that U and V are of degrees at most one by Corollary 4.3. The first equality in (12) implies that $RA_2 + RU = R$ by Lemma 2.3.

Likewise, the third equality implies that $RB_2 + RV = R$. So U and V can be written, respectively, as

$$U = wA_2 + u \quad \text{and} \quad V = \sigma(w)B_2 + v, \quad \text{where } u, v, w \in k \text{ and } uv \neq 0, \tag{13}$$

Note that the leading coefficient of V is equal to $\sigma(w)$ because of the second equality in (12). Consequently, $R(A_1u) = RA_2 \cap Ru = RA_2$ by the first equality in (12) and the last assertion of Lemma 2.1. Therefore, we have

$$A_1u = \sigma(u)A_2. \tag{14}$$

Expanding (14) yields (9). The third equality in (12) implies (10) in the same vein. To verify (11), we observe that $A_1U = (A_1w + \sigma(u))A_2$ by (14). Hence, the second equality in (12) implies $V = A_1w + \sigma(u)$, which, together with the definition of V given in (13), implies that $A_1w + \sigma(u) = \sigma(w)B_2 + v$. Equation (11) follows at once.

Conversely, suppose that there exist u, v and w in k with $uv \neq 0$ such that (9), (10) and (11) hold. Then (14) holds by (9). Consequently, $R(A_1u) = RA_2 \cap Ru$. In the same vein, $R(B_1v) = RB_2 \cap Rv$ by (10). Let

$$U = wA_2 + u \quad \text{and} \quad V = \sigma(w)B_2 + v.$$

Then the first and third equalities in (12) hold by the last assertion of Lemma 2.1. Furthermore, (11) implies that the second equality in (12) holds. Thus, $B_1A_1 \sim B_2A_2$ by Proposition 4.2. $\square$

Corollary 5.5 *Let $A_i = \partial - a_i$ and $B_i = \partial - b_i$ be in $k[\partial; \mathbf{1}, \delta]$ for $i = 1, 2$. If neither B_1A_1 nor B_2A_2 is completely reducible, then $B_1A_1 \sim B_2A_2$ if and only if there exist u, v and w in k with $uv \neq 0$ such that*

$$\delta(u) + (a_2 - a_1)u = 0 \tag{15}$$

$$\delta(v) + (b_2 - b_1)v = 0 \tag{16}$$

$$\delta(w) + (b_2 - a_1)w + u - v = 0. \tag{17}$$

Note that, in the differential case, if $L_1 = \partial^2 + f_1\partial + f_2$, $L_2 = \partial^2 + g_1\partial + g_2$, the mixed equation $L_1U \equiv 0 \pmod{L_2}$ leads to the following system of differential equations

$$\begin{cases} \delta^2(u_1) + (f_1 - 2g_1)\delta(u_1) + 2\delta(u_2) &= F_1u_1 + (g_1 - f_1)u_2 \\ \delta^2(u_2) - 2g_2\delta(u_1) + f_2\delta(u_2) &= F_2u_1 + (g_2 - f_2)u_2, \end{cases} \tag{18}$$

where $U = u_1\partial + u_2$, $F_1 = \delta(g_1) + f_1g_1 - g_1^2 - f_2 + g_2$ and $F_2 = \delta(g_2) + f_1g_2 - g_1g_2$. Corollary 5.5 takes advantage of the factors to decouple this system into two first-order scalar differential equations, and a Risch equation in w . If both (15) and (16) have nonzero solutions in k , we need to solve the parametric Risch equation:

$$\delta(w) + (b_2 - a_1)w = c_2v - c_1u$$

for w and unspecified nonzero constants c_1 and c_2 in k . Note that the parametric Risch equation has been studied intensively due to its fundamental importance in symbolic integration (see, [2, Chapter 7] and the references therein).

Corollary 5.6 *Let $A_i = \partial - a_i$ and $B_i = \partial - b_i$ be in $k[\partial; \sigma, \mathbf{0}]$ for $i = 1, 2$. If neither B_1A_1 nor B_2A_2 is completely reducible, then $B_1A_1 \sim B_2A_2$ if and only if there exist u, v and w in k with $uv \neq 0$ such that*

$$a_2\sigma(u) - a_1u = 0$$

$$b_2\sigma(v) - b_1v = 0$$

$$b_2\sigma(w) - a_1w + \sigma(u) - v = 0. \tag{19}$$

Equation (19) is the difference analog of Risch's equation. An algorithm for computing solutions of (19) in k can be found in [9].

Under the assumption that the solutions of parametric Risch's equation in k are computable, we present an algorithm for determine whether B_1A_1 and B_2A_2 are similar in the differential case.

Algorithm IsSimilar. Given a_1, a_2, b_1, b_2 in a differential field (k, δ) , determine whether

$$(\partial - b_1)(\partial - a_1) \sim (\partial - b_2)(\partial - a_2),$$

where the product is taken in the ring $k[\partial; \mathbf{1}, \delta]$.

[We set $L_1 = (\partial - b_1)(\partial - a_1)$ and $L_2 = (\partial - b_2)(\partial - a_2)$ for brevity.]

1. Determine whether L_1 and L_2 are completely reducible by the algorithm **IsCompletelyReducible**.
2. If one of them is completely reducible but the other is not, then return false [L_1 and L_2 are not similar].
3. If both L_1 and L_2 are completely reducible, then do the following.
  - 3.1. If both $\delta(z) + (a_2 - a_1)z = 0$ and $\delta(z) + (b_2 - b_1)z = 0$ have nonzero solutions in k , then return true [L_1 and L_2 are similar].
  - 3.2. If both $\delta(z) + (b_2 - a_1)z = 0$ and $\delta(z) + (a_2 - b_1)z = 0$ have nonzero solutions in k , then return true [L_1 and L_2 are similar].
  - 3.3. Return false.
4. If neither L_1 nor L_2 is completely reducible, then do the following
  - 4.1. Determine if the equation

$$\delta(z) + (a_2 - a_1)z = 0$$

has a nonzero solution in k . If there is none, then return false [L_1 and L_2 are not similar]. Otherwise, set u to be a solution of the above equation.

- 4.2. Determine if the equation

$$\delta(z) + (b_2 - b_1)z = 0$$

has a nonzero solution in k . If there is none, then return false [L_1 and L_2 are not similar]. Otherwise, set v to be a solution of the above equation.

- 4.3. Determine if there exists $w \in k$ and nonzero constants c_1, c_2 in k such that

$$\delta(w) + (b_2 - a_1)w + c_1u - c_2v = 0.$$

If there exist such w, c_1 and c_2 , then return true [L_1 and L_2 are not similar]. Otherwise false is returned.

The second step of this algorithm is obviously correct. The third step is correct by Proposition 3.2, and the fourth by Corollary 5.5. We remark that this algorithm can be modified to get an operator $U \in k[\partial; \mathbf{1}, \delta]$ such that $L_1U = \text{lcm}(L_2, U)$ if $L_1 \sim L_2$.

Similarly, Corollary 5.6 leads to an algorithm for determining whether two difference operators $(\partial - b_1)(\partial - a_1)$ and $(\partial - b_2)(\partial - a_2)$ in $k[\partial; \sigma, \mathbf{0}]$ are similar. But some special care should be taken when $a_1 = a_2 = 0$ or $b_1 = b_2 = 0$ in the difference case.

6. Examples

We present a few examples illustrating the methods described in Section 5. In some examples, we analyze the behavior of the right-hand divisors of Ore polynomials with respect to similarity; in the others, we compare our method with the method based on mixed equations.

The first example shows that two differential operators may be not similar even if their factors are similar in pairs.

Example 6.1 $R = \mathbb{Q}(x) [\partial; \mathbf{1}, \frac{d}{dx}]$, $A = \partial - x^2$, $B_1 = \partial - \frac{1}{x+1}$, $B_2 = \partial - \frac{1}{x}$. By the algorithm *IsCompletelyReducible*, neither B_1A nor B_2A is completely reducible. Solving the equations

$$\frac{du}{dx} = 0 \quad \text{and} \quad \frac{dv}{dx} + \left(\frac{1}{x} - \frac{1}{x+1} \right) v = 0$$

in $\mathbb{Q}(x)$ yields $u = 1$ and $v = \frac{x+1}{x}$, which implies that $B_1 \sim B_2$. But the equation

$$\frac{dw}{dx} + \left(\frac{1}{x} - x^2 \right) w + c_1 - c_2 \frac{x+1}{x} = 0$$

has no solution w in $\mathbb{Q}(x)$ for any c_1, c_2 in $\mathbb{Q} \setminus \{0\}$. So B_1A and B_2A are not similar.

A differential operator may have infinitely many second-order right-hand divisors that are dissimilar to each other, as given in the next example.

Example 6.2 Let R and A be the same as Example 6.1, and let $L = \partial^2 A$. It is easy to verify that $\left(\partial - \frac{1}{x+c} \right) A$ is a right-hand divisor of L for every $c \in \mathbb{Q}$. These divisors are not completely reducible, because

$$\frac{dz}{dx} + \left(\frac{1}{x+c} - x^2 \right) z = 1$$

has no solution in $\mathbb{Q}(x)$. We show further that these divisors are dissimilar to each other.

Put $P_c = \left(\partial - \frac{1}{x+c} \right) A$ and $P_d = \left(\partial - \frac{1}{x+d} \right) A$ with $c, d \in \mathbb{Q}$. Solving the equations

$$\frac{du}{dx} = 0 \quad \text{and} \quad \frac{dv}{dx} + \left(\frac{1}{x+d} - \frac{1}{x+c} \right) v = 0$$

in $\mathbb{Q}(x)$ yields $u = 1$ and $v = \frac{x+c}{x+d}$. Then $P_c \sim P_d$ if and only if the following equation

$$\frac{dw}{dx} + \left(\frac{1}{x+d} - x^2 \right) w + c_1 - c_2 \frac{x+c}{x+d} = 0 \tag{20}$$

has a solution w in $\mathbb{Q}(x)$ for some c_1, c_2 in $\mathbb{Q} \setminus \{0\}$. Substituting $w = \frac{y}{x+d}$ in (20) yields

$$\frac{dy}{dx} - x^2 y + (c_1 - c_2)x + c_1 d - c_2 c = 0, \tag{21}$$

which has no nonzero solution in $\mathbb{Q}(x)$. If $y = 0$, then $c_1 = c_2$ and, hence, $c = d$. Thus, $P_c \sim P_d$ if and only if $P_c = P_d$.

In the next two examples, we compare our algorithm with a naive use of the method based on mixed equations.

Example 6.3 Let $R = \mathbb{Q}(x) [\partial; \mathbf{1}, \frac{d}{dx}]$, $A_1 = \partial$, $A_2 = \partial + \frac{1}{x}$, and $B = \partial - x$. By the algorithm *IsCompletelyReducible*, BA_2 is completely reducible while BA_1 is not. Hence BA_1 and BA_2 are not similar.

If we solve the mixed equation

$$(B_1A_1)U \equiv 0 \pmod{B_2A_2}$$

by the function *Homomorphisms* in the Maple package *DEtools*, which is of form as (18), we get all the homomorphism ϕ_c from $R/R(B_1A_1)$ to $R/R(B_2A_2)$ given by

$$1 + R(B_1A_1) \mapsto \underbrace{c \left(\partial - \frac{x^2 - 1}{x} \right)}_{L_c} + R(B_2A_2),$$

where c is an arbitrary element in $\mathbb{Q}$. For any nonzero $c \in \mathbb{Q}$, ϕ_c is not an isomorphism, Since $RL_c + R(B_2A_2) = R \left(\partial - \frac{x^2 - 1}{x} \right)$. Consequently, B_1A_1 and B_2A_2 are not similar.

Example 6.4 Let $R = \mathbb{Q}(x) [\partial; \mathbf{1}, \frac{d}{dx}]$,

$$A_1 = \partial + \frac{1}{x}, B_1 = \partial - \frac{1}{x(x+1)}, A_2 = \partial, B_2 = \partial + \frac{1}{x+1} \in R.$$

Neither B_1A_1 nor B_2A_2 is completely reducible. Solving the equations

$$\frac{du}{dx} + \frac{1}{x}u = 0 \quad \text{and} \quad \frac{dv}{dx} - \frac{1}{x}v = 0$$

in $\mathbb{Q}(x)$, we obtain a solution $u = \frac{1}{x}$ and $v = x$, respectively. Next, we decide if

$$\frac{dw}{dx} + \frac{1}{x(x+1)}w + c_1 \frac{1}{x} - c_2x = 0$$

has a solution w in $\mathbb{Q}(x)$ for some c_1, c_2 in $\mathbb{Q} \setminus \{0\}$. Indeed, it has a solution

$$(c_1, c_2, w) = (2, 2, x^2 - x - 2).$$

Hence, $B_1A_1 \sim B_2A_2$.

If we solve the mixed equation

$$(B_1A_1)U \equiv 0 \pmod{B_2A_2}$$

by the function *Homomorphisms* in the Maple package *DEtools*, which is of form as (18), we get all the homomorphism ϕ_{d_1, d_2} from $R/R(B_1A_1)$ to $R/R(B_2A_2)$ given by the rule

$$1 + R(B_1A_1) \mapsto \underbrace{d_1 \frac{(x+1)\partial}{x} + d_2 \left((x+1)(x-2)\partial + \frac{2}{x} \right)}_{L_{d_1, d_2}} + R(B_2A_2)$$

where d_1 and d_2 are arbitrary elements in $\mathbb{Q}$. We need to decide if there exist d_1 and d_2 in $\mathbb{Q}$ such that ϕ_{d_1, d_2} is an isomorphism. Indeed, it is an isomorphism if and only if the gcd of $B_2 A_2$ and L_{d_1, d_2} is trivial, which can be done by a resultant-based method. We have that $\phi_{0,1}$ is an isomorphism.

The next example shows that our algorithm can be applied to operators with coefficients in a monomial extension over $\mathbb{Q}(x)$, thanks to the algorithms developed for Risch's equation (see [8, 2]).

Example 6.5 Let $R = \mathbb{Q}(x, \log(x)) [\partial; \mathbf{1}, \frac{d}{dx}]$,

$$A_1 = \partial, B_1 = \partial - \frac{\log(x) - 1}{x \log(x)}, A_2 = \partial - \frac{1 - \log(x)}{x \log(x)}, B_2 = \partial + \frac{1}{x \log(x)} \in R.$$

Neither $B_1 A_1$ nor $B_2 A_2$ is completely reducible by Algorithm *IsCompletelyReducible*. Solving the equations

$$\frac{du}{dx} + \frac{1 - \log(x)}{x \log(x)} u = 0 \quad \text{and} \quad \frac{dv}{dx} - \frac{1}{x} v = 0$$

in $\mathbb{Q}(x, \log(x))$ yields $u = \frac{x}{\log(x)}$ and $v = x$, respectively. The corresponding parametric Risch's equation

$$\frac{dw}{dx} - \frac{1}{x \log(x)} w + c_1 \frac{x}{\log(x)} - c_2 x = 0$$

has a solution $(c_1, c_2, w) = (1, 2, x^2)$. So $B_1 A_1 \sim B_2 A_2$.

The next example is for difference operators.

Example 6.6 Let $R = \mathbb{Q}(n, n!) [\partial; \sigma, \mathbf{0}]$ where $\sigma(n) = n + 1$, and $\sigma(n!) = (n + 1)!$.

$$A_1 = \partial - \frac{(n-1)n!}{n}, B_1 = \partial - \frac{n}{n+2}, A_2 = \partial - n!, B_2 = \partial - n \in R.$$

Neither $B_1 A_1$ nor $B_2 A_2$ is completely reducible by Lemma 5.2. Solving the equations

$$n! \sigma(u) - \frac{(n-1)n!}{n} u = 0 \quad \text{and} \quad n \sigma(v) - \frac{n}{n+2} v = 0$$

in $\mathbb{Q}(n, n!)$ yields a solution $u = \frac{1}{n-1}$ and $v = \frac{1}{(n+1)!}$, respectively. Next we decide whether the following parameterized linear difference equation

$$n \sigma(w) - \frac{(n-1)n!}{n} w + c_1 \frac{1}{n} - c_2 \frac{1}{(n+1)!} = 0$$

has a solution w in $\mathbb{Q}(n, n!)$ for some c_1, c_2 in $\mathbb{Q} \setminus \{0\}$ (see [9]). Solving this equation yields a solution $(c_1, c_2, w) = (1, 1, \frac{1}{(n-1)n!})$. So $B_1 A_1 \sim B_2 A_2$.

7. Concluding remarks

In this paper, we developed some techniques for determining similarities of Ore polynomials using their irreducible decompositions. A complete algorithm is given for reducible quadratic Ore polynomials. For cubic Ore polynomials, we need to know their maximal completely reducible (right-hand) divisors in order to apply Proposition 4.1. An interesting question is how to determine the maximal completely reducible (right-hand) divisor of a product of irreducible Ore polynomials.

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