Linear Differential Equations under Specialization

December 19, 2024

Let k be an algebraically closed field of characteristic zero and let B be a finitely generated k-algebra that is an integral domain. We consider the following linear differential equation

$$L(y) = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0,\ a_i \in B[x]$$

as a family of linear differential equations parametrized by the variety $X(k) = \mathrm{Hom}_k(B, k)$. Precisely, applying $c \in X(k)$ to the coefficients of each $a_i(x)$, one obtains a specialized differential equation

$$L^c(y) = a_n^c(x)y^{(n)} + a_{n-1}^c(x)y^{(n-1)} + \cdots + a_0^c(x)y = 0.$$

It is natural to ask how the algebraic properties of solutions of $L^c(y) = 0$ vary as c ranges over $X(k)$. For instance, one may ask for which $c \in X(k)$ $L^c(y) = 0$ has a basis of Liouvillian solutions, assuming that $L(y) = 0$ does not have such basis. We call the set of such c the exceptional set of $X(k)$. In this talk, generalizing a result of Hrushovski, we show that the exceptional set is indeed "small" in an appropriate sense. As an application, we prove Matzat's conjecture in full generality: The absolute differential Galois group of a one-variable function field over k, equipped with a non-trivial k-derivation, is the free proalgebraic group on a set of cardinality $|k|$.

This is joint work with Michael Wibmer from University of Leeds, UK.