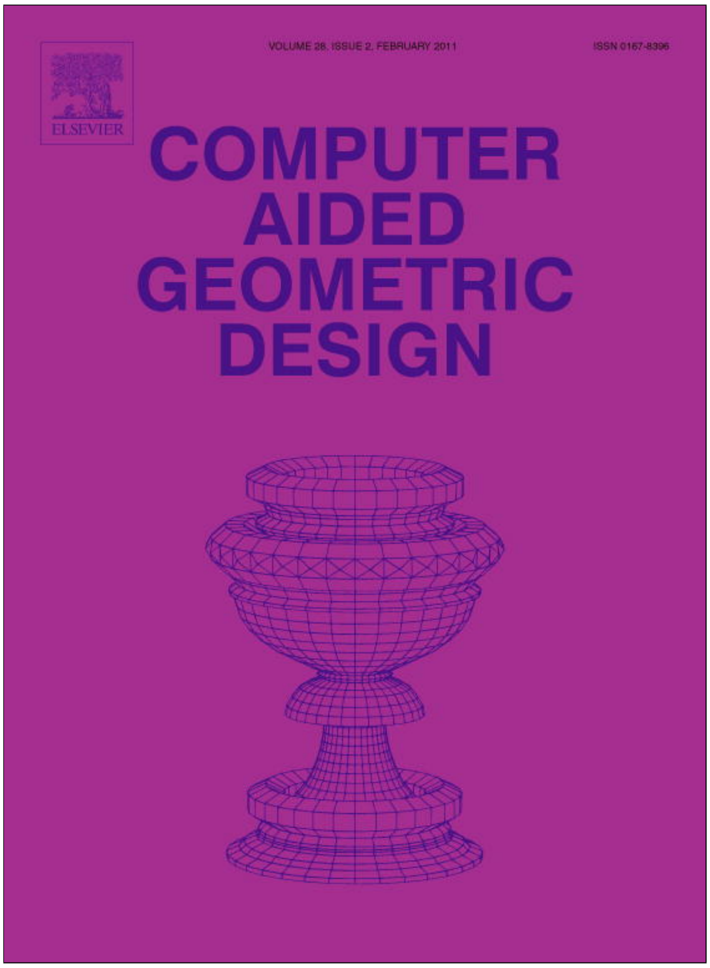
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Collision and intersection detection of two ruled surfaces using bracket method ☆

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ABSTRACT

Collision and intersection detection of surfaces is an important problem in computer graphics and robotic engineering. A key idea of our paper is to use the bracket method to derive the necessary and sufficient conditions for the collision of two ruled surfaces. Then the numerical intersection curve can be characterized. The cases for two bounded ruled surfaces are also discussed.

1. Introduction

Detecting collision of geometric objects has very important application in computer graphics, robotic engineering and computer animation. Some researchers have realized and studied the problem. Boyse (1979) discussed the detection of intersections among objects in fixed positions and collisions among objects moving along specified trajectories. Moore and Wilhelms (1988) divided the problem to a kinematic problem and a dynamic one. An algorithm for detecting intersection between n spheres was presented in Hopcroft et al. (1983). For high precision, the determining conditions for the relationship of geometric models are studied. For an instance, Wang et al. (2001) gave an algebraic condition for the separation of two ellipsoids.

The bracket can be defined as an algebraic tool to represent projective invariants symbolically (Hodge and Kromann, 1953). In this paper, we give the necessary and sufficient conditions for positional relationship of two space lines and two space line segments by bracket method. The representation with brackets offers a simple description for geometric relationship. That is to say, we can directly judge the intersection or the separation according the symbolic formula given in this paper. An advantage of our method is avoiding the redundant discussion and increasing the performance efficiency.

Rational ruled surfaces are an important class of algebraic surfaces which is widely used in computer aided geometry design. According to Chen (2003) and Li et al. (2008), people can find a simplified parametrization from a given ruled surface. In further considerations, it is necessary to determine the geometric relationship of two surfaces, including collision detection and intersection curve analysis. Heo et al. (1999) discussed the intersection of two parametric ruled surfaces, their idea is straightforward but the computation is a little complicated. Fioravanti et al. (2006) gave a way to compute the

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intersection by implicitizing one ruled surface and the topology analysis of a planar curve. The implicitization of the surface was based on the resultant and greatest common division (GCD) computations.

As an application of the bracket method, this paper discusses the positional relationship of two ruled surfaces. Based on some results on lines and line segments, we present the conditions for the collision of two ruled surfaces. We also give a characterization of intersection expressions with two parameters. Some conditions we obtain are similar to that in Heo et al. (1999), since we focus on the same problem. However, we simplify the analysis by dividing the intersection into two parts: overlapping rule lines and ordinary intersection points. And we need not to consider the degenerate cases which were discussed in Heo et al. (1999). The degree of the intersection expression is also decreased by setting certain specialized auxiliary points. Once the expression of the intersection is given, we can compute the numerical intersection using the typical process in Fioravanti et al. (2006) and we take a new algorithm for planar curves topology analysis from Cheng et al. (2009). But we do not need the implicitization process based on resultant and GCD computations as in Fioravanti et al. (2006). Furthermore, we discuss the conditions for collision of two ruled surface segments. This situation has more practical applications and the methods in Heo et al. (1999), Fioravanti et al. (2006) cannot be generalized to cover this problem easily. We reduce the collision detection to solving real solutions of a semi-algebraic system (SAS). If the time parameter is included, we can find the time interval of collision by solving SAS and quantifier elimination.

The rest of this paper is organized as follows. In Section 2, some notations and preliminaries are introduced. In Section 3, we give the conditions for characterizing positional relationship of two lines and line segments. In Section 4, we discuss the conditions for the intersection of two ruled surfaces and compute the numerical intersection. In Section 5, collision detection of two ruled surface segments are discussed. In Section 6, we summarize the paper.

2. Preliminaries

In this section, we introduce the notations needed in our discussion. Let $\mathbb{R}[u]$ be the ring of polynomials in u over the field of real numbers, and $\mathbb{R}[u]^4$ the set of column vectors of size four whose entries belong to $\mathbb{R}[u]$. A *rational ruled surface* is defined as a bi-degree $(n,1)$ tensor product rational surface:

$$(x,y,z)^T = \mathbf{P}(u,s) = \mathbf{P}_1(u)(1-s) + \mathbf{P}_2(u)s, \tag{2.1}$$

where $\mathbf{P}_i(u), i = 1,2,$ are rational curves, called the *directrices* of $\mathbf{P}(u,s)$, and $\mathbf{P}_1 \neq \mathbf{P}_2$. We assume that the rational parametrization (2.1) is nontrivial, that is, it defines a surface $f(x,y,z) = 0$.

For a fixed $u = u_0$, $\mathbf{P}(u_0,s) = \mathbf{P}_1(u_0)(1-s) + \mathbf{P}_2(u_0)s$ is a ruling line of the ruled surface. If another ruled surface $\mathbf{Q}(v,t) = \mathbf{Q}_1(v)(1-t) + \mathbf{Q}_2(v)t$ in the form of (2.1) intersects with $\mathbf{P}(u,s)$, then there exist u_0 and v_0 such that two ruling lines $\mathbf{P}(u_0,s)$ and $\mathbf{Q}(v_0,t)$ are intersected. Then the problem of computing intersection of two ruled surfaces is reduced to that of two moving lines.

We will apply the bracket method to analyze the intersection of lines effectively.

Definition 1. In $\mathbb{R}^n$, $n+1$ points $\mathbf{x}_1,\ldots,\mathbf{x}_{n+1}$ with the coordinates form $\mathbf{x}_i = (x_{i1},\ldots,x_{in})$ for $i = 1,\ldots,n+1$, the bracket $[\mathbf{x}_1\ldots\mathbf{x}_{n+1}]$ is defined as follows:

$$[\mathbf{x}_1\ldots\mathbf{x}_{n+1}] = \begin{vmatrix} x_{11} & \ldots & x_{(n+1)1} \\ \vdots & \ddots & \vdots \\ x_{1n} & \ldots & x_{(n+1)n} \\ 1 & \ldots & 1 \end{vmatrix}.$$

If the number of the points is $m+1 < n+1$, then they will determine a hyperplane with dimension less than $m+1$. In this situation, we will define the bracket of these points in a hyperplane of dimension m.

Definition 2. For a hyperplane $H \subset \mathbb{R}^n$, let its dimension be m, where $m < n$. Then there exist an $n \times n$ orthogonal matrix O_H and a vector $t_H \in \mathbb{R}^n$, for any $m+1$ points $\mathbf{x}_1,\ldots,\mathbf{x}_{m+1} \in H$ with the coordinates form $\mathbf{x}_i = (x_{i1},\ldots,x_{in})^T$ where $i = 1,\ldots,m+1$, such that

$$O_H \begin{pmatrix} x_{11} & \ldots & x_{(m+1)1} \\ \vdots & \ddots & \vdots \\ x_{1n} & \ldots & x_{(m+1)n} \end{pmatrix} + (t_H,\ldots,t_H) = \begin{pmatrix} x'_{11} & \ldots & x'_{(m+1)1} \\ \vdots & \ddots & \vdots \\ x'_{1m} & \ldots & x'_{(m+1)m} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}.$$

In this case, the bracket $[\mathbf{x}_1 \ldots \mathbf{x}_{m+1}]$ is defined as follows:

$$[\mathbf{x}_1 \ldots \mathbf{x}_{m+1}] = \begin{vmatrix} x'_{11} & \ldots & x'_{(m+1)1} \\ \vdots & \ddots & \vdots \\ x'_{1m} & \ldots & x'_{(m+1)m} \\ 1 & \ldots & 1 \end{vmatrix}.$$

As well known, for any $n+1$ points $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1} \in \mathbb{R}^n$ and their convex polytope $\mathrm{Q}(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})$ in $\mathbb{R}^n$, the determinant $|\bar{\mathbf{x}}_1, \ldots, \bar{\mathbf{x}}_{n+1}| = n!V(\mathrm{Q})$ where $V(\mathrm{Q})$ is the signed volume of Q and $\bar{\mathbf{x}}_i = (x_{i1}, \ldots, x_{in}, 1)^T$ is the homogeneous form of $\mathbf{x}_i$. Then $\det(\bar{\mathbf{x}}_1, \ldots, \bar{\mathbf{x}}_{n+1}) = 0$ if and only if $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}$ lie in a hyperplane of dimension less than n.

In Definition 2, the bracket is defined as the signed volume of the transformed points in $\mathbb{R}^n$. Since the transformation determined by O_H and t_H is a rigid transformation, the absolute value of volume is unchanged under this transformation. It means that the bracket gives the signed volume of their convex polytope can only change its sign with the different selection of O_H and t_H. Notice that, any linearly independent points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ define a hyperplane H passing through these points. Then we can give the following lemma.

Lemma 1. *A point $\mathbf{x}_{n+1} \in \mathbb{R}^n$ is on H if and only if $[\mathbf{x}_1 \ldots \mathbf{x}_n \mathbf{x}_{n+1}] = 0$. For two points $\mathbf{x}_{n+1}$ and $\mathbf{x}_{n+2} \in \mathbb{R}^n$ not on H, $[\mathbf{x}_1 \ldots \mathbf{x}_n \mathbf{x}_{n+1}]$ and $[\mathbf{x}_1 \ldots \mathbf{x}_n \mathbf{x}_{n+2}]$ have a different sign when $\mathbf{x}_{n+1}$ and $\mathbf{x}_{n+2}$ are in the different half space divided by H, otherwise, they have the same sign.*

That is to say, the position relationships do not depend on the selection of O_H, t_H. Furthermore, for any points $\mathbf{x}_1, \ldots, \mathbf{x}_n$, let their hyperplane passing through them be H. Once $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are selected and their permutation σ is given, $[\mathbf{x}_{\sigma(1)} \ldots \mathbf{x}_{\sigma(n)} \mathbf{x}_{n+1}]$ and $[\mathbf{x}_{\sigma(1)} \ldots \mathbf{x}_{\sigma(n)} \mathbf{x}_{n+2}]$ have the same sign if $\mathbf{x}_{n+1}$ and $\mathbf{x}_{n+2}$ are in the same half space divided by H, and vice versa.

Definition 3. We define a sign function

$$\begin{aligned} \delta : \mathbb{R} &\to \{-1, 0, 1\}, \\ a &\to a^\delta \end{aligned}$$

where a^δ is the sign value of a.

Since the problem is focused in $\mathbb{R}^3$, throughout the paper we assume that $\mathbf{a} = (a_x, a_y, a_z)^T \in \mathbb{R}^3$ and its homogeneous form is $\bar{\mathbf{a}} = (a_x, a_y, a_z, 1)^T$. By Definition 1 of bracket, the following lemma is true.

Lemma 2. $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2] = 0$ *if and only if* $\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1$ *and* $\mathbf{q}_2$ *are coplanar.*

By Definition 2, generally, one need to find the transformation matrices O_H and T_H. The following two lemmas give alternative methods to compute their brackets in $\mathbb{R}^3$ respectively, which can avoid transforming the coordinates from $\mathbb{R}^3$ to $\mathbb{R}^2$ or $\mathbb{R}$.

Lemma 3. *For any* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ *not collinear and a point* $\mathbf{d}$ *not on the plane* $\mathbf{abc}$*, we have*

$$[\mathbf{abc}] = 1/h_{\mathbf{d}}[\mathbf{abcd}],$$

where $h_{\mathbf{d}}$ is the signed distance from point $\mathbf{d}$ to plane $\mathbf{abc}$.

Proof. Since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are not linear, one has $[\mathbf{abc}] \neq 0$. By Definition 2, under some matrices O_H and t_H, the transformed coordinates of points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are $(a'_x, a'_y, 0)^T$, $(b'_x, b'_y, 0)^T$ and $(c'_x, c'_y, 0)^T$. Then

$$[\mathbf{abc}] = \begin{vmatrix} a'_x & b'_x & c'_x \\ a'_y & b'_y & c'_y \\ 1 & 1 & 1 \end{vmatrix} = 2S_{\mathbf{abc}},$$

where $S_{\mathbf{abc}}$ is the signed area of the triangle composing of points $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. And one has $|S_{\mathbf{abc}}| = \|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})\|$. On the other hand,

$$[\mathbf{abcd}] = \begin{vmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \\ 1 & 1 & 1 & 1 \end{vmatrix} = 6V_{\mathbf{abcd}},$$

where $V_{\mathbf{abcd}}$ is the signed volume of the tetrahedron composing of points $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$. Since $V_{\mathbf{abcd}}=\frac{1}{3}h_{\mathbf{d}}S_{\mathbf{abc}}$, where $h_{\mathbf{d}}$ is the signed distance from point $\mathbf{d}$ to plane $\mathbf{abc}$. Obviously,

$$[\mathbf{abc}]=\frac{1}{h_{\mathbf{d}}}[\mathbf{abcd}]. \qquad \square$$

Actually, if $[\mathbf{abc}]=0$, then $\mathbf{a},\mathbf{b},\mathbf{c}$ are collinear. We define $h_{\mathbf{d}}$ to be the signed distance of an orthogonal line segment from point $\mathbf{d}$ to line $\mathbf{abc}$. Use the notation above, we have $S_{\mathbf{abc}}=0$ and $V_{\mathbf{abcd}}=0$, hence the lemma is still true with the same formula.

Lemma 4. *If $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]$ and $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]$ are zeros simultaneously, then for any $\mathbf{a},\mathbf{b}\in\{\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}_1,\mathbf{q}_2\}$,*

$$[\mathbf{ab}]=\frac{1}{\lambda(\mathbf{a},\mathbf{b})}\Big(\sum(a_u-b_u)\Big)^{\delta}|\mathbf{a}-\mathbf{b}|, \quad u\in\{x,y,z\},$$

where λ is the direction sign function of two points. Letting the orientation of vector $\overrightarrow{\mathbf{p}_2\mathbf{p}_1}$ to be positive, we have $\lambda(\mathbf{p}_1,\mathbf{p}_2)=(p_{1x}+p_{1y}+p_{1z}-p_{2x}-p_{2y}-p_{2z})^{\delta}$. Otherwise, $\lambda(\mathbf{p}_1,\mathbf{p}_2)=(p_{2x}+p_{2y}+p_{2z}-p_{1x}-p_{1y}-p_{1z})^{\delta}$.

Proof. From $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]=[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]=0$ and hence $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2]=0$, we know that the four points $\mathbf{p}_i,\mathbf{q}_i$, $i=1,2$, are collinear. According to the definition of bracket, $[\mathbf{ab}]$ is the directional distance between $\mathbf{a}$ and $\mathbf{b}$. In other words,

$$[\mathbf{ab}]=\frac{1}{\lambda(\mathbf{a},\mathbf{b})}\Bigg(\sum_{u\in\{x,y,z\}}(a_u-b_u)\Bigg)^{\delta}|\mathbf{a}-\mathbf{b}|. \qquad \square$$

3. Brackets for lines and line segments

In application of engineering, curves and surfaces in three-dimensional Euclid space are of particular interest. In this section, we use the bracket algebra method to characterize the position relationship of two lines and two line segments respectively.

3.1. Two space lines

Let P and Q be two space lines defined by $\mathbf{p}_1,\mathbf{p}_2$ and $\mathbf{q}_1,\mathbf{q}_2$ respectively. Then we have the following lemmas.

Lemma 5. *Two space lines P and Q have only one intersection point if and only if $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2]=0$ and $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]\neq[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]$.*

Proof. By Lemma 2, the two lines are coplanar. Note that $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]=[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]$ if and only if P and Q are parallel or overlap each other. The conclusion follows. $\square$

Lemma 6. *Two space lines P and Q overlap with each other if and only if $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]=[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]=0$.*

Proof. The lemma follows from the linear independence of three vectors. $\square$

Theorem 1. *Two space lines P and Q intersect if and only if either $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2]=0$ with $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]\neq[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]$, or, $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]=[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]=0$.*

Proof. It can be summarized by Lemma 5 and Lemma 6. $\square$

From Theorem 1, we get the following formula of the intersection points.

Theorem 2. *Suppose that two space lines P and Q intersect. If $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]\neq[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]$, then they have only one intersection point $\mathbf{i}$ of the form*

$$\mathbf{i}=\frac{[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]\mathbf{q}_1-[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]\mathbf{q}_2}{[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]-[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]}.$$

Otherwise if $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]=[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]=0$, then the intersection is the line P.

Proof. In $\mathbb{R}^3$, for the plane H defined by $\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2$, by Definition 2, there exist a 3×3 orthogonal matrix O_H and a 3×3 matrix t_H, such that

$$O_H \begin{pmatrix} p_{1x} & p_{2x} & q_{1x} \\ p_{1y} & p_{2y} & q_{1y} \\ p_{1z} & p_{2z} & q_{1z} \end{pmatrix} + (t_H, t_H, t_H) = \begin{pmatrix} p'_{1x} & p'_{2x} & q'_{1x} \\ p'_{1y} & p'_{2y} & q'_{1y} \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2] = 0$, we have $O_H(q_{2x}, q_{2y}, q_{2z})^T + t_H = (q'_{2x}, q'_{2y}, 0)^T$. In $H \subset \mathbb{R}^2$, Let $\mathbf{p}'_1 = (p'_{1x}, p'_{1y})^T$, $\mathbf{p}'_2 = (p'_{2x}, p'_{2y})^T$, $\mathbf{q}'_1 = (q'_{1x}, q'_{1y})^T$, $\mathbf{q}'_2 = (q'_{2x}, q'_{2y})^T$. By Grassmann–Cayley algebra (Hodge, 1952), the intersection point $\mathbf{i}'$ of $\mathbf{p}'_1\mathbf{p}'_2$ and $\mathbf{q}'_1\mathbf{q}'_2$ can be regarded as the wedge product of $\mathbf{p}'_1\mathbf{p}'_2$ and $\mathbf{q}'_1\mathbf{q}'_2$:

$$\mathbf{i}' = \mathbf{p}'_1\mathbf{p}'_2 \wedge \mathbf{q}'_1\mathbf{q}'_2.$$

Expand the above expression with Shuffle Formula (Li and Wu, 2003), we have

$$\mathbf{i}' = \frac{[\mathbf{p}'_1\mathbf{p}'_2\mathbf{q}'_2]\mathbf{q}'_1 - [\mathbf{p}'_1\mathbf{p}'_2\mathbf{q}'_1]\mathbf{q}'_2}{[\mathbf{p}'_1\mathbf{p}'_2\mathbf{q}'_2] - [\mathbf{p}'_1\mathbf{p}'_2\mathbf{q}'_1]}. \tag{3.1}$$

Notice that

$$[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1] = \delta_1(O_H)\big[\mathbf{p}'_1\mathbf{p}'_2\mathbf{q}'_1\big], \qquad [\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2] = \delta_2(O_H)\big[\mathbf{p}'_1\mathbf{p}'_2\mathbf{q}'_2\big],$$

where $\delta_1(O_H)$ and $\delta_2(O_H)$ are sign functions with value -1 or 1 determined by O_H.

If $\mathbf{q}_1$ and $\mathbf{q}_2$ are (not) in the same half plane divided by $\mathbf{p}_1\mathbf{p}_2$, then $\mathbf{q}'_1$ and $\mathbf{q}'_2$ are (not) in the same half plane divided by $\mathbf{p}'_1\mathbf{p}'_2$ with the rigid transformation defined by O_H and t_H. Assume that $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]$ and $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]$ are nonzero, according to Lemma 1, $\delta_1(O_H) = \delta_2(O_H)$. It means that we can rewrite the intersection formula (3.1) as

$$\mathbf{i}' = \frac{[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]\mathbf{q}'_1 - [\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]\mathbf{q}'_2}{[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2] - [\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]}.$$

Since

$$O_H\mathbf{q}_1 + t_H = \begin{pmatrix} \mathbf{q}'_1 \\ 0 \end{pmatrix}, \qquad O_H\mathbf{q}_2 + t_H = \begin{pmatrix} \mathbf{q}'_2 \\ 0 \end{pmatrix},$$

we have

$$O_H\mathbf{i} + t_H = \begin{pmatrix} \mathbf{i}' \\ 0 \end{pmatrix}.$$

It follows that the intersection formula $\mathbf{i}$ can be obtained. And it is the trivial case for one of $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]$ and $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]$ is zero. □

3.2. Line segments

We now consider the geometric relationship of the two space line segments bounded by $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{q}_1, \mathbf{q}_2$ respectively.

Lemma 7. *If $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2] \neq 0$, line segments $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{q}_1\mathbf{q}_2$ are separated.*

Lemma 8. *If $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2] = 0$, $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]$ and $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]$ are not both zeros, then line segments $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{q}_1\mathbf{q}_2$ are separated if and only if*

$$[\mathbf{p}_1\mathbf{q}_1\mathbf{q}_2]^\delta[\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2]^\delta = 1 \quad \textit{or} \quad [\mathbf{q}_1\mathbf{p}_1\mathbf{p}_2]^\delta[\mathbf{q}_2\mathbf{p}_1\mathbf{p}_2]^\delta = 1.$$

Proof. Clearly, line segments $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{q}_1\mathbf{q}_2$ are coplanar. They are separated if and only if points $\mathbf{p}_1$ and $\mathbf{p}_2$ are on the same side of line segment $\mathbf{q}_1\mathbf{q}_2$, or, points $\mathbf{q}_1$ and $\mathbf{q}_2$ are on the same side of line segment $\mathbf{p}_1\mathbf{p}_2$. The latter condition is equivalent to $[\mathbf{p}_1\mathbf{q}_1\mathbf{q}_2]^\delta[\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2]^\delta = 1$, or, $[\mathbf{q}_1\mathbf{p}_1\mathbf{p}_2]^\delta[\mathbf{q}_2\mathbf{p}_1\mathbf{p}_2]^\delta = 1$, respectively. □

Lemma 9. *If $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]$ and $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]$ are zeros simultaneously, line segments $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{q}_1\mathbf{q}_2$ are separated if and only if*

$$\big|[\mathbf{p}_1\mathbf{q}_1]^\delta + [\mathbf{p}_1\mathbf{q}_2]^\delta + [\mathbf{p}_2\mathbf{q}_1]^\delta + [\mathbf{p}_2\mathbf{q}_2]^\delta\big| = 4.$$

Table 1
Intersection of line segments.

The necessary and sufficient condition	The overlap line segment
$\|[\mathbf{p}_1\mathbf{q}_1]\|+\|[\mathbf{q}_1\mathbf{q}_2]\|+\|[\mathbf{q}_2\mathbf{p}_2]\|=\|[\mathbf{p}_1\mathbf{p}_2]\|$ or $\|[\mathbf{p}_1\mathbf{q}_2]\|+\|[\mathbf{q}_2\mathbf{q}_1]\|+\|[\mathbf{q}_1\mathbf{p}_2]\|=\|[\mathbf{p}_1\mathbf{p}_2]\|$	$\mathbf{q}_1\mathbf{q}_2$
$\|[\mathbf{q}_1\mathbf{p}_1]\|+\|[\mathbf{p}_1\mathbf{p}_2]\|+\|[\mathbf{p}_2\mathbf{q}_2]\|=\|[\mathbf{q}_1\mathbf{q}_2]\|$ or $\|[\mathbf{q}_1\mathbf{p}_2]\|+\|[\mathbf{p}_2\mathbf{p}_1]\|+\|[\mathbf{p}_1\mathbf{q}_2]\|=\|[\mathbf{q}_1\mathbf{q}_2]\|$	$\mathbf{p}_1\mathbf{p}_2$
$\|[\mathbf{p}_2\mathbf{q}_1]\|+\|[\mathbf{q}_1\mathbf{p}_1]\|+\|[\mathbf{p}_1\mathbf{q}_2]\|=\|[\mathbf{p}_2\mathbf{q}_2]\|$	$\mathbf{p}_1\mathbf{q}_1$
$\|[\mathbf{p}_2\mathbf{q}_2]\|+\|[\mathbf{q}_2\mathbf{p}_1]\|+\|[\mathbf{p}_1\mathbf{q}_1]\|=\|[\mathbf{p}_2\mathbf{q}_1]\|$	$\mathbf{p}_1\mathbf{q}_2$
$\|[\mathbf{p}_1\mathbf{q}_1]\|+\|[\mathbf{q}_1\mathbf{p}_2]\|+\|[\mathbf{p}_2\mathbf{q}_2]\|=\|[\mathbf{p}_1\mathbf{q}_2]\|$	$\mathbf{p}_2\mathbf{q}_1$
$\|[\mathbf{p}_1\mathbf{q}_2]\|+\|[\mathbf{q}_2\mathbf{p}_2]\|+\|[\mathbf{p}_2\mathbf{q}_1]\|=\|[\mathbf{p}_1\mathbf{q}_1]\|$	$\mathbf{p}_2\mathbf{q}_2$

Proof. Clearly, line segments $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{q}_1\mathbf{q}_2$ are collinear. Then they are separated if and only if the four brackets: $[\mathbf{p}_1\mathbf{q}_1]$, $[\mathbf{p}_1\mathbf{q}_2]$, $[\mathbf{p}_2\mathbf{q}_1]$, and $[\mathbf{p}_2\mathbf{q}_2]$ have the same sign value. Thus, the equality $|[\mathbf{p}_1\mathbf{q}_1]^\delta+[\mathbf{p}_1\mathbf{q}_2]^\delta+[\mathbf{p}_2\mathbf{q}_1]^\delta+[\mathbf{p}_2\mathbf{q}_2]^\delta|=4$ holds. □

Conversely, we give the conditions for two line segments to intersect with each other.

Lemma 10. *The two line segments intersect at a point if and only if* $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2]=0$, $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]\neq[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]$, $[\mathbf{p}_1\mathbf{q}_1\mathbf{q}_2]^\delta[\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2]^\delta\in\{0,-1\}$ *and* $[\mathbf{q}_1\mathbf{p}_1\mathbf{p}_2]^\delta[\mathbf{q}_2\mathbf{p}_1\mathbf{p}_2]^\delta\in\{0,-1\}$.

Lemma 11. *The two line segments overlap if and only if* $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2]=0$, $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]=[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]=0$ *and* $|[\mathbf{p}_1\mathbf{q}_1]^\delta+[\mathbf{p}_1\mathbf{q}_2]^\delta+[\mathbf{p}_2\mathbf{q}_1]^\delta+[\mathbf{p}_2\mathbf{q}_2]^\delta|<4$.

We can now consider intersection of two line segments.

Theorem 3. *Suppose that line segments* $\mathbf{p}_1\mathbf{p}_2$ *and* $\mathbf{q}_1\mathbf{q}_2$ *intersect. If* $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]\neq[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]$, *then they have only one intersection point*

$$\frac{[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]\mathbf{q}_1-[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]\mathbf{q}_2}{[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]-[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]}.$$

One can find that Theorem 3 is similar to Theorem 2 in intersection formula but different with intersection conditions.

Theorem 4. *Suppose that line segments* $\mathbf{p}_1\mathbf{p}_2$ *and* $\mathbf{q}_1\mathbf{q}_2$ *intersect. If* $[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1]=[\mathbf{p}_1\mathbf{p}_2\mathbf{q}_2]=0$, *then their intersection are listed in Table* 1.

Proof. Obviously, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}_1$ and $\mathbf{q}_2$ are collinear. Without loss of the generality, we only discuss the case in which the intersection is a line segment $\mathbf{q}_1\mathbf{q}_2$. We see that $\mathbf{q}_1$ and $\mathbf{q}_2$ are all inside the line segment $\mathbf{p}_1\mathbf{p}_2$ in this case, which indicates $|[\mathbf{p}_1\mathbf{q}_1]|+|[\mathbf{q}_1\mathbf{q}_2]|+|[\mathbf{q}_2\mathbf{p}_2]|=|[\mathbf{p}_1\mathbf{p}_2]|$ or $|[\mathbf{p}_1\mathbf{q}_2]|+|[\mathbf{q}_2\mathbf{q}_1]|+|[\mathbf{q}_1\mathbf{p}_2]|=|[\mathbf{p}_1\mathbf{p}_2]|$. The other cases can be proved similarly. □

Bracket method provides a mathematical formulation for geometric relationship. This leads to useful symbolic conditions in determining the intersections of two line segments. The given formulas can be easily used and checked in programs, furthermore, they also simplify computations by bringing algebra to bear on geometry.

4. Collision detection and intersection curve

Now we apply the results in previous sections in the collision detection for two ruled surfaces.

4.1. Condition for collision

Consider the two ruled surfaces mentioned in Section 2:

$$\begin{aligned}&\mathbf{P}(u,s)=\mathbf{P}_1(u)(1-s)+\mathbf{P}_2(u)s,\\&\mathbf{Q}(v,t)=\mathbf{Q}_1(v)(1-t)+\mathbf{Q}_2(v)t.\end{aligned}\qquad(4.1)$$

Then (4.1) can be regarded as two moving lines defined by $\{\mathbf{P}_1(u),\mathbf{P}_2(u)\}$ and $\{\mathbf{Q}_1(v),\mathbf{Q}_2(v)\}$ with parameters u and v respectively. According to the discussion in Section 3, $\Delta(u,v)=[\mathbf{P}_1(u)\mathbf{P}_2(u)\mathbf{Q}_1(v)\mathbf{Q}_2(v)]$ having real solutions is the necessary condition for the two surfaces to intersect. Here $\mathbf{P}_i$ and $\mathbf{Q}_i$ should be column vectors, for brevity, we write them in row form in the following paper without the transposing mark. Under this condition, the two moving lines intersect with each other in two cases. Firstly, the intersection has overlapping lines. Then there exist real number pairs (u, v) such that $\Delta_1(u, v) = 0$ and $\Delta_2(u, v) = 0$, where

$$\Delta_1 = \big[\mathbf{P}_1(u)\mathbf{P}_2(u)\mathbf{Q}_1(v)\big] \quad \text{and} \quad \Delta_2 = \big[\mathbf{P}_1(u)\mathbf{P}_2(u)\mathbf{Q}_2(v)\big].$$

Similarly, we set $\Delta_3 = [\mathbf{P}_1\mathbf{Q}_1\mathbf{Q}_2]$ and $\Delta_4 = [\mathbf{P}_2\mathbf{Q}_1\mathbf{Q}_2]$. Secondly, there exist real number pairs (u, v) such that the intersection curve consisting of the ordinary intersection points of two lines, that is, $\Delta_1(u, v) \neq \Delta_2(u, v)$. According to Theorem 1, we have

Theorem 5. *Two ruled surfaces* (4.1) *have real intersection if and only if the following two sets are not both empty,*

$$S_1 = \big\{(u, v) \in \mathbb{R}^2 \bigm| \Delta_1(u, v) = \Delta_2(u, v) = 0\big\},$$
$$S_2 = \big\{(u, v) \in \mathbb{R}^2 \bigm| \Delta(u, v) = 0,\ \Delta_1(u, v) \neq \Delta_2(u, v)\big\}.$$

In Theorem 5, S_1 and S_2 correspond to the overlapping intersection and the ordinary intersection respectively.

The bracket $\Delta_1(u, v)$ of three points may be complicated. But we can use Lemma 3 to simplify this problem by computing $\tilde{\Delta}_1 = [\mathbf{P}_1\mathbf{P}_2\mathbf{Q}_1\mathbf{D}]$ where $\mathbf{D}$ is a point not on the plane $\mathbf{P}_1\mathbf{P}_2\mathbf{Q}_1$. Since the plane has two parameters, it is difficult or impossible to fix a constant point always not in the plane with parameters. Alternatively, we can fix four non-coplanar constant points $\mathbf{D}_i,\ i = 1, \ldots, 4$, then these points cannot be coplanar with $\mathbf{P}_1, \mathbf{P}_2$ and $\mathbf{Q}_2$ simultaneously. And it is not difficult to prove the following proposition.

Proposition 1. *Let* $\tilde{\Delta}_{1,i} = [\mathbf{P}_1\mathbf{P}_2\mathbf{Q}_1\mathbf{D}_i]$*, then* $\Delta_1(u, v) = 0$ *if and only if* $\tilde{\Delta}_{1,i}(u, v) = 0,\ i = 1, \ldots, 4$.

Similarly, $\Delta_2(u, v) = 0$ if and only if $\tilde{\Delta}_{2,i} = 0,\ i = 1, \ldots, 4$, where $\tilde{\Delta}_{2,i} = [\mathbf{P}_1\mathbf{P}_2\mathbf{Q}_2\mathbf{D}_i]$. In the computation, we select four non-coplanar constant points in homogeneous form $\bar{\mathbf{D}}_1 = (0, 0, 0, 1)^T,\ \bar{\mathbf{D}}_2 = (0, 0, 1, 0)^T,\ \bar{\mathbf{D}}_3 = (0, 1, 0, 0)^T$ and $\bar{\mathbf{D}}_4 = (1, 0, 0, 0)^T$. Then $[\mathbf{abcD}] = \det(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}, \bar{\mathbf{D}})$, where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{\mathbf{P}_1, \mathbf{P}_2, \mathbf{Q}_1, \mathbf{Q}_2\}$.

4.2. Numerical intersection curve

Finding numerical intersection curves is a necessary task in computer aided design and computer numerical control. It has been proved in Heo et al. (1999) that if two ruled surfaces overlap in a two-dimensional subset, then each of them must be a plane or a quadric. Here, we assume that the intersection is not a two-dimensional subset, which means that S_1 has only finite elements.

According to the above discussion, we divide the intersection into two parts. One part consists of the overlap lines $\mathbf{P}_1(u)(1 - s) + \mathbf{P}_2(u)s, u \in S_1$. Another part consists of ordinary intersection points, by Theorem 2, the intersection is

$$\frac{1}{\Delta_2 - \Delta_1}(\Delta_2\mathbf{Q}_1 - \Delta_1\mathbf{Q}_2), \quad (u, v) \in S_2. \tag{4.2}$$

In numerical computation, by Proposition 1, Δ_i can be replaced by the simpler expressions $\tilde{\Delta}_{i,k}$ for $i = 1, 2$ and $k \in \{1, 2, 3, 4\}$. We give an example to illustrate the process.

Example 1. Consider two ruled surfaces

$$\mathbf{P}(u, s) = \mathbf{P}_1(u)(1 - s) + \mathbf{P}_2(u)s,$$
$$\mathbf{Q}(v, t) = \mathbf{Q}_1(v)(1 - t) + \mathbf{Q}_2(v)t,$$

where

$$\mathbf{P}_1 = \left(\frac{1 - u^2}{1 + u^2}, \frac{2u}{1 + u^2}, 1\right), \qquad \mathbf{P}_2 = \left(\frac{2(1 - u^2)}{1 + u^2}, \frac{4u}{1 + u^2}, 2\right),$$
$$\mathbf{Q}_1 = \left(\frac{1 - v^2}{1 + v^2}, \frac{2v}{1 + v^2}, 1\right), \qquad \mathbf{Q}_2 = \left(\frac{1 - v^2}{1 + v^2}, \frac{2v}{1 + v^2} + 1, 2\right).$$

$\mathbf{P}$ is a cone and $\mathbf{Q}$ is cylinder (see Fig. 1), and there is a generator missing in each parametrization, corresponding to the limits $u \to \infty$ and $v \to \infty$, respectively. One can compute the condition functions Δ, Δ_1^2 and Δ_2^2, here $|\Delta_1|, |\Delta_2|$ are computed by the formula $\|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})\|$. We then give the numerator of condition equations, denoted by *Numer*$(\cdot)$, since their denominators are positive definite. Then

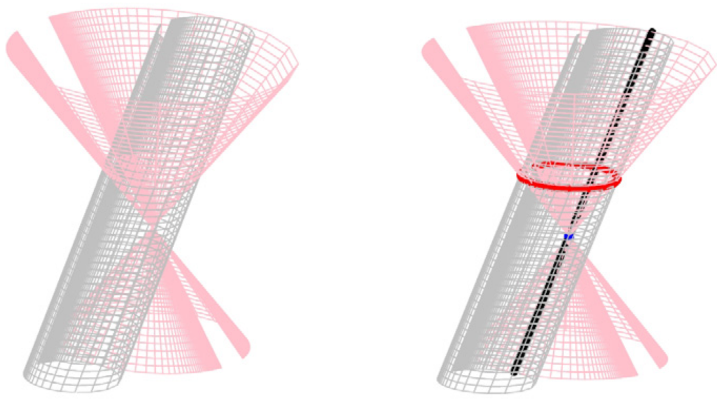


Fig. 1. A cone and a cylinder. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$Numer(\Delta) = -2(-1+u)(v-1)(v-u),$$
$$Numer\big(\Delta_1^2\big) = \big(2u^2v^2+v^2+2uv+2+u^2\big)(v-u)^2,$$

and $Numer(\Delta_2^2) = 8u^4v + 11u^4 + 3v^4u^4 + 18v^2u^4 + 8v^3u^4 - 16u^3v^2 - 8vu^3 - 24u^3v^3 - 4u^3 - 12u^3v^4 + 4u^2v^2 + 14v^4u^2 + 14u^2 - 12u - 24uv - 8uv^3 - 4uv^4 - 16uv^2 + 3 + 8v + 11v^4 + 8v^3 + 18v^2$.

Consider S_1, it is now the real intersection points of two planar curves $Numer(\Delta_1^2) = 0$ and $Numer(\Delta_2^2) = 0$. S_1 can only have finite real points by our assumption of the two ruled surfaces do not overlap in a two-dimensional subset. Actually, one can find that $S_1 = \{(1,1)\}$ using resultant computation. On the other hand, by Proposition 1, we can decrease the equation degrees in S_1. Compute $\{\tilde{\Delta}_{1,i}\}_{i=1}^4$ and get their numerators as $\{0,\ -(2(uv-1))(v-u),\ (2(v-u))(u+v), -(2(uv+1))(v-u)\}$. Similarly, the numerators of $\{\tilde{\Delta}_{2,i}\}_{i=1}^4$ are $\{0,\ 1+v^2+u^2+u^2v^2-4u-4uv^2+2v+2vu^2,\ 3v^2-u^2v^2+1-3u^2,\ (1+v)(-2uv+vu^2-v+u^2-1+2u)\}$. The same set $S_1 = \{(1,1)\}$ can be computed from the polynomial system with much lower degree.

Up to now, we find the overlapping intersection lines are

$$\mathbf{P}_1(1)(1-s) + \mathbf{P}_2(1)s = (0, s+1, s+1),$$

which is the tangent line of the two ruled surfaces (the black line in Fig. 1).

Another part of ordinary intersection points is given by (4.2). Here, to obtain the intersection expression with lower degree, we replace $\{\Delta_1, \Delta_2\}$ by $\{\tilde{\Delta}_{13}, \tilde{\Delta}_{23}\}$ which is the simplest nonzero pair of $\{\tilde{\Delta}_{1k}, \tilde{\Delta}_{2k}\}$, $k = 1\ldots4$. Then the intersection curve is

$$\begin{aligned} I(u,v) &= \frac{1}{\tilde{\Delta}_{23}-\tilde{\Delta}_{13}}(\tilde{\Delta}_{23}\mathbf{Q}_1 - \tilde{\Delta}_{13}\mathbf{Q}_2), \quad (u,v)\in S_2, \\ &= \left(-\frac{(v-1)(1+v)}{(1+v^2)}, \frac{2(v-1)(v+u^2)}{(-1+u)(1+u)(1+v^2)}, \frac{(1+u^2)(v-1)(1+v)}{(-1+u)(1+u)(1+v^2)}\right), \quad (u,v)\in S_2, \end{aligned}$$

where

$$\tilde{\Delta}_{13} = \frac{2(v-u)(v+u)}{(1+u^2)(1+v^2)}, \qquad \tilde{\Delta}_{23} = \frac{3v^2-u^2v^2+1-3u^2}{(1+u^2)(1+v^2)}.$$

Considering the parameter set S_2, we have

$$\begin{aligned} S_2 &= \big\{(u,v) \bigm| \Delta(u,v)=0,\ \Delta_1(u,v)\neq\Delta_2(u,v)\big\} \\ &= \big\{(u,v) \bigm| \Delta(u,v)=0\big\} \setminus \big\{(u,v) \bigm| \Delta_1(u,v)=\Delta_2(u,v)\big\} \\ &= \big\{(u,v) \bigm| (1,v),(v,v),(u,1)\big\} \setminus \big\{(u,v) \bigm| (1,v)\big\} \\ &= \big\{(u,v) \bigm| (v,v),(u,1)\big\} \setminus \big\{(u,v) \bigm| (1,1)\big\}. \end{aligned}$$

Substitute two topology set of S_2 into $I(u,v)$ to get

$$I_1(v) = \left(-\frac{v^2-1}{1+v^2}, \frac{2v}{1+v^2}, 1\right), \qquad I_2(u) = (0,0,0).$$

Then I_1 is a circle intersection curve (see Fig. 1, red circle without the points $(0,1,1)$ included in the overlap segments) and I_2 is apex of the cone (see Fig. 1, blue point).

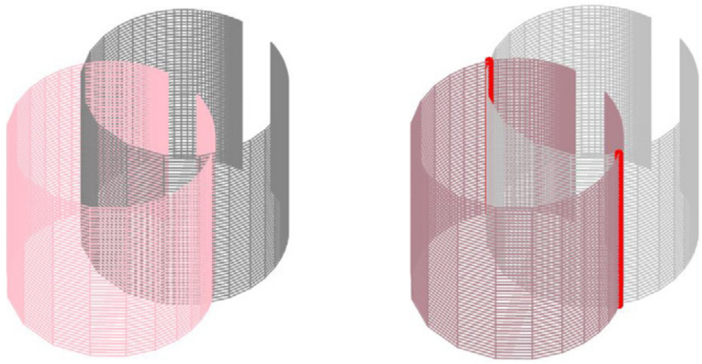


Fig. 2. Two cylinders. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

This example is also illustrated by Heo et al. and Fioravanti et al. The degree of the intersection expression in Heo et al. (1999) is $(3,2)$ w.r.t. (u,v), while ours is $(2,2)$. After substituting $\mathbf{Q}$ into the implicit equation of $\mathbf{P}$, the (v,t)-plane curve implicit equation $G(v,t)=t(1+v^2)(v-1)=0$ is proposed in Fioravanti et al. (2006). The line $t=0$ corresponds to the points in the intersection circle. The line $v=1$ corresponds to the common generator of the cone and the cylinder.

Considering in the preimage (u,v)-space, $\Delta_1(u,v)=0$ and $\Delta_2(u,v)=0$ define two planar curves. Then for the intersection of overlapping lines, we can compute S_1 by finding numerical real intersection points of these two planar curves. But if $\Delta_1(u,v)=\Delta_2(u,v)$, then the situation is a degenerated case mentioned in Heo et al. (1999) and the general procedure proposed there does not apply to this case. The following example shows that we can deal with this case by Proposition 1.

Example 2. Considering the two cylinders in Fig. 2 as

$$\mathbf{P}(u,s)=\left(\frac{1-u^2}{1+u^2},\frac{2u}{1+u^2},0\right)(1-s)+\left(\frac{1-u^2}{1+u^2},\frac{2u}{1+u^2},1\right)s,$$

$$\mathbf{Q}(v,t)=\left(\frac{1-v^2}{1+v^2}-1,\frac{2v}{1+v^2},0\right)(1-t)+\left(\frac{1-v^2}{1+v^2}-1,\frac{2v}{1+v^2},1\right)t.$$

Then we have $\Delta\equiv 0$ and

$$Numer\left(\Delta_1^2\right)=Numer\left(\Delta_2^2\right)=v^2u^2+9v^2-8uv+1+u^2.$$

Obviously, $S_2=\emptyset$ and that means two cylinders only have overlap lines if they intersect. But it is not obviously to determine the real solutions of

$$S_1=\left\{(u,v)\;\middle|\;v^2u^2+9v^2-8uv+1+u^2=0\right\}.$$

Fortunately, using Proposition 1, it is simple to compute S_1 from the real set formed by $\tilde{\Delta}_{1,i}(u,v)=0$ and $\tilde{\Delta}_{2,i}(u,v)=0$, i.e., $S_1=\{(u,v)\mid -2v(1-u^2+2uv)=0,\ -2(uv-1)(-u+v)=0,\ 1+3v^2-u^2+v^2u^2=0,\ 0\equiv 0\}$. Solving this system, we obtain $S_1=\{(\sqrt{3},\sqrt{3}/3),-(\sqrt{3},\sqrt{3}/3)\}$ (corresponding to the two overlap intersection lines (red) in Fig. 2).

In Fioravanti et al. (2006), the implicit equation and the parameter planar equation are computed as $x^2+y^2=1$ and $G(v,t)=v^4-2v^2-3=0$, while $v=\pm\sqrt{3}$ are corresponding to a generator common to both cylinders.

There are two main steps in computing the intersection curves consisting of ordinary intersection points. Firstly, we give the topology graph $\mathcal{G}$ of S_2 which is based on the previous work (Gonzalez-Vega and Necula, 2002; Cheng et al., 2009) of determining the topology of planar curves. We refer to compute the topology graph of S_2 using the method in Cheng et al. (2009) by removing $\Delta_1(u,v)-\Delta_2(u,v)=0$ from $\Delta(u,v)=0$. Secondly, we give the numerical points of the intersection corresponding to $\mathcal{G}$. This process is similar to computing the intersection curve of a parametric surface and an implicit surface (Fioravanti et al., 2006), but we do not need to implicitize the ruled surfaces. Finally, we summarize our process as an algorithm.

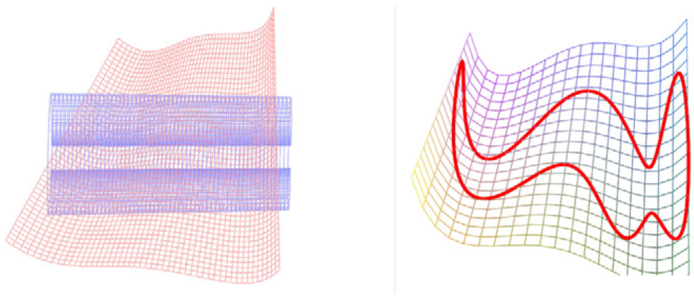


Fig. 3. Numerical intersection curve of two ruled surfaces.

Algorithm 1 *(Numerical intersection algorithm).*
Input: Ruled surfaces $\mathbf{P}(u,s)$ and $\mathbf{Q}(v,t)$.
Output: Their numerical intersections.
Steps:

1. Compute $\Delta(u,v)$, $\Delta_1(u,v)$ and $\Delta_2(u,v)$.
2. Compute the two intersection parts:
   overlap lines $\mathbf{P}(u,s), u \in S_1$ and output these lines;
   intersection curves $\frac{1}{\Delta_2-\Delta_1}(\Delta_2\mathbf{Q}_1 - \Delta_1\mathbf{Q}_2)$, $(u,v) \in S_2$.
3. Determine the topology graph $\mathcal{G}$ of S_2.
4. Output numerical curve intersection according to $\mathcal{G}$.

In the numerical intersection algorithm, we replace $\Delta_1(u,v)$ and $\Delta_2(u,v)$ by several lower degree functions $\tilde{\Delta}_{1i}(u,v)$ and $\tilde{\Delta}_{2i}(u,v)$ respectively. And actually, in the computation, we always represent the intersection with

$$\frac{1}{\tilde{\Delta}_{2k}-\tilde{\Delta}_{1k}}(\tilde{\Delta}_{2k}\mathbf{Q}_1 - \tilde{\Delta}_{1k}\mathbf{Q}_2), \quad (u,v) \in S_2,$$

where $\tilde{\Delta}_{1k}(u,v)$ and $\tilde{\Delta}_{2k}(u,v)$ are not both zeros. It is also simpler to find the real solutions of $\tilde{\Delta}_{1,i}(u,v) = \tilde{\Delta}_{2,i}(u,v) = 0,\ i = 1,\dots,4$.

Example 3. Here we give another example concisely. The two ruled surfaces (Fioravanti et al., 2006) are given as

$$\mathbf{P}(u,s) = \left(s+1, -\frac{-s+u^2s-60+60u^2}{60(1+u^2)}, \frac{u(s+60)}{30(1+u^2)}\right),$$
$$\mathbf{Q}(v,t) = \left(-vt/10+v, -vt-6v^3+11v^2-6v+v^4, t\right).$$

Then the condition functions can be found and the only the numerator is given $Numer(\Delta) = 295 - 295u^2 + 1800v^3 + 1800v^3u^2 - 3300v^2 - 3300u^2v^2 + 590vu + 1805v + 1795vu^2 - 300v^4 - 300v^4u^2 + 6v^4u - 11v^3u + 16uv^2 - v^5u$,

$$Numer(\Delta_1) = v(-1+v)(v-2)(v-3)u,$$
$$Numer(\Delta_2) = u^2 - 12v^3u + 22uv^2 - 14vu + 2v^4u - 1.$$

Following Algorithm 1, one can get the numerical intersection as the following Fig. 3. In the computation, the process of steps 3, 4 also has been shown in Fioravanti et al. (2006) similarly.

5. Collision detection with boundary condition

In practice, the collision detection of the surface segments has more application. This problem becomes more difficult but necessary when the time or position parameters are involved. In this section, we try our method on this problem.

Since a ruled surface is defined as $\mathbf{P}(u,s) = \mathbf{P}_0(u)(1-s) + \mathbf{P}_1(u)s$, the natural boundaries are the directrices $\mathbf{P}_0(u)$ and $\mathbf{P}_1(u)$ if we restrict $s \in [0,1]$. Similarly, $\mathbf{Q}(v,t)$ is bounded with $t \in [0,1]$. Then we can regard the ruled surfaces with boundary as a pencil of line segments. According to Lemma 10, the two ruled surfaces intersect with points if and only if $\Delta = 0$, $\Delta_1 \neq \Delta_2$, $\Delta_1\Delta_2 \leqslant 0$ and $\Delta_3\Delta_4 \leqslant 0$. We then write the conditions as a semi-algebraic system (Yang and Xia, 2005):

$$\mathcal{S}_1:\ \begin{cases} \Delta(u,v) = 0, \\ \Delta_1(u,v)\Delta_2(u,v) \leqslant 0, \\ \Delta_3(u,v)\Delta_4(u,v) \leqslant 0, \\ \Delta_1(u,v) - \Delta_2(u,v) \neq 0. \end{cases}$$

The system $\mathcal{S}_1$ has real solutions if and only if the two ruled surfaces collide. Here, Δ_i are the areas with directions for $i = 1, \ldots, 4$. If their degrees are high, we can replace Δ_i by $\tilde{\Delta}_{ij}$, which gives us semi-algebraic systems

$$\mathcal{S}_{1j}\colon\quad \begin{cases} \Delta(u,v) = 0, \\ \tilde{\Delta}_{1j}(u,v)\tilde{\Delta}_{2j}(u,v) \leqslant 0, \\ \tilde{\Delta}_{3j}(u,v)\tilde{\Delta}_{4j}(u,v) \leqslant 0, \\ \tilde{\Delta}_{1j}(u,v) - \tilde{\Delta}_{2j}(u,v) \neq 0 \end{cases}$$

for $j = 1, \ldots, 4$. We see that $\mathcal{S}_1 = \bigcap_{j=1}^{4} \mathcal{S}_{1j}$.

In engineering applications, we are interested in moving objects such as arms of the robot. This means that one or two of ruled surfaces are moving with parameters. In the following example, we will use a procedure `tofind` in a `Maple` package DISCOVERER (Xia, 2007) to solve the SAS with parameters. This package can be downloaded from http://www.is.pku.edu.cn/~xbc/discoverer.html.

For a given parametric semi-algebraic system $\mathcal{S}$ and an integer N, the function $\texttt{tofind}(\mathcal{S}, [\mathbf{X}], [\mathbf{Y}], N)$ is used to determine the necessary and sufficient conditions on $\mathbf{Y}$ such that the number of distinct real solutions $\mathbf{X}$ of $\mathcal{S}$ equals N with parameters $\mathbf{Y}$. Here, N can also be replaced by a positive range as $[N_1, N_2]$. The output of `tofind` is a quantifier-free formula Φ in parameters and border polynomials $BP(\mathbf{Y})$ which means that, provided $BP(\mathbf{Y}) \neq 0$, then the necessary and sufficient condition for $\mathcal{S}$ to have exactly N real solutions if Φ holds, assuming the parameters are not on boundary. To consider the parameter on the boundary $BP(\mathbf{Y}) = 0$, one should add the border polynomial as equations to $\mathcal{S}$ and call `tofind` repeatedly.

Example 4. Let $\mathbf{P}(u,s;T) = \mathbf{P}_1(u;T)(1-s) + \mathbf{P}_2(u;T)s,\ s \in [0,1]$ be a ruled surface segment with a parameter T such that $\mathbf{P}$ move along a parabola, where

$$\mathbf{P}_1 = \left(-1, \frac{1-u^2}{1+u^2} - T, \frac{2u}{1+u^2} - T^2\right),$$
$$\mathbf{P}_2 = \left(0, \frac{1-u^2}{1+u^2} - T, \frac{2u}{1+u^2} - T^2\right).$$

Another ruled surface segment is $\mathbf{Q} = \mathbf{Q}_1(v)(1-t) + \mathbf{Q}_2(v)t,\ t \in [0,1]$, where

$$\mathbf{Q}_1 = \left(\frac{1-v^2}{1+v^2} - 1, \frac{2v}{1+v^2}, \frac{1}{2}\right),$$
$$\mathbf{Q}_2 = \left(\frac{1-v^2}{1+v^2} - 1, \frac{2v}{1+v^2}, \frac{3}{2}\right).$$

By $\texttt{tofind}(\mathcal{S}_1, [u,v], [T], 1..\mathit{infinity})$, the necessary and sufficient condition for the collision of the two surfaces is

$$C(\mathbf{P}, \mathbf{Q}) = \{R_1 < 0,\ R_2 > 0,\ R_3 < 0,\ R_4 \neq 0,\ R_5 \neq 0\},$$

where $R_1 = T + Tu^2 - 2$, $R_2 = T + Tu^2 + 2u^2$, $R_3 = -4u + 2T^2 + 2T^2u^2 + 1 + u^2$, $R_4 = -1 + u^2 + T + Tu^2$, $R_5 = -4u + 2T^2 + 2T^2u^2 + 3 + 3u^2$ are border polynomials.

We detect the collision by verifying the condition for any fixed $T = T_0$. Then the problem is reduced to detecting real roots in a univariate semi-algebraic set. If $T = 0$, then one can check that $u = 1/2$ satisfies $C(\mathbf{P}, \mathbf{Q})$, which means $\mathbf{P}$ and $\mathbf{Q}$ are intersected (see Fig. 4.a). If $T = 1$, then obviously $R_3 = 3u^2 - 4u + 3$ is a definite positive function for u in $[0,1]$. Hence, $\mathbf{P}$ and $\mathbf{Q}$ are separated (Fig. 4.b).

For further consideration, there are two parameters in $C(\mathbf{P}, \mathbf{Q})$ and we will eliminate one by quantifier elimination. Eliminate u from $\{R_1 < 0,\ R_2 > 0,\ R_3 < 0\}$, one can obtain the $2T^2 < 1$ which is in the interval of the two segments are intersected, as showed above.

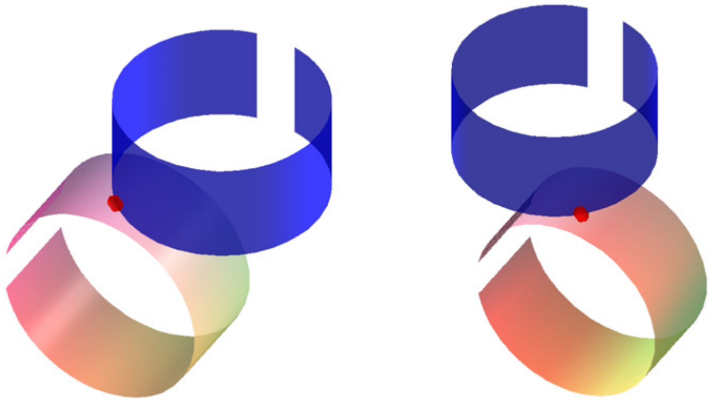
To decide the first instant of collision, we need to consider the situations on boundary, that means these situations occur with some border polynomials satisfied. Here, the border polynomials are the functions of boundaries. Add $R_3 = 0$ to the semi-algebraic system and solve it, we can get that the surface segments collided at $2T^2 = 1$. They are just the two instants for the two segments beginning or ending the collision. Actually, $\mathbf{P}$ and $\mathbf{Q}$ just meet at a point at $T = \frac{\sqrt{2}}{2}$ or $T = -\frac{\sqrt{2}}{2}$ (see Fig. 5).

There are no other collision situation. Finally, we claim that these two rule surface segments collide at $T \in [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ and they meet at $T = \pm\frac{\sqrt{2}}{2}$.

In this example, there exist no overlapping line segments in intersections. Hence we need not to consider the overlapping case.

a. Intersected at $T = 0$ b. Separated at $T = 1$

Fig. 4. Two cylinder segments.



a. $T = \frac{\sqrt{2}}{2}$ b. $T = -\frac{\sqrt{2}}{2}$

Fig. 5. Meet instant.

For the overlapping situations, by Lemma 11, the conditions for two ruled surface segments to overlap are $\Delta_1 = \Delta_2 = 0$ without $[\mathbf{P}_i\mathbf{P}_j] > 0$, $i, j = 1, 2$, or, $\Delta_1 = \Delta_2 = 0$ without $[\mathbf{P}_i\mathbf{P}_j] < 0$, $i, j = 1, 2$. We omit the details here.

When the two surface segments are intersected, it is also necessary to find the intersection curve in some situations. Based on the discussion in this paper, we can compute the intersection curve by two main parts: determine the intersecting conditions of parameters by solving SAS and find the intersection curve by Algorithm 1. The following example is given to illustrate the process.

Example 5. Following Example 4, we set $T = 0$ which corresponds to an intersection instant. The task is then to compute the intersection curve of $\mathbf{P}(u, s) = \mathbf{P}(u, s, 0)$ and $\mathbf{Q}(v, t)$ with $s, t \in [0, 1]$.

Similar to Example 1 and by Theorem 2, one can compute the intersection curve expression as

$$\left(-2\frac{v^2}{1+v^2}, 2\frac{v}{1+v^2}, -4\frac{vu}{(1+v^2)(u-1)(1+u)}\right), \quad (u, v) \in S_2,$$

where $S_2 = \{(u, v) \mid (v + vu + u - 1)(vu - v + 1 + u) = 0\} \backslash \{(u, v) \mid (1, v)\}$.

We should consider the restriction of segments $s, t \in [0, 1]$. Here, similar to Example 4, we find the conditions of parameter u and v for the collision are $u^2 - 4u + 1 \leqslant 0$ and $3v^2 - 1 \leqslant 0$ respectively. That means the parameters of the intersections curve should be in

$$S = S_2 \cap \left\{(u, v) \,\middle|\, u^2 - 4u + 1 \leqslant 0,\ 3v^2 - 1 \leqslant 0\right\}.$$

Since the topology of S is not complicated here, we can give the simple expression of the curve as

$$\left(-2\frac{v^2}{1+v^2}, 2\frac{v}{1+v^2}, -\frac{-1+v^2}{1+v^2}\right), \quad v \in \left[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right],$$

which is showed as red curve in Fig. 6.

Fig. 6. Intersection curve of segments. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

6. Conclusion

In this paper, we attempt to apply bracket method to the collision and intersection problem for ruled surfaces, and propose an efficient and robust intersection algorithm. After solving a semi-algebraic system, we obtain the necessary and sufficient algebraic conditions for collision detection of two ruled surface segments.

As future work, it is interesting and important to compute the numerical intersection of two ruled surfaces with arbitrary boundary. For further consideration, some singular points, e.g., cusp points, may be missing by the numerical approximation method, this leads to another future work: approximate the intersection curve preserving geometric features.

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