Arithmetic of
The Algebraic and Differential Generic Galois Group of
A q-difference Equation

(joint work with C. Hardouin)

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Let k be a perfect field and K be a finite extension of $k(q)$. The Grothendieck conjecture on p-curvatures asserts that the solutions of a linear differential equation L with coefficients in $K(x)$ with K a number field are algebraic if and only if the p-curvatures of the equation L equals zero for almost all prime p of K. We prove a discrete analog of this conjecture. In the case of q-difference equations, *i.e.* , we prove the equivalence among the following facts:

1. a q-difference module over $K(x)$ is trivial or equivalently a q-difference equation $Y(qx) = A(x)Y(x)$ where $A \in Gl_\nu(K(x))$;
2. its specialization at $q = \xi$ has zero curvature for almost all primitive roots of unity ξ;
3. its specialization at $q = \xi$ is endowed with a (necessarily trivial) structure of iterated ξ-difference module, for almost all primitive roots of unity ξ.

The equivalence between 1. and 3. is an analog of a Matzat-van der Put conjecture for differential equations over field of positive characteristic.

Then we consider two kinds of Galois groups (the second one only under the assumption that k has zero characteristic) attached to a q-difference module $\mathcal{M}$ over $K(x)$:

- the generic (also called intrinsic) Galois group in the sense of [Kat82] and [DV02], which is an algebraic group over $K(x)$. We refer to this group as the generic algebraic Galois group, or simply as the generic Galois group;
- the generic differential Galois group, which is a differential algebraic group in the sense of Kolchin, associated to the smallest differential tannakian category generated by $\mathcal{M}$, equipped of the forgetful functor.

The result above leads to an arithmetic description of the generic algebraic (resp. differential) Galois group: it is the smallest algebraic (resp. differential) group containing the curvatures of the q-difference module for almost all primitive roots of unity ξ. Although no general Galois correspondence holds in this setting, if the characteristic of k is positive and the generic Galois group is nonreduced, we can prove some devissage.

By specialization of the parameter q at 1 in the Galois group, we obtain an upper bound for the generic Galois group of the differential equation obtained by specialization. This upper bound has a curvature characterization in the spirit of the Grothendieck-Katz conjecture, but *via* different curvatures than the ones appearing in the conjecture.

References

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