Gröbner bases in difference-differential modules and their applications

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Abstract Let R be a commutative noetherian ring, $\Delta = \{\delta_1, \cdots, \delta_m\}$ and $\sigma = \{\alpha_1, \cdots, \alpha_n\}$ be set of derivations and automorphisms of the ring R, respectively, such that $\beta(x) \in R$ and $\beta(\gamma(x)) = \gamma(\beta(x))$ hold for any $\beta, \gamma \in \Delta \bigcup \sigma$ and $x \in R$. Then R is called a difference-differential ring with the basic set of derivations Δ and the basic set of automorphisms σ, or shortly a Δ-σ-ring. If R is a field, then it is called a Δ-σ-field. Λ will denote the commutative semigroup of elements of the form

$$\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \alpha_1^{l_1} \cdots \alpha_n^{l_n} \tag{1.1}$$

where $(k_1, \cdots, k_m) \in \mathbb{N}^m$ and $(l_1, \cdots, l_n) \in \mathbb{Z}^n$. This semigroup contains the free commutative semigroup Θ generated by the set Δ and free commutative semigroup Γ generated by the set σ. The subset $\{\alpha_1, \cdots, \alpha_n, \alpha_1^{-1}, \cdots, \alpha_n^{-1}\}$ of Λ will be denoted by σ^*.An expression of the form

$$\sum_{\lambda \in \Lambda} a_\lambda \lambda, \tag{1.2}$$

where $a_\lambda \in R$ for all $\lambda \in \Lambda$ and only finitely many coefficients a_λ are different from zero, is called a difference-differential operator (or shortly a Δ-σ-operator) over R. Two Δ-σ-operators $\sum_{\lambda \in \Lambda} a_\lambda \lambda$ and $\sum_{\lambda \in \Lambda} b_\lambda \lambda$ are equal if and only if $a_\lambda = b_\lambda$ for all $\lambda \in \Lambda$.The ring of all Δ-σ-operators over a Δ-σ-ring R is called the ring of difference-differential operators (or shortly the ring of Δ-σ-operators) over R, it will be denoted by D. A left D-module M is called a difference-differential module(or a Δ-σ-module). If M is finitely generated as a left D-module, then M is called a finitely generated Δ-σ-module.

In this talk we describe an extending of the theory of Gröbner bases to difference-differential modules. We present and verify algorithms for constructing these Gröbner bases counterparts. To this aim we introduce the concept of "generalized term order" on $\mathbb{N}^m \times \mathbb{Z}^n$ and on difference-differential modules. Also, we introduce a new concept, relative difference-differential Gröbner bases. Our notion of relative Gröbner basis is based on two generalized term orders on $\mathbb{N}^m \times \mathbb{Z}^n$. We define a special type of reduction for two generalized term orders in a free left module over a ring of difference-differential operators. Then the concept of relative Gröbner bases w.r.t. two generalized term orders is defined. An algorithm for constructing these Gröbner basis counterparts is presented and verified.

Using Gröbner bases on difference-differential modules and the relative Gröbner basis algorithm, it is possible to compute the difference-differential dimension polynomials of a difference-differential module and of a system of linear partial difference-differential equations, and difference-differential dimension polynomials in two variables.

The research is motivated by the difference-differential dimension polynomial theories of [Lev00]and [Lev07], in which the existence of the difference-differential dimension polynomial was proved via characteristic set.

In 2009 Christian Doench present the Maple implementations of two algorithms developed by M. Zhou and F. Winkler for computing a relative Gröbner basis of a finitely generated difference-differential module and use this to compute the bivariate difference-differential dimension polynomial of the module with respect to the natural bifiltration of the ring of difference-differential operators. And the implementations of the two algorithms are illustrated by some examples.

References

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