DART IV

The Fourth International Workshop on

Differential Algebra and Related Topics

Program and Abstracts

Academy of Mathematics and Systems Science

Beijing, China

October 27-30, 2010

Contents

Program

Note: All talks take place in the lecture hall of Siyuan Building, Academy of Mathematics and Systems Science. Posters and Registration are located in front of the lecture hall. The address of Siyuan Building is

No.55, Zhongguancun East Road

中关村东路 55 号(保福寺桥西南角).

Wednesday, October 27	
8:00-8:50	Registration
8:50-9:00	Opening Remarks
Morning Session I	Chair: Xiao-Shan Gao
9:00-9:50	**Claude Mitschi** Galoisian Approach to Monodromy Evolving Deformations with Regular Singularities
9:50-10:00	Tea Break
Morning Session II	Chair: Julia Hartmann
10:00-10:50	**Lucia Di Vizio** Arithmetic of the Algebraic and Differential Generic Galois Group of a q-Difference Equation
10:50-11:10	Tea Break
Morning Session III	Chair: V. Ravi Srinivasan
11:10-11:35	**Charlotte Hardouin** Non Linear q-Difference Equations and Curvatures
11:35-12:00	**Alexey Ovchinnikov** Differential Representations of SL_2
12:00-14:00	Lunch at Wuke Restaurant, fourth floor

Afternoon Session I Chair: Alexey Ovchinnikov	
14:00-14:50	**Chunming Yuan** Differential Chow Form
14:55-15:20	**Dmitry Trushin** Differential Schemes and Differential Algebraic Varieties
15:20-15:40	Tea Break
15:40-16:40	Poster Presentations
Afternoon Session II Chair: Chengming Bai	
16:40-17:05	**Qinghu Hou** Hypergeometric Series Solutions of Linear Operator Equations
17:05-17:30	**Ned Nedialkov** DAETS: a Differential-Algebraic Equation Code in C++ for High Index and High Accuracy
17:30-19:00	Dinner at Wuke Restaurant, fourth floor

Thursday, October 28	
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9:00-9:50	Sebastian Walcher Darboux Integrating Factors of Planar Polynomial Vector Fields: Inverse Problems
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Morning Session II Chair: Evelyne Hubert	
10:00-10:50	**Gloria Mari Beffa** Bi-Hamiltonian Flows and Geometric Realizations
10:50-11:10	Tea Break

Morning Session III Chair: Zhenya Yan	
11:10-11:35	**Alexander V. Mikhailov** Symmetries and Integrability Conditions for Difference Equations
12:00-14:00	Lunch at Wuke Restaurant, fourth floor
Afternoon Session I Chair: Chunming Yuan	
14:00-14:25	**Alexander Levin** Dimension of Finitely Generated Differential and Difference Field Extensions
14:25-14:50	**Xiaoping Xu** Methods of Solving Flag Partial Differential Equations
15:00-18:00	Excursion in Summer Palace
18:00-20:00	Banquet in Summer Palace (Ting Li Guan Restaurant)

Friday, October 29	
Morning Session I Chair: Li Guo	
9:00-9:50	**Chengming Bai** Classical Yang-Baxter Equation and Its Extensions
9:50-10:00	Tea Break
Morning Session II Chair: William Sit	
10:00-10:50	**Yuqun Chen** Gröbner-Shirshov Bases for Rota-Baxter Algebras and PBW Theorem for Dendriform Algebras
10:50-11:10	Tea Break

Morning Session III Chair: Qinghu Hou	
11:10-11:35	**Georg Regensburger** Integro-Differential Algebras, Operators, and Polynomials
11:35-12:00	**V. Ravi Srinivasan** Symbolic Integration
12:00-14:00	Lunch at Wuke Restaurant, fourth floor
Afternoon Session I Chair: François Ollivier	
14:00-14:50	**Jean Lévine** Differentially Flat Nonlinear Control Systems: an Overview of the Theory and Applications, and Differential Algebraic Aspects
14:50-15:00	Tea Break
Afternoon Session II Chair: Ziming Li	
15:00-15:25	**Zbigniew Bartosiewicz** Differential Universes of Control Systems on Time Scales
15:25-15:50	**Daniel Robertz** Parametrizing Linear Systems
15:50-16:00	Tea Break
16:00-16:40	Poster Presentations
Afternoon Session III Chair: Dongming Wang	
16:40-17:05	**François Lemaire** Application of Differential Algebra to the Quasi-Steady State Approximation in Biology and Physics
17:05-17:30	**Meng Zhou** Gröbner Bases in Difference-Differential Modules and Their Applications
17:30-19:00	Dinner at Wuke Restaurant, fourth floor

Saturday, October 30	
Morning Session I Chair: Meng Zhou	
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9:50-10:00	Tea Break
Morning Session II Chair: Sebastian Walcher	
10:00-10:50	**Irina Kogan** Invariants via Moving Frames: Computation and Applications
10:50-11:10	Tea Break
Morning Session III Chair: Ruyong Feng	
11:10-11:35	**Gabriela Jeronimo** A Geometric Index Reduction Method for DAE Systems
11:35-12:00	**Zhenya Yan** Exact Solutions to Three-Dimensional Generalized Nonlinear Schrödinger Equations with Varying Potential and Nonlinearities
12:00-12:10	Closing Remarks
12:10-14:00	Lunch at Wuke Restaurant, fourth floor

General Information

Notes for participants

- On the afternoon of Oct. 28, there will be an excursion to the Summer Palace. Shuttle buses are leaving for the Summer Palace in front of Siyuan Building at 15:00, Oct. 28, 2010.
- The banquet takes place at the Ting Li Guan Restaurant in the Summer Palace at 18:00, Oct. 28, 2010.
- Lunches and dinners are served at Wuke Hotel, fourth floor. If you have any dietary requirement, please contact Dr. Ruyong Feng (ryfeng@amss.ac.cn).
- Wireless connection is available in the conference hall.

Telephone Numbers for Emergency

Ambulance	120
Fire brigade	119
Police	110
Airport Beijing	86-10-6454-1111/6454-1100
Jade Palace Hotel	86-10-6262-8888
Wuke Hotel	86-10-8264-9140
Unis Center	86-10-6279-1888

Invited Speakers

Chengming Bai	Nankai University, China
Yuqun Chen	South China Normal University, China
Lucia Di Vizio	Universit é Paris 7, France
Irina Kogan	North Carolina State University, USA
Jean Lévine	Ecole Nationale Sup é rieure des Mines de Paris, France
Gloria Mari Beffa	University of Wisconsin-Madison, USA
Jorge Martín-Morales	University of Zaragoza, Spain
Claude Mitschi	Universit é Louis Pasteur et CNRS, France
Sebastian Walcher	RWTH Aachen University, Germany
Chunming Yuan	KLMM, Chinese Academy of Sciences, China
Zbigniew Bartosiewicz	Bialystok University of Technology, Poland
Charlotte Hardouin	Institut de math é matiques de Toulouse, France
Qinghu Hou	Nankai University, China
Gabriela Jeronimo	Universidad de Buenos Aires, Argentina
François Lemaire	Université Lille 1, France
Alexander Levin	The Catholic University of America, USA
Fang Li	Zhejiang University, China
Alexander V. Mikhailov	University of Leeds, UK
Ned Nedialkov	McMaster University, Canada
Alexey Ovchinnikov	The City University of New York, USA
Georg Regensburger	RICAM, Austria
Daniel Robertz	RWTH Aachen University, Germany
Ravi Srinivasan	Rutgers University at Newark, USA
Dmitry Trushin	Moscow State University, Russia
Xiaoping Xu	AMSS, Chinese Academy of Sciences, China
Zhenya Yan	KLMM, Chinese Academy of Sciences, China
Meng Zhou	Beihang University, China

Workshop Chair

Xiao-Shan Gao	KLMM, Chinese Academy of Sciences, China

Program Committee

Li Guo	Rutgers University at Newark, USA
Julia Hartmann	RWTH Aachen University, Germany
Evelyne Hubert	INRIA Sophia Antipolis, France
Ziming Li	KLMM, Chinese Academy of Sciences, China
Francois Ollivier	Ecole Polytechnique, France
Michael Singer	North Carolina State University, USA
William Sit	The City College of The City University of New York, USA

Local Arrangements

Ruyong Feng	KLMM, Chinese Academy of Sciences, China
Zhenya Yan	KLMM, Chinese Academy of Sciences, China
Chunming Yuan	KLMM, Chinese Academy of Sciences, China

Organized by

Key Laboratory of Mathematics Mechanization, CAS

Sponsored by

Academy of Mathematics and Systems Science, CAS

Chinese Society of Computer Mathematics

Institute of Systems Science, CAS

Key Lab of Mathematics Mechanization, CAS

National Natural Science Foundation of China

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Talk Abstracts

Classical Yang-Baxter Equation and Its Extensions

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We first give a brief introduction to the classical Yang-Baxter equation which emphasizes the relationship between the tensor and operator forms. Then we give two approaches to extend the classical Yang-Baxter equation. One is motivated by the study of Rota-Baxter algebras, which is also related to the study of integrable systems. Another is to get the analogues in the other algebras, which is related to the study of bialgebras. Moreover, there are some interesting algebraic structures behind these two approaches.

Differential Universes of Control Systems on Time Scales

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Let $\mathcal{F}_n$ be a family of real-valued functions defined on open subsets of $\mathbb{R}^n$ and let $\mathcal{F}$ be the disjoint union of all $\mathcal{F}_n$ for $n \in \mathbb{N}$. Let X be an arbitrary set. A set $\mathcal{U}$ of real-valued partially defined functions on X is called a *(function)* $\mathcal{F}$*-universe on* X (or just *universe* if $\mathcal{F}$ and X are fixed), if $\mathcal{U}$ contains the global 0 function, is closed with respect to amalgamation (i.e. glueing up functions that agree on the common domain) and is closed with respect to substitutions to functions from $\mathcal{F}$. If $\mathcal{F}$ consists of all analytic functions of finitely many variables, then $\mathcal{U}$ is an *analytic universe.* If $\mathcal{F}$ is the set of all polynomials of finitely many variables and $\mathcal{U}$ contains only global functions on X, then $\mathcal{U}$ is an algebra of real functions on X.

A *morphism* of two $\mathcal{F}$-universes $\mathcal{U}_1$ on X_1 and $\mathcal{U}_2$ on X_2 is a map $\tau : \mathcal{U}_1 \to \mathcal{U}_2$ that commutes with substitutions and amalgamation. A bijective morphism is an *isomorphism.*

Let $\mathcal{U}$ be an $\mathcal{F}$-universe and $\sigma : \mathcal{U} \to \mathcal{U}$ be a morphism. A map $\Delta : \mathcal{U} \to \mathcal{U}$ is a σ*-derivation* of $\mathcal{U}$ if there is $\mu > 0$ such that $\sigma = \mathrm{id} + \mu\Delta$ (thus

$\Delta = (\sigma - \mathrm{id})/\mu$). We extend this definition to $\mu = 0$ (so $\sigma = \mathrm{id}$) adding the standard requirement (the chain rule)

$$\Delta(F(\varphi_1, \ldots, \varphi_n)) = \sum_{k=1}^{n} \frac{\partial F}{\partial x_k}(\varphi_1, \ldots, \varphi_n)\Delta(\varphi_k).$$

Let $\varphi^\sigma := \sigma(\varphi)$ and $\varphi^\Delta := \Delta(\varphi)$.

A (σ-)*differential universe* is a universe $\mathcal{U}$ together with a σ-derivation Δ (for some σ). Differential universes $(\mathcal{U}_1, \Delta_1)$ and $(\mathcal{U}_2, \Delta_2)$, corresponding to the same μ, are *isomorphic*, if there is an isomorphism $\tau : \mathcal{U}_1 \to \mathcal{U}_2$ such that $\tau \circ \Delta_1 = \Delta_2 \circ \tau$.

Proposition 1 *Let $F \in \mathcal{F}_n$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{U}$. If F is of class C^1 then*

$$F(\varphi_1, \ldots, \varphi_n)^\Delta = \int_0^1 \sum_{k=1}^{n} \frac{\partial F}{\partial x_k}(\varphi_1 + s\mu\varphi_1^\Delta, \ldots, \varphi_n + s\mu\varphi_n^\Delta)\varphi_k^\Delta ds$$

for the σ-derivation Δ corresponding to μ.

Corollary 2 *If Δ is a σ-derivation then*

$$(\varphi\psi)^\Delta = \varphi^\sigma\psi^\Delta + \varphi^\Delta\psi = \varphi\psi^\Delta + \varphi^\Delta\psi^\sigma = \varphi\psi^\Delta + \varphi^\Delta\psi + \mu\varphi^\Delta\psi^\Delta,$$

for $\varphi, \psi \in \mathcal{U}$.

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. We shall consider here only homogeneous time scales, which are either $\mathbb{R}$ or $\mu\mathbb{Z}$ for $\mu > 0$. The constant μ is the *graininess* of $\mathbb{T}$. For $\mathbb{T} = \mathbb{R}$ we set $\mu = 0$. If $\alpha : \mathbb{T} \to \mathbb{R}$ then its Δ derivative at $t \in \mathbb{T}$, denoted by $f^\Delta(t)$, is either $f'(t)$ for $\mathbb{T} = \mathbb{R}$ (ordinary derivative) or $(\alpha(t + \mu) - \alpha(t))/\mu$ for $\mathbb{T} = \mu\mathbb{Z}$. Higher Δ derivatives are defined inductively. By $f^{[k]}$ we denote the Δ derivative of f of order k.

Two analytic control systems with output, on a time scale $\mathbb{T}$,

$$\Sigma :\ x^\Delta = f(x, u),\ y = h(x), \quad \tilde{\Sigma} :\ \tilde{x}^\Delta = \tilde{f}(\tilde{x}, \tilde{u}),\ \tilde{y} = \tilde{h}(\tilde{x})$$

are *externally dynamically equivalent*, if there exist dynamic transformations

$$y = \phi(\tilde{y}, \tilde{y}^{[1]}, \ldots, \tilde{y}^{[k]}, \tilde{u}, \tilde{u}^{[1]}, \ldots, \tilde{u}^{[k]}),\ u = \psi(\tilde{y}, \tilde{y}^{[1]}, \ldots, \tilde{y}^{[k]}, \tilde{u}, \tilde{u}^{[1]}, \ldots, \tilde{u}^{[k]})$$

and

$$\tilde{y} = \tilde{\phi}(y, y^{[1]}, \ldots, y^{[k]}, u, u^{[1]}, \ldots, u^{[k]}),\ \tilde{u} = \tilde{\psi}(y, y^{[1]}, \ldots, y^{[k]}, u, u^{[1]}, \ldots, u^{[k]})$$

that transform external trajectories $(\tilde{y}, \tilde{u})$ of $\tilde{\Sigma}$ onto external trajectories (y, u) of Σ and vice versa, and are mutually inverse on trajectories.

Let $\mathcal{U}$ denote the C^ω-universe of all analytic partially defined functions depending on finitely many variables from the set
$\{x_i, i = 1, \ldots, n,\ u_j^{[k]}, j = 1, \ldots, m; k \geq 0\}$. The map

$$\sigma(\varphi)(x, u^{[0]}, u^{[1]}, \ldots) := \varphi(f(x, u^{[0]}), u^{[1]}, \ldots)$$

is a morphism of $\mathcal{U}$ and an extension of Δ given by

$$\Delta(\varphi)(x, u^{[0]}, u^{[1]}, \ldots) := \int_0^1 \left[\frac{\partial \varphi}{\partial x}(x + s\mu f(x, u^{[0]}), u^{[0]} + s\mu u^{[1]}, \ldots) f(x, u^{[0]}) \right.$$
$$\left. + \sum_{k=0}^{\infty} \frac{\partial \varphi}{\partial u^{[k]}}(x + s\mu f(x, u^{[0]}), u^{[0]} + s\mu u^{[1]}, \ldots) u^{[k+1]} \right] ds$$

is a σ-derivation of $\mathcal{U}$.

Let $\mathcal{U}_\Sigma$ be the smallest C^ω-universe contained in $\mathcal{U}$, containing h_j, $j = 1, \ldots, r$, (the components of h) and u_i, $i = 1, \ldots, m$, (the components of u) and invariant with respect to the derivation Δ. Then $(\mathcal{U}_\Sigma, \Delta)$ is a σ-differential universe

Under some natural assumptions about the systems, the following can be shown.

Theorem 3 *The systems Σ and $\tilde{\Sigma}$ are externally dynamically equivalent if and only if the σ-differential universes $\mathcal{U}_\Sigma$ and $\mathcal{U}_{\tilde{\Sigma}}$ are isomorphic.*

Gröbner-Shirshov Bases for Rota-Baxter Algebras and PBW Theorem for Dendriform Algebras

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We establish the Composition-Diamond lemma for associative nonunitary Rota-Baxter algebras with weight λ. As applications, we obtain a linear basis of a free commutative Rota-Baxter algebra without unity, show that every countably generated Rota-Baxter algebra with weight 0 can be embedded into a two-generated Rota-Baxter algebra. Moreover, by using this lemma, we prove that every dendriform algebra over a field of characteristic 0 can be embedded into its universal enveloping Rota-Baxter algebra of weight 0.

Arithmetic of The Algebraic and Differential Generic Galois Group of A q-difference Equation

(Joint work with Charlotte Hardouin)

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Let k be a perfect field and K be a finite extension of $k(q)$. The Grothendieck conjecture on p-curvatures asserts that the solutions of a linear differential equation L with coefficients in $K(x)$ with K a number field are algebraic if and only if the p-curvatures of the equation L equals zero for almost all prime p of K. We prove a discrete analog of this conjecture. In the case of q-difference equations, *i.e.* , we prove the equivalence among the following facts:
1. a q-difference module over $K(x)$ is trivial or equivalently a q-difference equation $Y(qx) = A(x)Y(x)$ where $A \in Gl_\nu(K(x))$;
2. its specialization at $q = \xi$ has zero curvature for almost all primitive roots of unity ξ;
3. its specialization at $q = \xi$ is endowed with a (necessarily trivial) structure of iterated ξ-difference module, for almost all primitive roots of unity ξ.
The equivalence between 1. and 3. is an analog of a Matzat-van der Put conjecture for differential equations over field of positive characteristic.

Then we consider two kinds of Galois groups (the second one only under the assumption that k has zero characteristic) attached to a q-difference module $\mathcal{M}$ over $K(x)$:
- the generic (also called intrinsic) Galois group in the sense of [Kat82] and [DV02], which is an algebraic group over $K(x)$. We refer to this group as the generic algebraic Galois group, or simply as the generic Galois group;
- the generic differential Galois group, which is a differential algebraic group in the sense of Kolchin, associated to the smallest differential tannakian category generated by $\mathcal{M}$, equipped of the forgetful functor.
The result above leads to an arithmetic description of the generic algebraic

(resp. differential) Galois group: it is the smallest algebraic (resp. differential) group containing the curvatures of the q-difference module for almost all primitive roots of unity ξ. Although no general Galois correspondence holds in this setting, if the characteristic of k is positive and the generic Galois group is nonreduced, we can prove some devissage.

By specialization of the parameter q at 1 in the Galois group, we obtain an upper bound for the generic Galois group of the differential equation obtained by specialization. This upper bound has a curvature characterization in the spirit of the Grothendieck-Katz conjecture, but *via* different curvatures than the ones appearing in the conjecture.

References

[DV02] Lucia Di Vizio. Arithmetic theory of q-difference equations. The q-analogue of Grothendieck-Katz's conjecture on p-curvatures. *Inventiones Mathematicae*, 150(3):517–578, 2002. arXiv:math.NT/0104178.

[HS08] Charlotte Hardouin and Michael F. Singer. Differential Galois theory of linear difference equations. *Mathematische Annalen*, 342(2):333–377, 2008.

[Kat82] Nicholas M. Katz. A conjecture in the arithmetic theory of differential equations. *Bulletin de la Société Mathématique de France*, 110(2):203–239, 1982.

[vdPS97] Marius van der Put and Michael F. Singer. *Galois theory of difference equations*. Springer-Verlag, Berlin, 1997.

Non Linear q-difference Equations and Curvatures

(Joint work with Lucia Di Vizio)

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We show that the results in the talk of L. Di Vizio combined with [DV02] and [Hen96], allow to describe the generic (algebraic and differential) Galois groups of a complex q-difference equation, with $q \in \mathbb{C} \smallsetminus \{0, 1\}$, by means of curvatures (but not the same curvatures according that q is a root of unity; an algebraic number, not a root of unity; or a transcendental number).

There are many Galois theories for q-difference equations defined over fields such as $\mathbb{C}$, the field of elliptic functions, or the differential closure of $\mathbb{C}$. We give the comparison results between the two generic Galois groups that we have introduced and the other Galois groups in the literature, especially with the Hardouin-Singer Galois group. In particular, we show that the differential algebraic relations between the solutions of the equation are endowed by the differential algebraic relations satisfied by the curvatures.

Inspired by the work of B. Malgrange for non linear differential equations, A. Granier attach to a non linear q-difference equation $Y(qx) = F(Y(x))$ with $Y \in Gl_\nu$, a D-groupoid, *i.e.* , a differential ideal in the space of jets of the analytic variety $P^1\mathbb{C} \times \mathbb{C}^\nu$. The Malgrange-Granier D-groupoid was recently used by G. Casale and J. Roques to prove some non-integrability results and for a linear q-difference system with constant coefficients, A. Granier was able to show that the D-groupoid coincides with the usual Galois group of the system.

The arithmetic description of the differential generic Galois group plus the results of A. Granier imply that, in the linear case, the Malgrange-Granier D-groupoid of a linear q-difference system essentially coincides with the Kolchin closure of the dynamic of the system and that the group that fix the transversals in the Malgrange-Granier D-groupoid is the generic differential Galois group introduced in the previous talk. This result is the first attempt to relate the D-groupoid of B. Malgrange and the linear differential algebraic groups of Kolchin.

References

[DV02] Lucia Di Vizio. Arithmetic theory of q-difference equations. The q-analogue of Grothendieck-Katz's conjecture on p-curvatures. *Inventiones Mathematicae*, 150(3):517–578, 2002. arXiv:math.NT/0104178.

[Hen96] Peter Hendriks. *Algebraic Aspects of Linear Differential and Difference Equations.* PhD thesis, University of Groningen., 1996.

Hypergeometric Series Solutions of Linear Operator Equations

Qinghu Hou
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Let K be a field and $L\colon K[x] \to K[x]$ be a linear operator acting on the ring of polynomials in x over the field K. We provide a method to find a suitable basis $\{b_k(x)\}$ of $K[x]$ and a hypergeometric term c_k such that $y(x) = \sum_{k=0}^{\infty} c_k b_k(x)$ is a formal series solution to the equation $L(y(x)) = 0$. This method is applied to construct hypergeometric representations of orthogonal polynomials from the differential/difference equations or recurrence relations they satisfied. Both the ordinary cases and the q-cases are considered.

This is a joint work with Yan-Ping Mu of Tianjin University of Technology.

A Geometric Index Reduction Method for DAE Systems

Gabriela Jeronimo
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We will address the index reduction problem for quasi-regular DAE systems. We will show that any of these systems can be transformed into a generically equivalent first order DAE system consisting of a single purely algebraic (polynomial) equation plus an under-determined ODE (that is, a semi-explicit DAE system with differentiation index 1). Finally, we will describe a Kronecker-type algorithm with bounded complexity which computes this associated system.

Our approach makes use of the computation of successive derivatives of the equations, as many as the differentiation index of the input system, but, unlike previous methods, we deal with them in a purely algebraic way. Using a construction originally introduced by Kronecker, we parametrize the points of an algebraic variety associated with the system of all the new equations by means of the points of a hypersurface. In order to keep track of the differential structure, we use the parametrizations to construct a vector field over the hypersurface defining the semi-explicit DAE system.

This is a joint work with Lisi D'Alfonso, François Ollivier, Alexandre Sedoglavic and Pablo Solernó.

Invariants via Moving Frames: Computation and Applications

Irina Kogan
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I will describe algorithms for computing algebraic, differential and integral invariants that are based on modern formulations of Cartan's moving frame method.

I will discuss applications of invariants in differential and variational calculus, and also in classification problems that arise in computer image recognition.

Application of Differential Algebra to The Quasi-Steady State Approximation in Biology and Physics

François Lemaire
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The quasi-steady state approximation (QSSA) is a technics for approximating the evolution of a dynamical system which involves both slow and fast dynamics. It can be used when the fast dynamics tend to an equilibria which slowly drifts due to action of the slow dynamics.

I will show how to use differential algebra, and more precisely differential elimination, to solve the QSSA in the context of chemical reactions systems, as well as some examples taken from physics (pendulum, communicating vessels, diffusion, ...).

I will also present the Maple package called MABSys (Modeling and Analysis of Biological Systems) which provides tools for performing the QSSA in a transparent way.

Dimension of Finitely Generated Differential and Difference Field Extensions

Alexander Levin
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In 1964,E. Kolchin introduced the concept of a differential dimension polynomial associated with a finitely generated differential field extension L/K. Even though this polynomial, which measures the "size" of the extension, depends on the system of differential generators of L/K, it carries certain invariants that do not depend on such generators (one of these invariants is the differential transcendence degree of L/K).

In this talk we review basic facts about differential dimension polynomials and their analogs for difference field extensions, consider alternative characterizations of such polynomials and their invariants, and discuss applications of differential and difference dimension polynomials to the analysis of systems of algebraic differential and difference equations. We will also discuss the concept of limit degree, which characterizes the dimension of difference field extensions of zero difference transcendence degree.

Differentially Flat Nonlinear Control Systems: An Overview of The Theory and Applications, and Differential Algebraic Aspects

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Motion planning and trajectory tracking are among the main problems in Control Theory. These problems, though non trivial even in the class of linear systems, receive a simple solution in the particular class of systems called "Differentially flat".

For a finite dimensional system described by a set of differential equations, controlled by m independent inputs, it is called differentially flat if, and only if, there exists a vector of differentially independent generalized output (i.e. whose components are functions of the state, input variables and a finite number of their time derivatives) of the same dimension as the input vector, such that every system variable may be expressed as a function of this output and a finite number of its derivatives. Such an output, if it exists, is called "flat output". This definition may be rigorously expressed in the framework of manifolds of jets of infinite order and Lie-Bäcklund isomorphisms. As a direct consequence of this definition, all the trajectories generated by such a system may be parametrized by flat output ordinary curves in a suitable vector space (thus geometrically flat), without need to integrate the differential equations of the system. Moreover, it can be shown that the flatness property is equivalent to "endogeneous dynamic feedback linearizability", a property which means that the system may be transformed by Lie-Bäcklund isomorphism into a linear controllable system. The latter property is most helpful when one wants to stably track a reference trajectory by feedback, using standard linear control design methods.

The practical interest and simplicity of this approach, the so-called "flatness-based design", will be demonstrated by videos of various oscillating mechanical systems, showing how their oscillations may be significantly and robustly attenuated. Note also that, though non generic, differentially flat systems are often encountered in industrial applications

A major difficulty related to such systems concerns the determination of flat outputs if they exist. Computable necessary and sufficient conditions for differential flatness have been obtained, in the framework of module theory on non commutative polynomial rings of the operator d/dt. A system is differentially flat if and only if its variational system module admits an integrable basis, more precisely a basis of polynomial one-forms that admits a unimodular integrating factor. This integrating factor is determined by solving the so-called "generalized moving frame structure equations".

Computer algebra sequential procedures will be presented to compute its solutions, directly giving the flat outputs of the system if they exist. Simple examples, showing the computational complexity, and open problems will be presented, mostly related to the finiteness of these procedures.

References

[1] J. Lévine, Analysis and Control of Nonlinear Systems: A Flatness-based Approach, *Mathematical Engineering Series*, Springer, 2009.

[2] J. Lévine, On Necessary and Sufficient Conditions for Differential Flatness, to appear in *Applicable Algebra in Engineering, Communication and Computing.*

About the Lie Algebras of Differential Operators on A Path Algebra

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The aim of this article is to study the Lie algebra $\mathfrak{Diff}(\mathbf{k}\mathfrak{d})$ of differential operators on the path algebra $\mathbf{k}\Gamma$ of a quiver Γ and relate this Lie algebra to the algebraic and combinatorial properties of $\mathbf{k}\Gamma$. We first characterize when a linear operator on a path algebra is a differential operator and thus obtain a standard basis of $\mathfrak{Diff}(\mathbf{k}\mathfrak{d})$.

Moreover, mainly, we show that the Lie algebra $\mathfrak{Out}\mathfrak{Diff}(\mathbf{k}\mathfrak{d})$ of outer differential operators, defined to be the quotient of $\mathfrak{Diff}(\mathbf{k}\mathfrak{d})$ modulo $\mathfrak{In}\mathfrak{Diff}(\mathbf{k}\mathfrak{d})$, is closely related to the topological and graph theoretic properties of a finite connected planar Γ, such as the genus 0 and Euler's characteristic of the Riemann sphere which the quiver Γ is embedded into rather than into the plane. In particular, from a careful analysis of the conection matrix and boundary matrix of a quiver, a canonical basis of $\mathfrak{Out}\mathfrak{Diff}(\mathbf{k}\mathfrak{d})$ is given.

Bi-Hamiltonian Flows and Geometric Realizations

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In this talk, I will describe how bi-Hamiltonian completely integrable systems appear naturally in certain geometric backgrounds via the natural definition of a pair of associated Hamiltonian structures. We will use these natural constructions to obtain geometric realizations for integrable systems; that is, associated geometric curve flows whose behavior is closely tied to that of the integrable systems. Throughout the talk I will emphasize the algebraic part of these constructions and will describe open problems that can be algebraically addressed. The main one is the classification of normal forms for Maurer-Cartan matrices associated to group-based moving frames.

Introduction to D-module Theory: Algorithms for Computing Bernstein-Sato Polynomials

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Go directly to the second page if you want to skip the details.

1 Basic notations

Assume $\mathbb{K}$ to be a field of characteristic 0. By R_n we denote the ring of polynomials $\mathbb{K}[x_1, \ldots, x_n]$ in n variables over $\mathbb{K}$ and by D_n we denote the ring of $\mathbb{K}$-linear partial differential operators with coefficients in R_n, that is the n-th Weyl algebra. The ring D_n is the associative $\mathbb{K}$-algebra generated by the partial differential operators ∂_i and the multiplication operators x_i subject to relations

$$\{\partial_i x_j = x_j \partial_i + \delta_{ij},\ x_j x_i = x_i x_j,\ \partial_j \partial_i = \partial_i \partial_j \mid 1 \leq i, j \leq n\}.$$

That is, the only non-commuting pairs of variables are (x_i, ∂_i); they satisfy the relation $\partial_i x_i = x_i \partial_i + 1$. Finally, denote $D_n[s] = D_n \otimes_{\mathbb{K}} \mathbb{K}[s]$.

2 Main object to study

Let us recall Bernstein's construction. Given $f \in R_n$ a non-zero polynomial, we consider $M = R_n[s, \frac{1}{f}] \cdot f^s$ which is by definition the free $R_n[s, \frac{1}{f}]$-module of rank one generated by the formal symbol f^s. Then M has a natural structure of left $D_n[s]$-module. Here the differential operators act in a natural way,

$$\partial_i\big(g(s,x) \cdot f^s\big) = \left(\frac{\partial g}{\partial x_i} + s g(s,x) \frac{\partial f}{\partial x_i} \frac{1}{f}\right) \cdot f^s \ \in M.$$

Theorem 4 (Bernstein) *For every non-constant polynomial $f \in R_n$, there exists a non-zero polynomial $b(s) \in \mathbb{K}[s]$ and a differential operator $P(s) \in$*

$D_n[s]$ *such that*

$$P(s)f \cdot f^s = b(s) \cdot f^s \; \in \; R_n\Big[s, \frac{1}{f}\Big] \cdot f^s = M. \tag{1}$$

The monic polynomial $b(s)$ of minimal degree satisfying (1) is called the **Bernstein-Sato polynomial** or the **global b-function**.

3 Abstract

Bernstein-Sato polynomial of a hypersurface is an important invariant in Singularity Theory with numerous applications. It is known, that it is complicated to obtain it computationally, as a number of open questions and challenges indicate. In this talk we overview the main properties of this polynomial as well as several well-known algorithms to compute it.

In the second part of the talk, we propose a family of algorithms called `checkRoot` for optimized check of whether a given rational number is a root of Bernstein-Sato polynomial and the computations of its multiplicity. These algorithms are applied in numerous situations.

1. Computation of the b-functions via upper bounds. We use several techniques to find such upper bounds.
    - Embedded resolutions.
    - Topologically equivalent singularities.
    - A'Campo's formula / Spectral numbers.
2. Integral roots of b-functions. They are important in very different settings. We consider these two problems.
    - Logarithmic comparison problem.
    - Intersection homology D-module.
3. Stratification associated with local b-functions. Notably, the algorithm we propose does not employ primary decomposition.
4. Bernstein-Sato polynomials for varieties.

These methods have been implemented in SINGULAR:PLURAL as libraries `dmod.lib` and `bfun.lib`. All the examples that we present have been computed with this implementation.

Preprint available at arXiv:1003.3478v1 [math.AG].

Symmetries and Integrability Conditions for Difference Equations

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We set up an algebraic framework for symmetries and conservation laws of difference equations. With a difference equation we associate a difference field of fractions. Then symmetries and conservation laws can be regarded as exceptional properties of the field. Existence of an infinite hierarchy of symmetries is taken as a definition of integrability. The concept of recursion operator, which generates the hierarchy of symmetries is adapted to the case of partial difference equations. We have constructed an infinite sequence of integrability conditions for a given difference equation, which are necessary conditions for the existence of a formal recursion operator. These conditions are presented in the form of a canonical sequence of conservation laws for a difference equation. If time permits, I am planning to discuss the concepts of co-symmetries and co-recursion operators as well.

Galoisian Approach to Monodromy Evolving Deformations with Regular Singularities

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The parametrized Picard-Vessiot theory recently developed by Cassidy and Singer, building on previous work by Kolchin and Landesman, provides appropriate tools to study algebraic and topological properties of parameterized linear differential systems. We will in particular define parameterized regular singularities of such systems. Isomonodromic deformations with only regular singularities are characterized by their parameterized Galois group being conjugate, as a linear differential algebraic group, to a constant linear algebraic group, We similarly characterize a special type of "projective" monodromy evolving deformations by algebraic properties of their parameterized Galois group. This is joint work with Michael F. Singer.

DAETS: a Differential-Algebraic Equation Code in C++ for High Index and High Accuracy

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Ned Nedialkov and John Pryce are the authors of DAETS, a C++ code for solving differential-algebraic equations (DAEs). It uses Pryce's structural analysis theory, and expands the solution in Taylor series using automatic differentiation. DAETS is very effective when high accuracy is required, and at solving problems of high index-we have solved artificial DAEs of index up to 47. DAETS is versatile: higher-order systems do not have to be cast in first-order form; it can solve explicit and implicit ODEs; it can solve purely algebraic problems, by simple or by arc-length continuation.

This talk will outline the theory and algorithms behind DAETs and the code structure of DAETS. We give examples of code's performance on standard test problems, an index-15 DAE, and a pure algebraic system solved as an implicit index-1 DAE using continuation. We also present current work on event location for systems of high-index DAEs, which is crucial for integrating hybrid systems of DAEs.

Differential Representations of SL_2

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Linear differential algebraic groups (LDAGs) were introduced by P. Cassidy and are now used to find differential algebraic relations among solutions of linear differential and difference equations with parameters. This is done via the parametrized differential and difference Galois theory of P. Cassidy, C. Hardouin, and M. Singer.

The representation theory LDAGs and the knowledge of differential algebraic subgroups of a given LDAG will be used to develop algorithms that calculate Galois groups of differential and difference equations with parameters. W. Sit characterized differential algebraic subgroups of SL_2. In this talk, we will be discussing the representation theory of SL_2.

Integro-Differential Algebras, Operators, and Polynomials

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Motivated by boundary problems for linear ordinary differential equations (LODEs), the notion of integro-differential algebra combines a differential algebra with an integral operator. As in a differential Rota-Baxter algebra, the integral should be a right inverse of the derivation but we additionally require a suitable version of integration by parts. This so-called differential Baxter axiom allows us to define an "evaluation" in any integro-differential algebra, which is also the starting point for treating initial and boundary conditions in an algebraic setting.

We first review basic properties and examples of integro-differential algebras. Then we discuss the construction and some algebraic and algorithmic aspects of the associated ring of integro-differential operators. Integro-differential operators provide in particular an algebraic structure for computing with boundary problems for LODEs as well as their solution operators (Green's operators). As a second basic algebraic structure, we also outline canonical forms for integro-differential polynomials generalizing the usual differential polynomials.

This talk is based on joint work with Markus Rosenkranz.

Parametrizing Linear Systems

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Given a system of (homogeneous) linear equations, an adequate way to represent the space of solutions is as the image of an operator to be constructed from the equations. In case the equations have coefficients in a field, Gaussian elimination achieves this objective. If the system is underdetermined, then the corresponding Gauss-reduced matrix singles out some variables of the system as parameters and specifies how all other variables are expressed (linearly) in terms of the parameters. This procedure can be viewed as identifying the kernel of the linear map induced by the system matrix as the image of another linear map.

More generally, if the equations have coefficients in a ring, that is not necessarily a (skew-) field, the question arises whether it is still possible to construct an operator which is defined over the same ring and whose image equals the space of solutions. For instance, a system of (homogeneous) linear partial differential equations may be written as an equation whose left hand side is a matrix differential operator applied to the vector of unknown functions and whose right hand side is zero. Is it possible to *parametrize* the system, i.e. to construct another matrix differential operator whose image equals the kernel of the given one? In general, the answer is negative.

This talk presents recent joint work with F. Chyzak and A. Quadrat addressing several aspects of the above problem, in particular the algorithmic decision of existence and the construction of parametrizations for the case of systems of linear partial differential equations (with constant, polynomial, rational, or analytic coefficients).

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Symbolic Integration

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In this talk, I will discuss the structure of liouvillian differential field extensions obtained by adjoining antiderivatives alone. A differential field extension of this kind need not be always Picard-Vessiot over its base field and if it is, then it is known that the differential Galois group is isomorphic to a unipotent algebraic group. To this end, I will illustrate a procedure to construct Picard-Vessiot extensions whose differential Galois group is isomorphic to a full unipotent subgroup of the general linear group. We will also compute differential equations for these extensions.

Differential Schemes and Differential Algebraic Varieties

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We consider the following three problems of differential algebra.

1) Differential spectrum of global sections. For an arbitrary Keigher ring we show that the differential spectrum of the ring of global sections coincides with the differential spectrum of the initial ring. If we change derivations by iterative derivations the result holds for every differential ring.

2) We introduce the notion of differential integral dependence and use this notion to describe all universally closed affine differential schemes. We use this machinery to give an example of complete irreducible affine differential algebraic variety.

3) We improve the Rosenfeld result about catenary property. We extend the set of points for which the catenary property holds.

Darboux Integrating Factors of Planar Polynomial Vector Fields: Inverse Problems

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Given a planar polynomial differential equation

$$\begin{aligned} x' &= P(x,y) \\ y' &= Q(x,y). \end{aligned}$$

One says that this system admits a Darboux integrating factor if there exist (irreducible, pairwise relatively prime) polynomials $f_1, \cdots, f_r$ and (nonzero) constants $d_1, \cdots, d_n$ such that

$$\mathrm{div} f_1^{-d_1} \cdots f_r^{-d_r} \begin{pmatrix} P \\ Q \end{pmatrix} = 0.$$

Integrating factors of this type are interesting for several reasons. For instance, they are (by a result of S. Lie) directly related to certain nonlinear symmetries of the differential equation, and by a now classical result of Prelle and Singer the existence of a Darboux integrating factor with rational exponents is necessary for the existence of an elementary first integral.

One verifies that the existence of a Darboux integrating factor $f_1^{-d_1}, \cdots, f_r^{-d_r}$ implies for every i the existence of a polynomial L_i such that

$$Pf_{i,x} + Qf_{i,y} = L_i f_i;$$

in other words the (complex) zero set of f_i is invariant for the differential equation. Thus, any discussion of Darboux integrating factors includes a discussion of algebraic invariant curves.

There have been a number of results on existence and nonexistence of invariant algebraic curves for a given polynomial vector field; most notably a recent computational approach by Coutinho and Schechter.

Inverse problems are also of interest, for computational and for structural reasons: Thus, given $f_1, \cdots, f_r$ and $d_1, \cdots, d_r$ determine all polynomial vector fields with the corresponding prescribed integrating factor; as a preliminary problem, determine all polynomial vector fields with prescribed invariant curves given by $f_1 = 0, \cdots, f_r = 0$. These inverse problems were essentially solved by C. Christo- pher, J. Lllibre, C. Pantazi and the speaker in recent years, in the following sense: (i) In the linear space of all vector fields admitting prescribed invariant curves, there exists a subspace of "trivial" vector fields such that the quotient modulo this subspace has finite dimension. The quotient can be determined, in principle, by standard Groebner base methods. (ii) In the linear space of all vector fields admitting a prescribed Darboux integrating factor, there exists a subspace of trivial vector fields such that the quotient modulo this subspace has finite dimension. To determine the quotient computationally, one has to work with standard problems in commutative algebra and, at a critical point, with the first Weyl algebra.

The talk will focus on these inverse problems, and outline some of the methods and open questions.

Methods of Solving Flag Partial Differential Equations

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Flag partial differential equations naturally appear in geometry, physics and the representation theory of Lie algebras (groups). In this talk, we present the methods of using a higher-order Campbell-Hausdorff formula and Lie algebra grading technique to solve them. In particular, we find a family of new special functions by which we are able to explicitly give the solutions of the initial value problems of a large family of constant-coefficient linear partial differential equations in terms of coefficients.

Exact Solutions to Three-Dimensional Generalized Nonlinear Schrödinger Equations with Varying Potential and Nonlinearities

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It is shown that using the similarity transformations, a set of three-dimensional generalzied nonlinear Schrödinger (NLS) equations with inhomogeneous coefficients can be reduced to one-dimensional stationary NLS equation with constant or varying coefficients, thus allowing for obtaining exact localized and periodic wave solutions. In the suggested reduction the original coordinates in the $(1+3)$ space are mapped into a set of one-parametric coordinate surfaces, whose parameter plays the role of the coordinate of the one-dimensional equation. We describe the algorithm of finding solutions and concentrate on power (linear and nonlinear) potentials presenting a number of case examples. Generalizations of the method are also discussed.

Differential Chow Form

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In this talk, an intersection theory for generic differential polynomials is presented. The intersection of an irreducible differential variety of dimension d and order h with a generic differential hypersurface of order s is shown to be an irreducible variety of dimension $d-1$ and order $h+s$. As a consequence, the dimension conjecture for generic differential polynomials is proved. Based on the intersection theory, the Chow form for an irreducible differential variety is defined and most of the properties of the Chow form in the algebraic case are extended to its differential counterpart. Furthermore, the generalized differential Chow form is defined and its properties are proved. As an application of the generalized differential Chow form, the differential resultant of $n+1$ generic differential polynomials in n variables is defined and properties similar to that of the Sylvester resultant of two univariate polynomials are proved.

This is joint work with Xiao-Shan Gao and Wei Li.

Gröbner Bases in Difference-Differential Modules and Their Applications

(Joint work with Franz Winkler)

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Let R be a commutative noetherian ring, $\Delta = \{\delta_1, \cdots, \delta_m\}$ and $\sigma = \{\alpha_1, \cdots, \alpha_n\}$ be set of derivations and automorphisms of the ring R, respectively, such that $\beta(x) \in R$ and $\beta(\gamma(x)) = \gamma(\beta(x))$ hold for any $\beta, \gamma \in \Delta \bigcup \sigma$ and $x \in R$. Then R is called a difference-differential ring with the basic set of derivations Δ and the basic set of automorphisms σ, or shortly a Δ-σ-ring. If R is a field, then it is called a Δ-σ-field. Λ will denote the commutative semigroup of elements of the form

$$\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \alpha_1^{l_1} \cdots \alpha_n^{l_n} \tag{1.1}$$

where $(k_1, \cdots, k_m) \in \mathbb{N}^m$ and $(l_1, \cdots, l_n) \in \mathbb{Z}^n$. This semigroup contains the free commutative semigroup Θ generated by the set Δ and free commutative semigroup Γ generated by the set σ. The subset $\{\alpha_1, \cdots, \alpha_n, \alpha_1^{-1}, \cdots, \alpha_n^{-1}\}$ of Λ will be denoted by σ^*.An expression of the form

$$\sum_{\lambda \in \Lambda} a_\lambda \lambda, \tag{1.2}$$

where $a_\lambda \in R$ for all $\lambda \in \Lambda$ and only finitely many coefficients a_λ are different from zero, is called a difference-differential operator (or shortly a Δ-σ-operator) over R. Two Δ-σ-operators $\sum_{\lambda \in \Lambda} a_\lambda \lambda$ and $\sum_{\lambda \in \Lambda} b_\lambda \lambda$ are equal if and only if $a_\lambda = b_\lambda$ for all $\lambda \in \Lambda$.The ring of all Δ-σ-operators over a Δ-σ-ring R is called the ring of difference-differential operators (or shortly the ring of Δ-σ-operators) over R, it will be denoted by D. A left D-module M is called a difference-differential module(or a Δ-σ-module). If M is finitely generated as a left D-module, then M is called a finitely generated Δ-σ-module.

In this talk we describe an extending of the theory of Gröbner bases to difference-differential modules. We present and verify algorithms for constructing these Gröbner bases counterparts. To this aim we introduce the concept of "generalized term order" on $\mathbb{N}^m \times \mathbb{Z}^n$ and on difference-differential modules. Also, we introduce a new concept, relative difference-differential Gröbner bases. Our notion of relative Gröbner basis is based on two generalized term orders on $\mathbb{N}^m \times \mathbb{Z}^n$. We define a special type of reduction for two generalized term orders in a free left module over a ring of difference-differential operators. Then the concept of relative Gröbner bases w.r.t. two generalized term orders is defined. An algorithm for constructing these Gröbner basis counterparts is presented and verified.

Using Gröbner bases on difference-differential modules and the relative Gröbner basis algorithm, it is possible to compute the difference-differential dimension polynomials of a difference-differential module and of a system of linear partial difference-differential equations, and difference-differential dimension polynomials in two variables.

The research is motivated by the difference-differential dimension polynomial theories of [Lev00]and [Lev07], in which the existence of the difference-differential dimension polynomial was proved via characteristic set.

In 2009 Christian Doench present the Maple implementations of two algorithms developed by M. Zhou and F. Winkler for computing a relative Gröbner basis of a finitely generated difference-differential module and use this to compute the bivariate difference-differential dimension polynomial of the module with respect to the natural bifiltration of the ring of difference-differential operators. And the implementations of the two algorithms are illustrated by some examples.

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Poster Abstracts

On the Structure of Multivariate Hyperexponential-hypergeometric Functions

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August 31, 2010

A hyperexponential-hypergeometric function is a nonzero solution of a first-order linear homogeneous differential-difference system over the field of multivariate rational functions. We present a structure theorem for multivariate hyperexponential-hypergeometric functions. This theorem is derived from the integrability conditions for the certificates of a multivariate hyperexponential-hypergeometric function.

Let k be a field of characteristic zero. For brevity, $\mathbf{t}$ stands for $(t_1, \ldots, t_m)$, $\mathbf{x}$ for $(x_1, \ldots, x_n)$, and $k(\mathbf{t}, \mathbf{x})$ for the field of rational functions in $t_1, \ldots, t_m$ and $x_1, \ldots, x_n$. Assume that

$$\delta_i(f) = \frac{\partial(f)}{\partial t_i} \quad \text{and} \quad \sigma_j(f) = f(\mathbf{t}, x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_n)$$

for all $f \in k(\mathbf{t}, \mathbf{x})$. By a *first-order fully integrable system* over $k(\mathbf{t}, \mathbf{x})$, we mean a system of the form

$$\delta_1(z) = a_1 z, \ \ldots, \ \delta_m(z) = a_m z, \ \sigma_1(z) = b_1 z, \ \ldots, \ \sigma_n(z) = b_n z \tag{1}$$

with $a_1, \ldots, a_m, b_1, \ldots, b_n \in k(\mathbf{t}, \mathbf{x})$, $b_1 \cdots b_n \neq 0$,

$$\delta_i(a_p) = \delta_p(a_i) \quad \text{for all } i, p \text{ with } 1 \leq i < p \leq m, \tag{2}$$

$$\frac{\sigma_q(b_j)}{b_j} = \frac{\sigma_j(b_q)}{b_q} \quad \text{for all } j, q \text{ with } 1 \leq j < q \leq n, \tag{3}$$

and

$$\frac{\delta_i(b_j)}{b_j} = \sigma_j(a_i) - a_i \quad \text{for all } i, j \text{ with } 1 \leq i \leq m \text{ and } 1 \leq j \leq n. \tag{4}$$

The equalities in (2), (3) and (4) are called *integrability conditions* for (1). For the first-order fully integrable system (1), there exists a simple differential-difference extension of $k(\mathbf{t}, \mathbf{x})$ such that it contains a nonzero solution of this system [2]. Such a nonzero solution h is called a *hyperexponential-hypergeometric function* over $k(\mathbf{t}, \mathbf{x})$. The rational functions a_i and b_j are called the certificates of h with respect to t_i and x_j, respectively.

In the case when $m = 0$, Abramov and Petkovšek [1] have shown that there exists a rational function f in $k(\mathbf{x})$ such that the certificate b_j of h with respect to x_j is of the form $b_j = \frac{\sigma_j(f)}{f} g_j$, where g_j is a rational function whose numerator and denominator factor into integer-linear factors for all j with $1 \leq j \leq n$. Their proof of the multivariate discrete case of the Wilf–Zeilberger conjecture is based on this result. In the case when $m = n = 1$, Proposition 5 in [3] shows that

$$a_1 = \frac{\delta_1(f)}{f} + x_1 \frac{\delta_1(\beta)}{\beta} + \gamma \quad \text{and} \quad b_1 = \frac{\sigma_1(f)}{f} \beta\alpha,$$

for some $f \in k(t_1, x_1)$, $\beta, \gamma \in k(t_1)$ and $\alpha \in k(x_1)$. The above result is used to develop algorithms for finding Liouvilian solutions of prime order linear difference-differential equations.

In this poster, we present an idea proving the following lemma, which generalizes the above formula for a_1 and b_1 to the multivariate case.

Lemma 1. *Assume that $a_1, \ldots, a_m, b_1, \ldots, b_n \in k(\mathbf{t}, \mathbf{x})$ are the certificates of a hyperexponential-hypergeometric function, then there exist $f \in k(\mathbf{t}, \mathbf{x})$, $\beta_1, \ldots, \beta_n \in k(\mathbf{t})$, $\gamma_1, \ldots, \gamma_m \in k(\mathbf{t})$ and $\alpha_1, \ldots, \alpha_n \in k(\mathbf{x})$ such that*

$$a_i = \frac{\delta(f)}{f} + \sum_{j=1}^{n} x_j \frac{\delta_i(\beta_j)}{\beta_j} + \gamma_i \quad \textit{and} \quad b_j = \frac{\sigma_j(f)}{f} \beta_j \alpha_j, \tag{5}$$

for all i with $1 \leq i \leq m$ and j with $1 \leq j \leq n$. Moreover, all the equalities in (2), (3) and (4) remain valid if a_i is replaced by γ_i and b_j is replaced by α_j.

According to Lemma 1, a hyperexponential-hypergeometric function can be decomposed into a multiplicative form.

Theorem 2. *Assume that k is algebraically closed and $h(\mathbf{t}, \mathbf{x})$ is hyperexponential-hypergeometric over $k(\mathbf{t}, \mathbf{x})$. Then h is of the form*

$$h = c f \beta_1^{x_1} \cdots \beta_n^{x_n} h',$$

where $c \in k$, $\beta_1, \ldots, \beta_n \in k(\mathbf{t})$ and the certificates of h' are $\gamma_1, \ldots, \gamma_m, \alpha_1, \ldots, \alpha_n$ with $\gamma_i \in k(\mathbf{t})$ and $\alpha_j \in k(\mathbf{x})$ satisfying the integrability conditions.

Example 3 (Jacobi polynomial)**.** *$P_n^{(m_1,m_2)}(x) = \sum_\ell h(x, \ell, n, m_1, m_2)$, where*

$$h = \frac{1}{2^n} \binom{n + m_1}{n - \ell} \binom{n + m_2}{\ell} (x - 1)^\ell (x + 1)^{n-\ell}.$$

Then h is a hyperexponential-hypergeometric function with a multiplicative form

$$h = \left(\frac{x - 1}{x + 1}\right)^\ell (x + 1)^n h', \quad \textit{where} \quad h' = \frac{1}{2^n} \binom{n + m_1}{n - \ell} \binom{n + m_2}{\ell}.$$

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Matrix representations for generalized term orders

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2010 Mathematics Subject Classification. 12H05,12H10

Dealing with polynomial rings in computer algebra is strongly connected with the concept of Gröbner bases due to the fact that they are understood to provide a possibility of performing algorithms suitable for most essential computations in polynomial rings. However a big drawback is the time complexity which in the worst case can be double exponential in the number of solutions of the system of equations in concern. In fact, the complexity depends on the term order used for the computations. In applications it is sometimes favorable to use a so-called elimination order to compute a Gröbner bases which makes it easy to solve the given system of equations. Unfortunately especially these orders usually lead to more complex computations than graded orders. By introducing concepts form combinatorics and polyhedral geometry the theory was advanced by creating the concepts of Gröbner fan and Gröbner walk which are used for converting a given Gröbner basis to one with respect to a different term order. The point is that for certain cases it can be more efficient to compute a Gröbner basis with respect to a graded term order and transform it to a Gröbner basis with respect to an elimination order using the Gröbner walk than to compute the later directly using Buchberger's algorithm. The trigger for these developments was Robbiano's classification of term orders stating that every term order can be represented by a matrix over the reals [Rob85]. In rings of difference-differential operators there are several approaches to compute Gröbner bases. Zhou and Winkler [ZW06, ZW08] extended techniques used for Gröbner basis computations in Laurent polynomial rings including the concept of generalized term orders in order to provide a generalized view of the situation. A crucial point for reasoning about the comlexity of their algorithm is a better understanding of generalized term orders. We are going to show that for any generalized term order on the set of difference-differential terms $[\Delta, \Sigma] = [\delta_1, \ldots, \delta_m, \sigma_1, \ldots, \sigma_n]$ there exists $t > m + n$, a term order on the set of terms $[X] = [x_1, \ldots, x_t]$ and a map $\phi : [\Delta, \Sigma] \to [X]$ such that the order of any two difference-differential terms is preserved by ϕ. Hence, given ϕ and the matrix representing the term order on $[X]$ we have a representation of the generalized term order on $[\Delta, \Sigma]$.

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†This work has been supported by the Austrian Science Fund FWF project DIFFOP, no. P20336-N18

Reducing Second-Order Input-Output Equations

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A continuous-time input-output equation is of the form

$$y^{(n)}(t) = h\left(y(t), \ldots, y^{(n-1)}(t), u(t), u^{(1)}(t), \ldots, u^{(n-1)}\right), \tag{1}$$

where h is a meromorphic function, $y^{(i)}(t)$ and $u^{(j)}(t)$ stand for the ith derivative of $y(t)$ and jth derivative of $u(t)$, respectively.

This poster is motivated by studying accessibility of continuous-time and discrete-time input-output equations [1, 2, 3, 4].

To begin with, we consider continuous-time input-output equations of second-order, and assume that h in (1) is a rational function. We are concerned with the following question:

When $n = 2$, can one rewrite (1) *as:*

$$\left\{ \begin{array}{c} \phi(z, z^{(1)}) = 0 \\ z = g(y, y^{(1)}, u) \end{array} \right. \tag{2}$$

where $\phi \in \mathbb{C}[Y_0, Y_1]$ and $g \in \mathbb{C}\left(y, y^{(1)}, u\right)$.

We take a module-theoretic approach to studying this question. We associate a differential field K to (1), and let $\mathcal{S}$ be the ring of linear differential operators over K. Moreover, let (Ω, d) be the K-linear space of $\mathbb{C}$-differentials of K. The K-space Ω is also a left module over $\mathcal{S}$. This module structure connects the derivation of K with $\mathbb{C}$-differentials.

Definition 1 *We say that $g \in K$ is an* autonomous element *of (K, d) if $\mathrm{d}g$ is a torsion element of Ω.*

Observe that g is an autonomous element if and only if (1) can be rewritten as (2). Hence, we study the following two questions:

- deciding if Ω is torsion-free;
- deciding if there exists an autonomous element, and finding one when it is existent.

The first question can be settled by a gcld-computation over K; while the second appears difficult.

Example 2 *Consider a continuous-time equation* $y^{(2)} = u^{(1)}y + uy^{(1)}$. *Its associated field is* $K = \mathbb{C}(y, y^{(1)}, u, u^{(1)} \ldots)$ *with the derivation operator* $\delta(y^{(1)}) = u^{(1)}y + uy^{(1)}$. *Then* $\omega = \mathrm{d}y^{(1)} - u\mathrm{d}y - y\mathrm{d}u$ *is a torsion element of the module of Kähler differentials of* K *over* $\mathbb{C}$. *In this case,* 1 *happens to be an integrating factor of* ω. *Hence, the given equation can be rewritten as* $\delta(y^{(1)} - uy) = 0$.

As the second question is difficult, we describe a differential field extension $\widehat{K}$ of K, and a $\mathbb{C}$-derivation $\hat{\mathrm{d}}$ from $\widehat{K}$ to a module $\widehat{V}$ such that there exists an autonomous element of $(\widehat{K}, \hat{\mathrm{d}})$ if Ω is not torsion-free.

This work grew out of discussions with M. Halás, Ü. Kotta and C. Yuan. We thank them for helpful comments and suggestions.

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On the Classification of Differential Type and Rota-Baxter Type Operators

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The problem of classifying linear operators on associative algebras was raised by Rota. We use the language of operated algebras and bracketed words to study some of these linear operators, namely operators of differential type and Rota-Baxter type.

Convolution Surfaces Generated by Basic $1D$ and $2D$ Skeletons

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We present general closed form formulae for the convolution surfaces around sets of polygonal lines and planar polygons.

Convolution is a technique used in computer graphics to generate smooth $3D$ volumes around a skeleton of lower dimension. One- dimensional skeletons create tubular like volumes which are well suited for modeling organic shapes. For general shapes one needs to consider $2D$ skeletons as well.

Convolution surfaces are defined as level set of a function obtained by integrating a kernel function along this skeleton. To allow for interactive modeling, the technique has relied on closed form formulae for integration obtained through symbolic computation software.

We consider families of kernels indexed by an integer that controls either the smoothness or the sharpness of the shape created. Generality is achieved by exhibiting the recurrence relationship for the convolution functions generated by line segments. The convolution functions for polygons are then expressed in terms of the convolution functions generated by the bounding polygonal line by application of Green's theorem. This approach does not require prior triangulation and simplifies a great deal the geometrical computations previously needed when dealing with compact support kernels.

Part of this work is in collaboration with M-P. Cani (EPI EVASION) in the framework of the RTRA and ARC project PlantScan3D.

Differential Krull Dimension

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We consider the differential Krull dimension of a differential polynomial ring over an ordinary Ritt algebra. We generalize the Johnson result on differential fields of characteristic zero to the following three classes of ordinary differential Ritt algebras

1. Noetherian rings of finite Krull dimension
2. differential Prüffer domains
3. differential rings with locally nilpotent derivation

Keywords: Differential rings, Differential spectra, Differential Krull dimension.

Differential Chow Form

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In this poster, an intersection theory for generic differential polynomials is presented. The intersection of an irreducible differential variety of dimension d and order h with a generic differential hypersurface of order s is shown to be an irreducible variety of dimension $d-1$ and order $h+s$. As a consequence, the dimension conjecture for generic differential polynomials is proved. Based on the intersection theory, the Chow form for an irreducible differential variety is defined and most of the properties of the Chow form in the algebraic case are extended to its differential counterpart. Furthermore, the generalized differential Chow form is defined and its properties are proved. As an application of the generalized differential Chow form, the differential resultant of $n+1$ generic differential polynomials in n variables is defined and properties similar to that of the Sylvester resultant of two univariate polynomials are proved.

This is joint work with Xiao-Shan Gao and Chunming Yuan.

Rational Curves and Differential Equations *

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Abstract

The solutions of a non-linear ordinary differential equation (ODE) have been studied from geometric point of view. Recently, it have been systematically developed by R. Feng and X-S. Gao in their papers on rational general solutions of an autonomous ODE of order 1 ([FG04], [FG06]). We have continued this approach to study the rational general solutions of a parametrizable non-autonomous ODE of order 1 ([NgoW]). This is a natural extension of the autonomous ODEs of order 1 with rational solutions and it leads to studying a system of autonomous ODEs of order 1 and of degree 1. We call the associated system of the original ODE. It turns out that an autonomous ODE with rational solutions has a simple associated system of ODEs. In fact, it is again an autonomous ODE in one indeterminate of order 1 and of degree 1 and we already had a degree bound for its rational solutions. However, we do not have a degree bound for rational solutions of the associated system in general.

The associated system is a planar rational system, whose rational solutions form rational parametrization of the rational invariant algebraic curves of the corresponding polynomial system. The problem is how to find an effective degree bound for its rational solutions. This problem is known as Poincaré problem and it has been solved in the case without dicritical singularities.

The presentation is to describe the whole process and some open questions on finding rational solutions of non-linear algebraic ODEs of order 1, especially on finding rational solutions of the associated system.

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*This work has been supported by the Austrian Science Foundation (FWF) via the Doctoral Program "Computational Mathematics" (W1214), project DK11.

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Simultaneous proof of the dimensional conjecture and of Jacobi's bound

Abstract

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Avec le soutien du projet ANR LÉDA (Logistique des Équations Différentielles Algériques).

Introduction

Jacobi's bound, probably formulated by Jacobi around 1840 [2] is an upper bound on the order of a system of n differential equations f_i in n variables x_j, which is expressed as the *tropical determinant* of the order matrix $A := (a_{i,j} := \mathrm{ord}_{x_j} f_i)$, with the convention $\mathrm{ord}_{x_j} f_i := -\infty$ when f_i is free from x and its derivatives :

$$\max_{\sigma \in S_n} \sum_{i=1}^{n} a_{i,\sigma(i)}.$$

The result is known to hold for *quasi-regular* systems [3], but remains conjectural in the general case. Cohn was the first to relate it to the dimensional conjecture [1], which claims that the differential codimension of a system of r equations is at most r, showing that Jacobi's bound implies the dimensional conjecture.

During my talk at DART III, I proposed a scheme of proof for the bound, using first a proof of the dimensional conjecture, trying to generalize Ritt's proof for codimension 1 [6], which is based on Puiseux series computations. Then, some kind of reduction process was to be used, mostly based of Jacobi's reduction methods. The dimension properties was crucial there to withdraw components defined by two few equations, or equations satisfying relations.

Working to complete this proof, I had trouble with the computation of Puiseux series that required to introduce some change of variables... allowing to get both results at the same time, thanks to the same reduction process. The result we prove is in fact a little more general.

We define here the order of a component of positive dimension s as the order of its intersection with s generic hyperplanes [5]. Working with a system of r equations is not easy. In fact, it is better to consider an *prime algebraic ideal* of codimension r.

THEOREM 1. — *Let f_i, $1 \le i \le r$ be a characteristic set of an algebraic ideal $\mathcal{I}$ of codimension r in $\mathcal{F}\{x_1, \ldots, x_n\}$, where $\mathcal{F}$ is a differential field of chatacteristic 0. Let us denote by $S_{r,n}$ the set of injections from $[1, r]$ to $[1, n]$.*

If $\mathcal{P}$ be a prime component of $\{\mathcal{I}\}$, then :
i) the differential codimension s of $\mathcal{P}$ is at most $r+p$;
ii) if the differential codimension of $\mathcal{P}$ is equal to $r+p$, the order of $\mathcal{P}$ is at most the strong Jacobi number

$$\mathcal{O} := \max_{\sigma \in S_{r,n}} \sum_{i=1}^{s} a_{i,\sigma(i)}.$$

If the f_i are just arbitrary equations, we easily reduce to the hypotheses of the theorem by considering the prime components of $\sqrt{[f]}$.

1 Main ideas of the proof.

The most concise way of presenting the proof is to replace the recursive reduction process by a *reductio ab absurdum*. Let us assume that i) or ii) is false. There exist counter-examples such that $n-r$ is minimal, and among them counter-examples with minimal Jacobi number. We will try to work out a contradiction.

Let $B := (\lambda_i + a_{i,j})$ be a *minimal canon* [2, 5]for the order matrix, meaning that (λ_i) is the smallest vector of integers such that B has elements maximal in their columns and located in all differents lines and columns. We define $\Lambda := \max_i \lambda_i$, $\alpha_i := \Lambda - \lambda_i$ and $\beta_j := \max_i a_{i,j} - \alpha_i$. We say that some ordering $\prec$ on derivatives is a *Jacobi ordering* if $k_1 - \beta_{j_1} < k_2 - \beta_{j_2}$ implies $x_{j_1^{(k_1)}} \prec x_{j_2^{(k_2)}}$.

We may assume that the f_i are ordered by increasing α_i ; let ϖ be the smallest integer such that :
A) $f_1, \ldots, f_{n-\varpi}$ is a characteristic set of a prime differential ideal $\mathcal{Q}$, for a Jacobi ordering ;
B) $\mathcal{Q} \cap \mathcal{F}[x_j^{(k)} | 1 \le j \le n,\ 0 \le k \le \beta_j + \alpha_{n-\varpi}] \subset \mathcal{P}$.

Now, we may assume further that the system f has been chosen with minimal ϖ, among those with minimal $n-r$ and Jacobi number.

Lemma 2. — *Under the above hypotheses, if ϖ is equal to 0, then $\mathcal{Q} \not\subset \mathcal{P}$; if $\varpi > 0$, then $\varpi < n$ and $\mathcal{J} := \mathcal{Q} \cap \mathcal{F}[x_j^{(k)} | 1 \le j \le n,\ 0 \le k \le \beta_j + \alpha_{n-\varpi+1}] \not\subset \mathcal{P}$, which is equivalent to saying that condition B) does not stand for $\varpi - 1$.*

PROOF. — As f_1 is a prime polynomial, it must be the char. set of a prime differential ideal, so $\varpi < n$. If the i_0 first equations f_i are such that $\alpha_i = \alpha_1$, then $f_1, \ldots, f_{i_0}$ is also a char. set of a differential prime ideal.

Assume that $\varpi > 0$, that condition B) does not stand for $\varpi - 1$ and that $\alpha_{n-\varpi+1} = \cdots = \alpha_{i_0}$. Then, some prime component of the radical of $[f_{n-\varpi+1}, \ldots, f_{i_0}] + \mathcal{J}$ must be contained in $\mathcal{P}$. For some Jacobi ordering it has a char. set, of which one may extract a char. set of a prime differential ideal, of the form $f_1, \ldots, f_{n-\varpi}, g_{n-\varpi+1}, \ldots, g_{i_0}$—we use here the minimality of $n-r$ and of the Jacobi number. Replacing the corresponding f_i by the g_i, we get a new char. set with a smaller value of ϖ, equal to $n - i_0$: a contradiction to the minimality hypothesis.

Assume now that ϖ is equal to 0. If $\mathcal{Q} \subset \mathcal{P}$, the prime component $\mathcal{P}$ would be equal to $\mathcal{Q}$, of which f is a characteristic set for a Jacobi ordering. This would imply that i) the order of $\mathcal{P}$ is $\mathcal{O}$ and ii) its dimension equal to $n - r$. So, $\mathcal{Q} \not\subset \mathcal{P}$. □

We have shown that possible couter-examples are related to *singular* components of the system $\mathcal{I}$.

2 The singular case.

The idea used to achieve the proof in the singular case is to reduce to the regular one by a suitable change of variables. It may be illustrated by the most simple example of equation $f(x) = x'^2 - 4x = 0$. We introduce a change of variables defined by $y = z'$. The new system $x - y^2 = 0, y(y'-1) = 0$ is equivalent to $f(x) = 0$, but the main and singular components of this system are now respectively associated to the factors $y' - 1 = 0$ and $y = 0$.

As $\mathcal{P}$ is a component that does not contain $\mathcal{Q}$, we may find a minimal n-uple of integer μ_i such that $[f_i^{(\nu)} | 1 \le i \le n - \varpi,\ 0 \le \nu \le \mu_i] : H$[1] contains a polynomial that does not belong to $\mathcal{P}$.

We may now assume that the system has been chosen among those with minimal $n - r$, then minimal Jacobi number, then minimal ϖ, in such a way that $\gamma := \max_{i=1}^{r} \mu_i$ is minimal. Obviously, it must be greater than 0. We may assume that $\mu_i > 0$ for $1 \le i \le s$ and that the leading derivatives of f_i is $x_i^{(\alpha_i+\beta_i)}$. We increase the ground field with r arbitrary functions ζ_i : the new ground field is $\mathcal{F}\langle\zeta\rangle$. We introduce new variables $y_i = x_i' + \zeta_i x_i$.

We start with the prime ideal $\mathcal{I} + [y_i - x_i' - \zeta_i x_i]$ for which f_i, $1 \le i \le n - \varpi$ and $y_i - x_i' - \zeta_i x_i$, $1 \le i \le r$ is a characteristic set.
The first step is to rewrite the equations defining the new variables as $x_i' = y_i - \zeta_i x_i$, to differentiate them $\alpha_i + \beta_i - 1$ times, so that each derivatives $x_i', \ldots, x_i^{(\alpha_i+\beta_i)}$ are expressed as a linear combinations of $y_i, \ldots, y_i^{(\alpha_i+\beta_i-1)}$ and x_i, that may be substituted in the elements of $\mathcal{I}$ to get a new ideal $\mathcal{I}_1$. This does not change the components of the system.
The next step is to eliminate x_i' in $\mathcal{I}_1'$ using $x_i' = y_i - \zeta_i x_i$, which gives $\mathcal{I}_2$—that is not and ideal ! But one gets a new ideal $\mathcal{I}_1 + \mathcal{I}_2$.

The components are again preserved, but the ideal need not be prime. If $\gamma > 1$, then we just keep the "main" component, that is included in $\mathcal{P}$. The values of $n-r$, the Jacobi number and ϖ are preserved, but it is easilly seen that the new value of γ is $\gamma - 1$, which contradicts the minimality of γ.

If $\gamma = 1$, then the components of $\sqrt{\mathcal{Q}_1 + \mathcal{Q}_2}$ that are included in $\mathcal{P}$ have the same $n - r$, but strictly smaller Jacobi number, as they correspond to the former singular components, where H did vanish : a final contradiction that completes the proof.

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1. Where H_f is the product of initials and separants of $f_1, \ldots, f_{n-\varpi}$.

An overview of the SCIEnce Project

Abstract

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With the support of the SCIEnce (Symbolic Computation Infrastructure for Europe) Project 026133, *Integrated Infrastructure Initiative*, European Commission, Research Infrastructures Action, Framework 6.

Introduction

It should be first stated that this presentation has no claim for originality and that its author has no personnal merits in the works that are described here, his team being mostly involved in other tasks of the project.

The aims of SCIEnce are to allow sharing components of computer algebra systems, to make CAS interoperable through suitable Web services and to make them ready for the use of Grid computing. The project started on april 9^{th} 2006 for 5 years. It involves developpers of four major CAS : GAP, KANT, Maple and MuPAD.

As the DART community may be interested by computation tools that are not available in a single CAS, and also in specialized softwares, such as BLAD or Lépisme, I thought interesting to take advantage of this conference to present tools dedicated to software interoperability.

Moreover, we know that differential algebraic system solving, that is worse than algebraic system solving, may be a task of a great complexity, so that we could perhaps take advantage of Grid computing. These are the main motivations of this poster, hoping that specialists will forgive the inaccuracies, the goal being to bring attention to direct and better sources.

1 Software composability

The work on software composability is mostly centered on SCSCP (Symbolic Computation Software Composability Protocol)[5], which is a remote procedure call framework with two main specificities : it relies on OpenMath[1] [3], for both protocol messages and data, and it is implemented in the computer algebra systems, instead of using wrappers.

At this stage, support for OpenMath and SCSCP has been developped in GAP, by Alexander Konovalov and Steve Linton, Marco Costantini, Andrew Solomon ; KANT by Sebastien Freundt and Sylla Lesseni ; MUPAD by Peter Horn.

2 Related tools

A Java library has also been developped, that supports OpenMath representation and also offers LaTeXexport.

OpenMath has been designed for communication between computers, not humans. So, an OpenMath representation convenient for direct user interaction, Popcorn, which stands for "Only Practical Convenient OpenMath Replacement Notation", has been developped. The Java library mentioned above also supports Popcorn.

WUPSI (Universal Popcorn SCSCP Interface) is a command line that can be used to access an arbitrary number of SCSCP servers, possibly in parallel and to exchange data between them. It can also be used to retrieve information on OpenMath symbols or be used as a manual SCSCP sever.

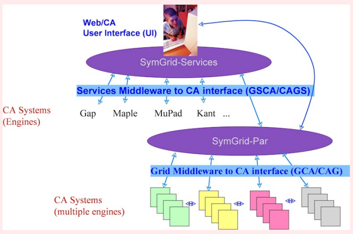
1. OpenMath is a standard to represent mathematical object with their semantics that can be used for their storage on databases, exchanges between computer programs or publication on web pages. It is strongly related to the MathML recommendation of the Worldwide Web Consortium.

With the long term goal of proving or certifying algorithms used in computer algebra systems, a Computer algebra object internalisation in Coq proof assistant has been provided[2]

3 Grid computing

A new grid framework, SymGrid has been developped. Maple, GAP, Kant and Mupad are initially integrated into the project. These heterogenous symbolic components may be used together, possibly in parallel.

The project includes two main components : SymGrid services, a generic interface to grid services, provides an interface to Grid and Web services that relies on OpenMath. SymGrid-Par is built around GRID-GUM, a system designed for parallel computation on the Grid, with adaptations for symbolic engines, using again OpenMath.



4 Differential equations in OpenMath

I will try here to give a few examples of the OpenMath syntax. The most basic definitions are to be found in OpenMath CD (Content Dictionary) `calculus1`. This is how

$$\frac{\partial^2}{\partial x \partial y}(xyz) = z$$

looks like in OpenMath.

```
<math xmlns="http://www.w3.org/1998/Math/MathML">
 <apply><csymbol cd="relation1">eq</csymbol>
  <apply>
   <apply><csymbol cd="calculus1">partialdiffdegree</csymbol>
    <apply><csymbol cd="list1">list</csymbol><cn>1</cn><cn>0</cn><cn>1</cn></apply>
    <cn>2</cn>
    <bind><csymbol cd="fns1">lambda</csymbol>
     <bvar><ci>x</ci></bvar>
     <bvar><ci>y</ci></bvar>
     <bvar><ci>z</ci></bvar>
     <apply><csymbol cd="arith1">times</csymbol><ci>x</ci><ci>y</ci><ci>z</ci></apply>
    </bind>
   </apply>
   <ci>x</ci>
   <ci>y</ci>
   <ci>z</ci>
  </apply>
  <ci>y</ci>
 </apply>
</math>
```

We see that the Popcorn notation is easier to handle.

```
calculus1.partialdiffdegree([1 , 0 , 1], 2, fns1.lambda[$x, $y, $z -> $x * $y * $z])($x, $y, $z) = $y
```

Obviously, we are still missing many objects required for differential algebra. We may however notice the existence of the CD `weylalgebra1`. The important question of data structure does not seem to be taken in account in most cases. However, the CD `equations1` privides predicates "dense" and "sparse". Some CD, such as `polyd1` provide definitions for multivariate polynomial, adapted for Gröbner bases computations, condidered in `polygb1` and `polygb2`. I found nothing for differential polynomials, or characteristic sets, even in the pure algebraic case.

Besides computer algebra, OpenMath could also be used to search information on the Web, provided that people actually use it as a standard ! The paper of Draheim *et al.*[1] considers the issue of looking for possible occurences of a given differential equations on the WEB.

Conclusion

It is not clear that the success of a standard is due to its quality, nor that it fails to be adopted because of its technical drawbacks. It seems rather in many cases that it is just a question of critical mass and initial success, for unknown reasons. People develop the standard because they feel it will become a reference and such a process is self-sustained.

Obviously, many tools are still lacking in OpenMath, mostly for specialised fields of research such as differential algebra, but enough has been done to consider the development of new definitions with a limited amount of extra work.

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Integration in finite terms for Liouvillian functions

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Introduction

Computing integrals is a common task in many areas of science, antiderivatives are one way to accomplish this. The problem of integration in finite terms can be stated as follows. Given a differential field (F, D) and $f \in F$, compute g in some elementary extension of (F, D) such that $Dg = f$ if such a g exists.

This problem has been solved for various classes of fields F. For rational functions $(C(x), \frac{d}{dx})$ such a g always exists and algorithms to compute it are known already for a long time. In 1969 Risch [3] published an algorithm that solves this problem when (F, D) is a transcendental elementary extension of $(C(x), \frac{d}{dx})$. Later this has been extended towards integrands being Liouvillian functions by Singer et. al. [4] via the use of regular log-explicit extensions of $(C(x), \frac{d}{dx})$. Our algorithm extends this to handling transcendental Liouvillian extensions (F, D) of $(C, 0)$ directly without the need to embed them into log-explicit extensions. For example, this means that $\int (z - x)x^{z-1}e^{-x}\, dx = x^z e^{-x}$ can be computed without including $\log(x)$ in the differential field.

Problem overview

Given (F, D) a transcendental Liouvillian extension of its subfield of constants C and $f_0, \ldots, f_m \in F$, compute (a basis of) all linear combinations $f \in \mathrm{span}_C\{f_0, \ldots, f_m\}$ that have an elementary integral over F together with corresponding g's such that $Dg = f$.

We present a decision procedure for this parametric problem. The algorithm follows the general recursive structure of its precursors proceeding through the transcendental extensions one by one. Integrands from $F =: K(t)$ are reduced to integrands from the differential subfield K. Then a refined version of Liouville's theorem has to be proven for reducing the question of having an elementary integral over F to having an elementary integral over K. A special case is already implicitly contained in [4]. When dealing with non-elementary extensions this naturally leads to a parametric version of the problem of integration in finite terms even when we started with just one single integrand.

This refinement is crucial to obtain a decision procedure for Liouvillian extensions. Also Bronstein [1] presented generalizations of parts of Risch's algorithm to certain types of non-elementary extensions, but he did not consider the appropriate parametric versions needed. So, for example, with the results given there one does not find the integral

$$\int \frac{(x+1)^2}{x\log(x)} + \mathrm{li}(x)\,dx = (x+2)\mathrm{li}(x) + \log(\log(x)).$$

Considering the parametric problem is not merely a side-effect, but is also useful in its own right. Definite integrals can not only be computed via the evaluation of antiderivatives. If the integral depends on a parameter one can try to compute linear difference/differential equations that are satisfied by the parameter integral even when no antiderivative of the integrand is available. E.g. for $I(x) = \int_0^{\pi/2}(1 - x\sin(t))^r dt$ one obtains the ODE $2(x-1)xI''(x) + ((3-2r)x-2)I'(x) - rI(x) = 0$; $r = \frac{1}{2}$ gives the elliptic integral $E(x)$.

Algorithm

In what follows our algorithm is compared to the previous algorithms in more detail. In some sense the algorithm can be viewed as unification of the algorithms presented in [4, Theorem A1] and [1]: On the one hand it is a decision procedure for parametric integration over transcendental Liouvillian extensions and also decides the auxiliary parametric logarithmic derivative problem. On the other hand it minimizes the computations done in algebraic extensions and tries to avoid factorization into irreducibles as much as possible.

In addition to what has been mentioned so far the main improvement compared to the other algorithms is the following. In order to determine the necessary restrictions for the linear combinations of the integrands [4] relies on irreducible factorization of the denominator over some algebraically extended coefficient domain. The algorithm for the single-integrand case from [1] – a generalization of [2] – avoids unnecessary algebraic extensions and complete factorization, but does not carry over to the parametric case. However, reformulating the Rothstein-Trager resultant appropriately we obtained an algorithm with the desired properties, relying on the extended euclidean algorithm.

The last phase of one step in the recursion mentioned above consists of bounding the degree of the remaining part and solving for the coefficients, which requires solving auxiliary problems such as the parametric logarithmic derivative problem and the Risch differential equation. Here the parametric logarithmic derivative heuristic from [1] has been turned into a decision procedure along the idea sketched in [3].

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Differential Algebraic Groups and Factorization of Partial Differential Operators

(joint work with Phyllis Cassidy)

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An ordinary differential operator can be factored as a product of irreducible operators and any two such factorizations have the same number of factors and, after a possible permutation, these factors are equivalent in a suitable sense. Examples showing that such a result is not true for partial differential operators have been known for over 100 years.

Solutions of systems of homogeneous linear partial differential equations form a group under addition and are an example of a differential algebraic group. We show that a Jordan-Hölder type theorem holds for such groups, that is, any such group can be filtered by a finite subnormal series of differential algebraic groups such that successive quotients are "almost simple". Furthermore, any two such series have the same length and, after a possible permutation, successive quotients are "isogenous". This allows us to recover a version of unique factorization for partial differential operators. Many examples will be shown.