Integration in finite terms for Liouvillian functions

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Introduction

Computing integrals is a common task in many areas of science, antiderivatives are one way to accomplish this. The problem of integration in finite terms can be stated as follows. Given a differential field (F, D) and $f \in F$, compute g in some elementary extension of (F, D) such that $Dg = f$ if such a g exists.

This problem has been solved for various classes of fields F. For rational functions $(C(x), \frac{d}{dx})$ such a g always exists and algorithms to compute it are known already for a long time. In 1969 Risch [3] published an algorithm that solves this problem when (F, D) is a transcendental elementary extension of $(C(x), \frac{d}{dx})$. Later this has been extended towards integrands being Liouvillian functions by Singer et. al. [4] via the use of regular log-explicit extensions of $(C(x), \frac{d}{dx})$. Our algorithm extends this to handling transcendental Liouvillian extensions (F, D) of $(C, 0)$ directly without the need to embed them into log-explicit extensions. For example, this means that $\int (z - x)x^{z-1}e^{-x}\, dx = x^z e^{-x}$ can be computed without including $\log(x)$ in the differential field.

Problem overview

Given (F, D) a transcendental Liouvillian extension of its subfield of constants C and $f_0, \ldots, f_m \in F$, compute (a basis of) all linear combinations $f \in \mathrm{span}_C\{f_0, \ldots, f_m\}$ that have an elementary integral over F together with corresponding g's such that $Dg = f$.

We present a decision procedure for this parametric problem. The algorithm follows the general recursive structure of its precursors proceeding through the transcendental extensions one by one. Integrands from $F =: K(t)$ are reduced to integrands from the differential subfield K. Then a refined version of Liouville's theorem has to be proven for reducing the question of having an elementary integral over F to having an elementary integral over K. A special case is already implicitly contained in [4]. When dealing with non-elementary extensions this naturally leads to a parametric version of the problem of integration in finite terms even when we started with just one single integrand.

This refinement is crucial to obtain a decision procedure for Liouvillian extensions. Also Bronstein [1] presented generalizations of parts of Risch's algorithm to certain types of non-elementary extensions, but he did not consider the appropriate parametric versions needed. So, for example, with the results given there one does not find the integral

$$\int \frac{(x+1)^2}{x\log(x)} + \mathrm{li}(x)\,dx = (x+2)\mathrm{li}(x) + \log(\log(x)).$$

Considering the parametric problem is not merely a side-effect, but is also useful in its own right. Definite integrals can not only be computed via the evaluation of antiderivatives. If the integral depends on a parameter one can try to compute linear difference/differential equations that are satisfied by the parameter integral even when no antiderivative of the integrand is available. E.g. for $I(x) = \int_0^{\pi/2}(1 - x\sin(t))^r dt$ one obtains the ODE $2(x-1)xI''(x) + ((3-2r)x-2)I'(x) - rI(x) = 0$; $r = \frac{1}{2}$ gives the elliptic integral $E(x)$.

Algorithm

In what follows our algorithm is compared to the previous algorithms in more detail. In some sense the algorithm can be viewed as unification of the algorithms presented in [4, Theorem A1] and [1]: On the one hand it is a decision procedure for parametric integration over transcendental Liouvillian extensions and also decides the auxiliary parametric logarithmic derivative problem. On the other hand it minimizes the computations done in algebraic extensions and tries to avoid factorization into irreducibles as much as possible.

In addition to what has been mentioned so far the main improvement compared to the other algorithms is the following. In order to determine the necessary restrictions for the linear combinations of the integrands [4] relies on irreducible factorization of the denominator over some algebraically extended coefficient domain. The algorithm for the single-integrand case from [1] – a generalization of [2] – avoids unnecessary algebraic extensions and complete factorization, but does not carry over to the parametric case. However, reformulating the Rothstein-Trager resultant appropriately we obtained an algorithm with the desired properties, relying on the extended euclidean algorithm.

The last phase of one step in the recursion mentioned above consists of bounding the degree of the remaining part and solving for the coefficients, which requires solving auxiliary problems such as the parametric logarithmic derivative problem and the Risch differential equation. Here the parametric logarithmic derivative heuristic from [1] has been turned into a decision procedure along the idea sketched in [3].

References

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