Differential Chow Form

Chun-Ming Yuan
Joint work with X.S. Gao and W. Li

Key Laboratory of Mathematics Mechanization，
Institute of Systems Science，AMSS, Chinese Academy of Sciences

2010.10.27

Motivations

- **Differential Chow Form**
  Algebraic chow form is a powerful tool in constructive algebraic geometry with fruitful applications in elimination theory, transcendental number theory, complexity analysis, etc.
  Differential Chow form is not studied before.

- **Differential Resultant**
  Resultant as a powerful tool is not fully explored in differential case.

- **Differential dimension conjecture Proposed by Ritt (1950)**

Outline of the Talk

- Preliminaries

- Intersection Theory for Generic Differential Polynomials

- Chow Form for an Irreducible Differential Variety

- Differential Chow Variety

- Generalized Differential Chow Form and Differential Resultant

- Summary

Main Tools Used in This Talk:

- **Differential CS**

- **Differential Specialization**

- **Algebraic Chow Form**

Preliminaries   Intersection Theory   Differential Chow Form   Differential Chow variety   Generalized Differential Chow Form   Summa

●○○○   ○○○   ○○○○○○○○○○   ○○   ○○○○○○○○

Preliminaries

Ordinary differential field: $(\mathcal{F}, \delta)$ and $char(\mathcal{F}) = 0$.

Θ: semigroup with unit generated by δ.

Indeterminates: $\mathbb{Y} = \{y_1, \ldots, y_n\}$. **Notation:** $y_i^{(k)} = \delta^k y_i$.

Differential polynomial ring: $\mathcal{F}\{\mathbb{Y}\} = \mathcal{F}[\Theta(\mathbb{Y})]$

Ranking $\mathscr{R}$**:** a total order over $\Theta(\mathbb{Y})$ satisfying
1) $\delta u > u$ and 2) $u > v \Longrightarrow \delta u > \delta v$ for any $u, v \in \Theta(\mathbb{Y})$.

Example:

1) **Elimination ranking**:
$y_i > y_j \Rightarrow \delta^k y_i > \delta^l y_j$ for any $k, l \geq 0$

2) **Orderly ranking**:
$k > l \Rightarrow \delta^k y_i > \delta^l y_j$ for any $i, j \in \{1, \ldots, n\}$.

Given a diff poly p, then
Leader: the greatest derivative w.r.t. $\mathscr{R}$, denoted by u_p or $\mathrm{ld}(p)$.
$p = I_d u_p^d + I_{d-1} u_p^{d-1} + \ldots + I_0$.
Initial: $\mathbf{I}_p = I_d$
Separant: $\mathrm{S}_p = \frac{\partial p}{\partial u_p}$
Order of p: $\mathbf{ord}(p) = \max_i \mathbf{ord}(p, y_i)$ with
$\mathbf{ord}(p, y_i) := \max\{k : y_i^{(k)}\text{effectively appears in } p\}$.

Autoreduced set: $\mathcal{A} = A_1, \ldots, A_t$ with $\mathrm{ld}(A_i) = y_{c_i}^{(o_i)}$.
Parametric set of $\mathcal{A}$: $\mathrm{Pm}(\mathcal{A}) = \mathbb{Y} \setminus \{y_{c_1}, \ldots, y_{c_t}\}$.
Order of $\mathcal{A}$: $ord(\mathcal{A}) = \sum_{i=1}^t o_i$.

Characteristic set (CS) of $\mathbb{P}$: the smallest autoreduced set in
$\mathbb{P}$.

Dimension and order of a prime differential ideal

$\mathcal{I}$: a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$.

$\mathcal{A}$: a CS of $\mathcal{I}$ w.r.t. any **orderly ranking**.

$\mathcal{B}$: a CS of $\mathcal{I}$ w.r.t. any ranking and $\mathrm{Pm}(\mathcal{B}) = U$.

Known Facts:

Dimension of $\mathcal{I}$: dim$(\mathcal{I}) = n - |\mathcal{A}|$.

Order of $\mathcal{I}$: ord$(\mathcal{I}) = $ ord$(\mathcal{A})$.

Relative order of $\mathcal{I}$ w.r.t. U: ord$_U(\mathcal{I}) \doteq $ ord$(\mathcal{B})$.

Lemma (Relation between order and relative order)

ord$(\mathcal{I})$ *is the maximum of all the relative orders of $\mathcal{I}$, that is,*
ord$(\mathcal{I}) = \max_{\mathbb{U}}$ ord$_{\mathbb{U}}(\mathcal{I})$, *where $\mathbb{U}$ is a parametric set of $\mathcal{I}$.*

A Property on Differential Specialization

Lemma (Differential dependence under specialization)

$P_i(\mathbb{U}, \mathbb{Y}) \in \mathcal{F}\{\mathbb{U}, \mathbb{Y}\}$: *diff polys in* $\mathbb{U}$, $\mathbb{Y} = (y_1, \ldots, y_n)$.

$\mathbb{Y}^0 = (y_1^0, \ldots, y_n^0)$: y_i^0 *in an extension field of* $\mathcal{F}$.

If $P_i(\mathbb{U}, \mathbb{Y}^0)$ *are differentially dependent over* $\mathcal{F}\langle\mathbb{U}\rangle$, *then for any specialization* $\mathbb{U}$ *to* $\mathbb{U}^0$, $P_i(\mathbb{U}^0, \mathbb{Y}^0)$ *are differentially dependent over* $\mathcal{F}$.

Remark. This is a key tool of our theory.

The algebraic analog is also a basic tool in algebraic elimination theory. But, the differential case needs an essentially new proof.

Preliminaries   **Intersection Theory**   Differential Chow Form   Differential Chow variety   Generalized Differential Chow Form   Summa

0000   000   0000000000   00   00000000

Intersection Theory for Generic Differential Polynomials

Determine the Differential Dimension and Order of the intersection variety by a Generic Differential hypersurface.

Dimension of the Intersection Variety

Generic diff poly: Complete diff poly with given order, degree, and indeterminate coeffs.

Quasi-generic diff poly: Sparse generic diff poly contains **constant term** and a term in $\mathcal{F}\{y_i\}\backslash\mathcal{F}$ for each i.

$\mathcal{I}$: a prime diff ideal in $\mathcal{F}\{\mathbb{Y}\}$ with dimension d.

In algebraic case, **dim**$((\mathcal{I}, f)) \geq d - 1$ always holds.

In differential case, **dim**$([\mathcal{I}, f]) < d - 1$ (Ritt 1950).

Theorem

f: a quasi-generic diff poly with $\deg(f) > 0$ and coefficients $\mathbf{u}_f$.

If $d > 0$, then $[\mathcal{I}, f]$ is a prime diff ideal in $\mathcal{F}\langle\mathbf{u}_f\rangle\{\mathbb{Y}\}$ with dimension $d - 1$.

And if $d = 0$, then $\mathcal{I}_1 = [\mathcal{I}, f]$ is the unit ideal.

Order of the Intersection Variety

$\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\}$: a prime diff ideal with dimension $d > 0$ and order h.

Theorem

f: a generic diff poly of order s with $\mathbf{u}_f$ the set of its coefficients.

Then $\mathcal{I}_1 = [\mathcal{I}, f]$ is a prime diff ideal in $\mathcal{F}\langle \mathbf{u}_f \rangle \{\mathbb{Y}\}$ with dimension $d - 1$ and order $h + s$.

Corollary (Basis for differential Chow form)

The intersection of V with a generic prime is of dimension **dim**(*V*) − 1 *and order* **ord**(*V*).

$\mathcal{I}_1 = [\mathcal{I}, u_0 + u_1 y_1 + \ldots + u_n y_n]$ *is a prime diff ideal in* $\mathcal{F}\langle u_0, u_1, \ldots, u_n \rangle \{\mathbb{Y}\}$ *with dimension d − 1 and order h.*

Dimension Conjecture in Generic Case

Dimension conjecture: $f_1, \ldots, f_r$ d.p. with $r < n$, is every component of $[f_1, \ldots, f_r]$ of dimension at least $n - r$?

Theorem (Generic Dimension Theorem)

$f_1, \ldots, f_r$: quasi-generic diff polys in diff variables $\mathbb{Y}$ with $r \leq n$.

Then $[f_1, \ldots, f_r]$ is a prime diff ideal with dimension $n - r$.

Together with order considered, the above results can be strengthened as follows.

Theorem

$f_1, \ldots, f_r$ $(r \leq n)$: generic diff polys with **ord**$(f_i) = s_i$.

Then $[f_1, \ldots, f_r]$ is a prime diff ideal with dimension $n - r$ and order $\sum_{i=1}^{r} s_i$.

Differential Chow Form

Definition of Differential Chow Form

$\mathbf{I} \subset \mathcal{F}\{\mathbb{Y}\}$: Prime differential ideal of dimension d.

Differential Primes:

$$\mathbb{P}_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n \, (i = 0, \ldots, d).$$

Theorem

As a consequence of our intersection theory,

$$[\mathbf{I}, \mathbb{P}_0, \ldots, \mathbb{P}_d] \cap \mathcal{F}\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d\} = \mathbf{sat}(F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d))$$

is a prime ideal of co-dimension one, where
$\mathbf{u}_i = (u_{i0}, \ldots, u_{in}) \, (i = 0, \ldots, d)$

We call $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, \ldots, u_{d0})$ the **differential Chow form** of I or $\mathbb{V}(I)$.

Example

$\mathcal{I} = \textbf{sat}(p) \subset \mathcal{F}\{\mathbb{Y}\}$: a prime diff ideal of dimension $n - 1$.

Then its differential Chow form is
$F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{n-1}) = D^m p(\frac{D_1}{D}, \ldots, \frac{D_n}{D})$, where

$$
D = \begin{vmatrix}
u_{01} & u_{02} & \ldots & u_{0n} \\
u_{11} & u_{12} & \ldots & u_{1n} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
u_{n-1,1} & u_{n-1,2} & \ldots & u_{n-1,n}
\end{vmatrix}
$$

and $D_i(i = 1, \ldots, n)$ is the determinant of the matrix formed by replacing the i-th column of D by the column vector $(-u_{00}, -u_{10}, \ldots, -u_{n-1,0})^T$, and m is the minimal integer such that $D^m p(\frac{D_1}{D}, \ldots, \frac{D_n}{D}) \in \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$.

Order of Differential Chow Form

Chow form of $\mathcal{I}$: $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, u_{10}, \ldots, u_{d0})$

Property of Chow form.

- $F(\ldots, \mathbf{u}_\sigma, \ldots, \mathbf{u}_\rho, \ldots) = (-1)^{r_{\sigma\rho}} F(\ldots, \mathbf{u}_\rho, \ldots, \mathbf{u}_\sigma, \ldots)$.
- $\mathbf{ord}(F, u_{00}) \neq 0$, $\mathbf{ord}(F, u_{00}) = \mathbf{ord}(F, u_{ij})$ if u_{ij} occurs in F

Order of Chow form: $\mathbf{ord}(F) = \mathbf{ord}(f, u_{00})$.

Theorem (Order of Chow Form)

$\mathbf{ord}(F) = \mathbf{ord}(\mathcal{I})$.

Definition (An equivalent definition for the order of $\mathcal{I}$)

Order of a prime diff ideal is defined to be the order of its Chow form.

Degree of Differential Chow Form

Differentially homogenous diff poly of degree m:

$$p(ty_0, ty_1 \ldots, ty_n) = t^m p(y_0, y_1, \ldots, y_n)$$

Theorem

$F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$: differential Chow form of V.

Then $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ is differentially homogenous of degree r in each set $\mathbf{u}_i$ and F is of total degree $(d+1)r$.

Definition (Differential degree)

r as above is defined to be the **differential degree** of $\mathcal{I}$, which is an invariant of $\mathcal{I}$ under invertible linear transformations.

Preliminaries   Intersection Theory   **Differential Chow Form**   Differential Chow variety   Generalized Differential Chow Form   Summa

oooo   ooo   ooooo●ooooo   oo   oooooooo

Factorization of Differential Chow Form

V: a diff irreducible variety of dimension d and order h.
$F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$: the differential Chow form of V.

Theorem ($F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ can be uniquely factored)

$$
\begin{aligned}
F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) &= A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{g} (u_{00} + \sum_{\rho=\mathbf{1}}^{\mathbf{n}} \mathbf{u}_{\mathbf{0}\rho} \xi_{\tau\rho})^{(h)} \\
&= A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{g} \mathbb{P}_0(\xi_{\tau 1}, \ldots, \xi_{\tau n})^{(h)}
\end{aligned}
$$

where $g = \deg(F, u_{00}^{(h)})$ and $\xi_{\tau\rho}$ are in an extension field of $\mathcal{F}$.
And the points $(\xi_{\tau 1}, \ldots, \xi_{\tau n})$ $(\tau = 1, \ldots, g)$ are **generic points**
of the variety V.

Leading Differential Degree

Differential primes:

$$\mathbb{P}_i := u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n \, (i = 1, \ldots, d),$$

Algebraic primes:

$$^a\mathbb{P}_0 := u_{00} + u_{01}y_1 + \cdots + u_{0n}y_n,$$
$$^a\mathbb{P}_0^{(s)} := u_{00}^{(s)} + \sum_{j=1}^n \sum_{k=0}^s \binom{s}{k} u_{0j}^{(k)} y_j^{(s-k)} \, (s = 1, 2, \ldots)$$

Theorem

$(\xi_{\tau 1}, \ldots, \xi_{\tau n}) \, (\tau = 1, \ldots, g)$ *are the* **only elements** *of V which lie on* $\mathbb{P}_1, \ldots, \mathbb{P}_d$ *as well as on* $^a\mathbb{P}_0, {}^a\mathbb{P}_0', \ldots, {}^a\mathbb{P}_0^{(h-1)}$.

Definition

Number g is defined to be the **leading diff degree** *of V or I.*

Example (2)

V: the general component of $(y')^2 - 4y$ in $\mathbf{Q}(t)\{y\}$.

$\mathbf{dim}(V) = 0$ and the diff Chow form of V is
$F(\mathbf{u}_0) = u_1^2(u_0')^2 - 2u_1 u_1' u_0 u_0' + (u_1')^2 u_0^2 + 4u_1^3 u_0$, where
$\mathbf{u}_0 = (u_0, u_1)$.

Then
$F(\mathbf{u}_0) = u_1^2(u_0' + \xi_1 u_1' + 2\sqrt{-1}\sqrt{u_0 u_1})(u_0' + \xi_2 u_1' - 2\sqrt{-1}\sqrt{u_0 u_1}) = u_1^2(u_0 + \xi_1 u_1)'(u_0 + \xi_2 u_1)'$ where
$\xi_i = -u_0/u_1\big|_{u_0' = \frac{u_0}{u_1}u_1' \mp 2\sqrt{-1}\sqrt{u_0 u_1}}$ $(i = 1, 2)$.

Note that both ξ_i $(i = 1, 2)$ satisfy ${}^a\mathbb{P}_0 = u_0 + u_1\xi_i = 0$, but
${}^a\mathbb{P}_0^{(1)} = (u_0 + u_1\xi_i)' \neq 0$. And from the Chow form, we have
$\xi_i'^2 + 4u_0/u_1 = 0$.

Relations between Diff Chow Form and the Variety

$\mathcal{I} \in \mathcal{F}\{\mathbb{Y}\}$: a prime diff ideal of dimension d.
$F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$: the differential Chow form of $\mathcal{I}$.
$\zeta = (\zeta_0, \ldots, \zeta_d)$: a generic point of F as a diff poly in
$\mathcal{F}\{u_{00}, \ldots, u_{d0}\}$.

Theorem (Recover the generic point of $\mathcal{I}$ from diff Chow form)

Let

$$\xi_\rho = \overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}} \Big/ \overline{\frac{\partial f}{\partial u_{00}^{(h)}}}, \rho = 1, \ldots, n$$

where each $\overline{\frac{\partial f}{\partial u_{0i}^{(h)}}} = \frac{\partial f}{\partial u_{0i}^{(h)}}\big|_{(u_{00}, \ldots, u_{d0})=(\zeta_0, \ldots, \zeta_d)}.$

Then $(\xi_1, \ldots, \xi_n)$ *is a generic point of* $\mathcal{I}$.

$F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$: differential Chow form of V and $S_F = \frac{\partial F}{\partial u_{00}^{(h)}}$.

When u_{ij} specialize to $v_{ij} \in \mathcal{E}(\mathcal{F})$, $\mathbb{P}_i$ specialize to $\overline{\mathbb{P}}_i$.

Theorem

If $\overline{\mathbb{P}}_i = 0 (i = 0, \ldots, d)$ meet V, then $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$.

Furthermore, if $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$ and $S_F(\mathbf{v}_0, \ldots, \mathbf{v}_d) \neq 0$, then the $d + 1$ primes $\overline{\mathbb{P}}_i = 0$ $(i = 0, \ldots, d)$ meet V.

Let $S = (s_{ij})$ be an $(n+1) \times (n+1)$ skew-symmetric matrix with $s_{ij}(i < j)$ independent differential indeterminates.
A generic prime passing through a point x is of the form Sx.

Theorem

V: a differential variety of dimension d and $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ its differential Chow form.

Then $V \setminus \mathbb{V}(\mathcal{D}) = \mathbb{V}(\mathcal{P}) \setminus \mathbb{V}(\mathcal{D}) \neq \emptyset$, where $\mathcal{P}, \mathcal{D}$ are the diff poly sets obtained from $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ and $S_F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ by substituting $\mathbf{u}_l$ with $S^l y$ and collecting coeffs in s_{ij}^l.

Differential Chow Variety

Differential Chow variety

V is said to be **of index** (n, d, h, g, m) if V is in $\mathcal{E}^n$, with dim d, order h, leading diff degree g, and diff degree m.

$F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$: a diff poly with index (n, d, h, g, m) and coefficients a_i.

Definition

A quasi-variety $\mathbb{CV}$ in a_i is the **Chow Variety** with index (n, d, h, g, m) if

$\quad (\bar{a}_i) \in \mathbb{CV}$

$\Leftrightarrow \bar{F}$: Chow form with index (n, d, h, g, m_1) with $m_1 \leq m$.

$\Leftrightarrow V$: order-unmixed var of index (n, d, h, g, m_1) with $m_1 \leq m$.

Theorem

In the case $g = 1$, the differential Chow variety exists.

Theorem (Sufficient condition for Chow form)

$F(\mathbf{u}_0, \ldots, \mathbf{u}_d) \in \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$: *an irreducible diff polynomial.*
If F satisfies the following conditions, then it is the Chow form for an irreducible diff variety of dimension d and order h.

1. $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ is differentially homogenous of the same degree in each $\mathbf{u}_i$ and $\mathbf{ord}(F, u_{ij}) = h$ for all u_{ij} occurring in F.

2. $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ can be factored uniquely into the following form

$F(\mathbf{u}_0, \ldots, \mathbf{u}_d) = A \prod_{\tau=1}^{g} (u_{00}^{(h)} + \sum_{\rho=1}^{n} u_{0\rho}^{(h)} \xi_{\tau\rho} + t_\tau)$
$= A(\mathbf{u}_0, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{g} (u_{00} + \sum_{\rho=1}^{n} u_{0\rho} \xi_{\tau\rho})^{(h)}.$

3. $\Xi_\tau = (\xi_{\tau 1}, \ldots, \xi_{\tau n}) \, (\tau = 1, \ldots, g)$ are on the differential primes $\mathbb{P}_\sigma = 0 \, (\sigma = 1, \ldots, d)$ as well as on the algebraic primes ${}^a\mathbb{P}_0^{(\delta)} = 0 \, (\delta = 0, \ldots, h-1)$.

4. For each τ, if $v_{i0} + v_{i1}\xi_{\tau 1} + \cdots + v_{in}\xi_{\tau n} = 0 \, (i = 0, \ldots, d)$, then $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$, where $\mathbf{v}_i = (v_{i0}, \ldots, v_{in})$ and $v_{ij} \in \mathcal{E}(\mathcal{F})$.

Generalized Differential Chow Form and Differential Resultant

Generalized differential Chow form

$\mathcal{I}$: a prime differential ideal of dimension d and order h.

Generic differential polynomials:

$$\mathbb{P}_i = \mathbf{u_{i0}} + \sum_{j=1}^{n} \sum_{k=0}^{s_i} u_{ijk} y_j^{(k)} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_i+1)} \\ 1 < |\alpha| \leq m_i}} u_{i\alpha} (\mathbb{Y}^{(s_i)})^{\alpha}, \ (i = 0, \ldots, d)$$

where $(\mathbb{Y}^{(s_i)})^{\alpha} = \prod_{j=1}^{n} \prod_{k=0}^{s_i} (y_j^{(k)})^{\alpha_{jk}}$ and $|\alpha| = \sum_{j=1}^{n} \sum_{k=0}^{s_i} \alpha_{jk}$

Clearly, **ord**$(\mathbb{P}_i) = s_i$ and $\deg(\mathbb{P}_i) = m_i$.

Definition

As a consequence of our intersection theory,

$$[\mathcal{I}, \mathbb{P}_0, \ldots, \mathbb{P}_d] \cap \mathcal{F}\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d\} = \mathbf{sat}(G(\mathbf{u}_0, \ldots, \mathbf{u}_d))$$

is a prime ideal of co-dimension one, where

$$\mathbf{u}_i = (u_{i0}, \ldots, u_{ijk}, \ldots, u_{i\alpha}, \ldots) \, (i = 0, \ldots, d)$$

We call $G(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = g(\mathbf{u}; u_{00}, \ldots, u_{d0})$ the **generalized differential Chow form** of $\mathcal{I}$ or $\mathbb{V}(\mathcal{I})$.

Generalized Diff Chow Form Has Similar Properties

$\mathcal{I}$: a prime differential ideal with dimension d and order h.

$G(\mathbf{u}_0, \ldots, \mathbf{u}_d) = g(\mathbf{u}; u_{00}, \ldots, u_{d0})$ its generalized Chow form.

Theorem

For each i, $\mathbf{ord}(g, u_{i0}) = h + s - s_i$ with $s = \sum_{l=0}^{d} s_l$. Moreover, $\mathbf{ord}(G, \mathbf{u}_i) = h + s - s_i$.

Theorem

There exist $\xi_{\tau\rho}(\rho = 1, \ldots, n; \tau = 1, \ldots, t_0)$ such that

$$G(\mathbf{u}_0, \ldots, \mathbf{u}_d) = A(\mathbf{u}_0, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{t_0} \mathbb{P}_0(\xi_{\tau 1}, \ldots, \xi_{\tau n})^{(h_0)},$$

where $A(\mathbf{u}_0, \ldots, \mathbf{u}_d) \in \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$ and $t_0 = \deg(g, u_{00}^{(h_0)})$.

Theorem

The points $(\xi_{\tau 1}, \ldots, \xi_{\tau n})$ $(\tau = 1, \ldots, t_0)$ are generic points of $\mathcal{I}$, and lie on the d generic differential primals $\mathbb{P}_1 = 0, \ldots, \mathbb{P}_d = 0$.

Theorem

$G(\mathbf{u}_0, \ldots, \mathbf{u}_d)$: generalized differential Chow form of V with $\mathbf{ord}(G, u_{00}) = h_0$.
$S_G = \frac{\partial G}{\partial u_{00}^{(h_0)}}$.

When $\mathbf{u}_i$ specialize to $\mathbf{v}_i \in \mathcal{E}(\mathcal{F})$, $\mathbb{P}_i$ specialize to $\overline{\mathbb{P}}_i$.

If $\overline{\mathbb{P}}_i = 0 (i = 0, \ldots, d)$ meet V, then $G(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$.

Furthermore, if $G(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$ and $S_G(\mathbf{v}_0, \ldots, \mathbf{v}_d) \neq 0$, then the $d + 1$ primals $\overline{\mathbb{P}}_i = 0$ $(i = 0, \ldots, d)$ meet V.

Differential Resultant

Existence Work on Differential Resultant:

• Ritt (1932) studied differential resultant for $n = 1$.

• Ferro (1997, 1997) defined differential resultant as algebraic Macaulay resultant. (Existence not proved, from algebraic to differential is difficult.)

• Rueda-Sendra (2010) differential resultant of linear systems.

$\mathbb{P}_i$ ($i = 0, \ldots, n$): generic diff polynomials with $\mathbf{u}_i$ its coefficients.

Definition

The **differential resultant** of $\mathbb{P}_i$ is defined to be the generalized Chow form of [0], denoted by $\delta Res(\mathbb{P}_0, \ldots, \mathbb{P}_n) = R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$.

Properties of Differential Resultant

The differential resultant has the following properties:

a) $R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ is differentially homogeneous in each $\mathbf{u}_i$ and is of order $h_i = s - s_i$ in $\mathbf{u}_i$ $(i = 0, \ldots, n)$ where $s = \sum_{l=0}^{n} s_l$.

b) **Algebraic case:** $\mathrm{Res}(A(x), B(x)) = c \prod_{\xi, B(\xi)=0} A(\xi)$.

Differential case: There exist $\xi_{\tau\rho}$ such that
$R(\mathbf{u}_0, \ldots, \mathbf{u}_n) = A(\mathbf{u}_0, \ldots, \mathbf{u}_n) \prod_{\tau=1}^{t_0} \mathbb{P}_{\mathbf{0}}(\xi_{\tau\mathbf{1}}, \ldots, \xi_{\tau\mathbf{n}})^{(\mathbf{h_0})}$.

And $(\xi_{\tau 1}, \ldots, \xi_{\tau n})\, (\tau = 1, \ldots, t_0)$ are generic points of the prime ideal $[\mathbb{P}_1, \ldots, \mathbb{P}_n]$.

Preliminaries Intersection Theory Differential Chow Form Differential Chow variety Generalized Differential Chow Form Summa

0000 000 0000000000 00 0000000000

c) Algebraic case: $\mathrm{Res}(A(x), B(x)) = A(x)T(x) + B(x)W(x)$, where $\deg(T) < \deg(B), \deg(W) < \deg(A)$.

Differential case: $R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ can be written as a linear combination of $\mathbb{P}_i$ and the derivatives of $\mathbb{P}_i$ up to the order $s - s_i$. Precisely, we have

$$R(\mathbf{u}_0, \ldots, \mathbf{u}_n) = \sum_{i=0}^{n} \sum_{j=0}^{s-s_i} h_{ij} \mathbb{P}_i^{(j)}.$$

And $\deg(h_{ij}) \leq (sn + n)^2 D^{sn+n} + D(sn + n)$ with $D = \max_i \{m_i\}$.

d) Algebraic case: $\mathrm{Res}(A(x), B(x)) = 0 \Leftrightarrow A(x), B(x)$ have a common solution.

   Differential case: When $\mathbf{u}_i$ specialize to $\mathbf{v}_i \in \mathcal{E}(\mathcal{F})$, $\mathbb{P}_i$ specialize to $\overline{\mathbb{P}}_i$.

   If $\overline{\mathbb{P}}_i = 0$ have a common solution, then $R(\mathbf{v}_0, \ldots, \mathbf{v}_n) = 0$.

   And if $R(\mathbf{v}_0, \ldots, \mathbf{v}_n) = 0$ and $S_R(\mathbf{v}_0, \ldots, \mathbf{v}_n) \neq 0$, then $\overline{\mathbb{P}}_i = 0 (i = 0, \ldots, n)$ have a common solution, where $S_R = \frac{\partial R}{\partial u_{00}^{(h_0)}}$.

Summary

- An intersection theory for generic differential polynomials is developed.

  As a consequence, the differential dimension conjecture for generic differential polynomials is proved.

- The differential Chow form is defined and most of the properties of the Chow form in the algebraic case are extended to its differential counterpart.

- The generalized diff Chow form is defined and its properties are proved.

  As an application, the differential resultant is defined and properties similar to that of the Sylvester resultant of two univariate polynomials are proved.

Thanks !

Reference. X.S. Gao, W. Li, C.M. Yuan. *Intersection Theory for Generic Differential Polynomials and Differential Chow Form*. arxiv20100901, 1-50.