Differential schemes and differential algebraic varieties

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$\widehat{R} = \mathcal{O}_R(X) \quad \widehat{X} = \operatorname{Spec}^{\Delta} \widehat{R}$
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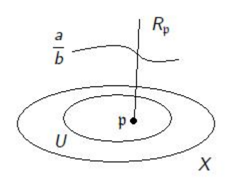

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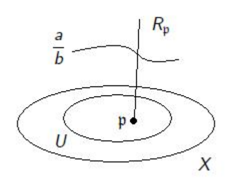

Conjecture

$\iota^*\colon \widehat{X} \to X$ *is a homeomorphism*

Auxiliary sheaf

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$\mathcal{O}'_R(U) = \textit{regular functions in } U$
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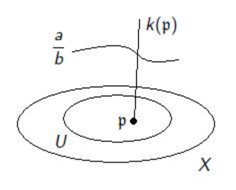

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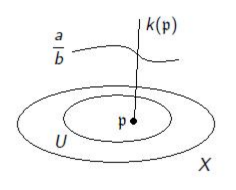

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$\iota_r \colon R \to D \subseteq \widehat{R}'$. *Then* $\iota_r^* \colon \mathrm{Spec}^{\Delta} D \to \mathrm{Spec}^{\Delta} R$ *is a homeomorphism.*

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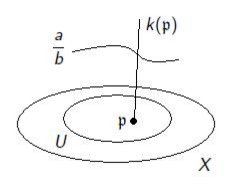

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Corollary

The mapping $\iota_r^* \colon \widehat{X}' \to X$ *is a homeomorphism.*

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Definition (Keigher ring)

$I \subseteq R$ is a Δ-ideal $\Rightarrow r(I)$ is a Δ-ideal.

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Definition

Let $\delta = \{\delta^k\}_{k\geqslant 0}$, $\delta_k \colon R \to R$:

1) $\delta^0(x) = x$
2) $\delta^k(a+b) = \delta^k(a) + \delta^k(b)$
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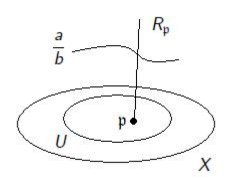

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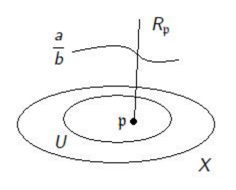

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The structure sheaves

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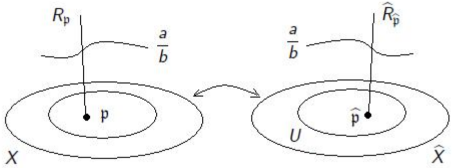

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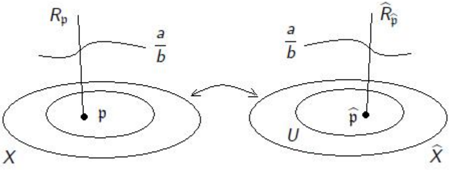

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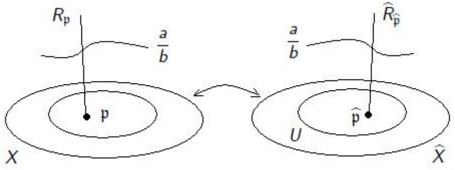

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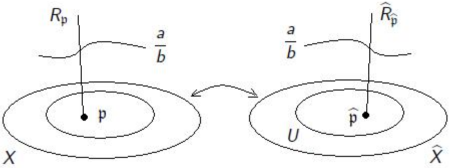

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Fact

If R is reduced. Then $\mathcal{O}_R = \mathcal{O}_{\widehat{R}}$.

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Properties of the integral dependence in commutative case:
(Integral dependence = ID)

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From now all differential rings are **Ritt algebras** ($\mathbb{Q} \subseteq R$)

Valuation rings and integral dependence

Let K be a field and $A, B \subseteq K$ be local rings with maximal ideals $\mathfrak{m}, \mathfrak{n}$

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Let $A \subseteq K$. The following condition are equivalent:

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Fact

B is integral over A iff $B \subseteq \bigcap A_\alpha$, where A_α are all valuation rings in K containing A.

Differential integral dependence

The notion of differential valuation ring.

Definition

- $A \subseteq K$ is an extremal ring if A is a maximal local Δ-ring with respect to $\leqslant$ and $\mathfrak{m}$ is differential
- $A \subseteq K$ is Δ-valuation if $\exists\ L \supseteq K$ and extremal $A' \subseteq L$ such that $A = A' \cap K$

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The notion of differential integral dependence.

Definition

- $A \subseteq B$ is Δ-integral if $B \subseteq A'$ whenever $A \subseteq A'$ and A' is a Δ-valuation ring
- $A \to B$ is Δ-integral if $\forall \mathfrak{p} \subseteq B$, $A/\mathfrak{p}^c \subseteq B/\mathfrak{p}$ is Δ-integral

Properties of differential integral dependence

Theorem

Let $f\colon A \to B$ be Δ-integral. Then

1. $\mathfrak{b} \subseteq B$, $a = \mathfrak{b}^c \Rightarrow A/\mathfrak{a} \to B/\mathfrak{b}$ *is* Δ*-integral.*
2. $S \subseteq A \Rightarrow S^{-1}A \to S^{-1}B$ *is* Δ*-integral.*
3. A, B, C *are* D*-algebras* $\Rightarrow A \otimes_D C \to B \otimes_D C$ *is* Δ*-integral.*
4. $f^*\colon \operatorname{Spec}^\Delta B \to \operatorname{Spec}^\Delta A/\ker f$ *is surjective.*
5. *The* **going up** *property holds for* f*.*
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Theorem

$f\colon A \to B$ *is* Δ*-integral iff*
$\forall$ *A-algebra* C : $(f \otimes 1)^*\colon \operatorname{Spec}^\Delta B \otimes_A C \to \operatorname{Spec}^\Delta C$ *is closed.*

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Let $A \to B$ be reduced Δ-rings.
Then $\operatorname{Spec}^\Delta B \to \operatorname{Spec}^\Delta A$ is universally closed iff $A \to B$ is Δ-integral.

Examples

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Theorem

- $C \subset L$, $\operatorname{trdeg}_C L = 1$
- $\forall c \in C \Rightarrow c' = 0$
- $A_i \subset L$ *are valuation rings such that* $\mathfrak{m}_i$ *are differential*

Then $\overline{C}$ *is either* C *or* $\cap_i A_i$.

Complete varieties

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Example (Kolchin)

If $C \subseteq K$ is a constant subfield. Then the constant projective space $\mathbb{P}^1_C$ is differentially complete.

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Definition

- $\dim^{\Delta} B = \sup\{k \mid \mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \dots \subsetneqq \mathfrak{p}_k,\ \mu(\mathfrak{p}_i, \mathfrak{p}_{i+1}) = m\}$
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Theorem (Johnson)

$\operatorname{coht}^\Delta \mathfrak{p} = \dim^\Delta B/\mathfrak{p} = \operatorname{trdeg}^\Delta{}_K B.$

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Conjecture

B is **differentially catenary***:*
$\forall\ \mathfrak{p} \subseteq \mathfrak{q}$ *and* $\forall$ *saturated chain* $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_k = \mathfrak{q}$ *we have*
$k = \operatorname{ht}^\Delta(\mathfrak{q}/\mathfrak{p}) = \dim^\Delta B/\mathfrak{p} - \dim^\Delta B/\mathfrak{q}$.

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Theorem (Rosenfield)

If $\mathfrak{p}$ is regular with respect to some ranking. Then the inequality holds.

Regular points

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- $B = K\{x_1, \dots, x_n\}$ and $\mathfrak{p} \subseteq B$
- $A = B_{\mathfrak{p}}$ and $\mathfrak{m} \subseteq A$
- $G_{\mathfrak{m}}(A) = \oplus_{k \geqslant 0} \mathfrak{m}^k / \mathfrak{m}^{k+1}$
- $K_{\mathfrak{m}} = A/\mathfrak{m}$
- $S_K(V)$ is the symmetric algebra on V over K

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Theorem (Johnson)

- $\mathfrak{p}$ *is regular with respect to some ranking* $\Rightarrow$ $\mathfrak{p}$ *is regular*
- $\mathfrak{p}$ *is regular then* $G_{\mathfrak{m}}(A) = S_{K_{\mathfrak{m}}}(\mathfrak{m}/\mathfrak{m}^2)$
- B *is reduced* $\Rightarrow$ *The set of all regular primes is open and not empty*

Completions and Catenarity

Theorem

If $G_{\mathfrak{m}}(A) = S_{K_{\mathfrak{m}}}(\mathfrak{m}/\mathfrak{m}^2)$ *holds. Then the inequality*

$$\operatorname{ht}^{\Delta}\mathfrak{p} \geqslant \dim^{\Delta} A - \dim^{\Delta} A/\mathfrak{p}$$

holds.

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$$\begin{tikzcd}
K\{y_1,\ldots,y_k\} \arrow[r] & A \arrow[r] & \widehat{A} \\
\mathfrak{p}_\alpha \arrow[rr, dashed, bend right] & \mathfrak{q}_\alpha & \mathfrak{q}'_\alpha \arrow[l]
\end{tikzcd}$$