Lie Algebra of Differential Operators on Path Algebras

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Differential Algebra

- A **differential algebra** is an associative algebra R together with a linear operator $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$.
- Differential algebra originated from the algebraic study of differential equations (Ritt and Kolchin) and is a natural yet profound extension of commutative algebra and the related algebraic geometry. Differential algebra has also found important applications, such as to arithmetic geometry, logic and computational algebra, especially in the work of W. T. Wu on mechanical theorem proving in geometry.
- More recently, ideas of differential Galois theory has been applied in the work of Connes and Marcolli on renormalization of QFT and motivic Galois groups.

Integral algebra and Rota-Baxter algebra

- In opposite to the differential operator, there is the **integral operator** $P : R \to R$ such that $P(x)P(y) = P(xP(y)) + P(P(x)y), \quad \forall x, y \in R$.
- In the Lie algebra context, this is the operator form of the classical Yang-Baxter equation:
$$[P(x), P(y)] = P[x, P(y)] + P[P(x), y], \forall x, y \in \mathfrak{g}.$$
- There is also the more general **Rota-Baxter operator** $P : R \to R$:
$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \quad \forall x, y \in R,$$
where λ is a fixed constant, and Baxter is an American mathematician.
- This concept has appeared in the Connes-Kreimer Hopf algebra approach to renormalization of QFT (Connes and Kreimer, Comm. Math. Phy. (1999-2003), Ebrahimi-Fard, Guo and Kreimer, Integrable renormalization I,II, *J. Math. Phy.* (2004), *Ann. H. Poincare* (2005), *Comm. Math. Phy.* (2006), Guo and Zhang, *J. Algebra* (2008)).
- Even more general is the concept of an $\mathcal{O}$-operator, again applicable to integrable systems (Bai, *J. Phy. A* (2007), Bai, Guo and Ni: *Comm. Math. Phy.* (to appear)).

Back to differential algebra

- Recently, differential algebra has found combinatorial connections. For instance, differential structures were found on heap ordered trees (Grossman and Larson, Adv. Appl. Math, 2005) and on decorated rooted trees (Guo and Keigher, J. Pure and Appl. Algebra, 2008).
- In the current work, we consider differential algebra structures on another combinatorially defined objects, namely the path algebras of quivers (Fang Li, J. Algebra (2009), $\cdots$).
- This gives a natural class of differential algebras of finite and infinite dimensions.
- Most of the study on differential algebra up to date have been for commutative algebras and fields. This paper can be regarded as a first step to extended the study to noncommutative algebras by studying their differential aspects.
- Through the realization of basic algebras and Artin algebras as quotients of path algebras (Gabriel Theorem) and generalized path algebras, we hope this study will lead to the study of differential basic algebras and Artin algebras.

Quivers and their path algebras

- Recall that a quiver Γ is (algebraically) defined to be a quadruple $(\Gamma_0, \Gamma_1, h, t)$ where Γ_0 is a set (of vertices), Γ_1 (of edges) and map $h, t : \Gamma_1 \to \Gamma_0$, giving the head $h(p)$ and the tail $t(p)$ of p. An arrow is a triple $(t(p), p, h(p))$, or more intuitively, $\bullet_{t(p)} \xrightarrow{p} \bullet_{h(p)}$.
- A path p in Γ is either a vertex $p = v \in \Gamma_0$ or a sequence (composition) of arrows

$$p := \bullet_{t(p_1)} \xrightarrow{p_1} \bullet_{h(p_1)=t(p_1)} \xrightarrow{p_2} \bullet_{t(p_2)=t(p_3)} \cdots \bullet_{h(p_{k-1})=t(p_k)} \xrightarrow{p_k} \bullet_{h(p_k)}.$$

  Define $t(p) = t(p_1), h(p) = t(p_k)$ and $\ell(p) = k$.
- An oriented cycle is a path p with $h(p) = t(p)$.
- Let $\mathcal{P}$ denote the set of paths of Γ. Define the product of two paths p and q to be the composition pq if $h(p) = t(q)$ and to be zero otherwise. This defines an associative algebra structure on the linear space $\mathbf{k}\Gamma := \mathbf{k}\mathcal{P}$, called the path algebra of Γ.

Differential operators on a path algebra

- For an algebra A, we have two Lie algebras:
  (i) The Lie algebra $Lie(A) = (A, [,])$ with $[x, y] = xy - yx \quad \forall x, y \in A$.
  (ii) The derivation Lie algebra

$$Der(A) = \{D \in End(A) \mid D(uv) = uD(v) + D(u)v \quad \forall u, v \in A\}$$

  with the Lie multiplication $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$.
- For $a \in A$, we have an inner derivation

$$D_a : A \to A, \quad D_a(b) = (ad_a)(b) := ab - ba, \quad b \in A.$$

  This gives a Lie algebra homomorphism

$$\mathfrak{D} : Lie(A) \to Der(A), \quad \mathfrak{D}(a) = D_a, \quad a \in A.$$

- $IDiff(A) := \mathrm{im}\mathfrak{D} \subseteq Der(A)$ is a Lie ideal and $\ker \mathfrak{D}$ is precisely the center $C(A)$ of A. Denote $EDiff(A) = Der(A)/IDiff(A)$.
- Our main interest is $Diff(\mathbf{k}\Gamma) = \mathrm{Der}(\mathbf{k}\Gamma)$ for a quiver Γ.
- It is well-known that $C(\mathbf{k}\Gamma) = \mathbf{k}[x]$ if Γ is an oriented loop $\bullet_v \xrightarrow{p} \bullet_v$, $C(\mathbf{k}\Gamma) = \mathbf{k}$ otherwise.
- Thus unless Γ is a loop, we have $IDiff(\mathbf{k}\Gamma) = \mathbf{k}\Gamma/\mathbf{k} \hookrightarrow Der(\mathbf{k}\Gamma)$.

Questions to consider

- Existence of non-zero differential operators on $\mathbf{k}\Gamma$;
- Structure of $Der(\mathbf{k}\Gamma)$;
- Relationship between the combinatorial structure of Γ and $Der(\mathbf{k}\Gamma)$ and $ODiff(\mathbf{k}\Gamma)$.
- Structure on $ODiff(\mathbf{k}\Gamma)$;
- We will see that all the answers depend on an explicitly given basis of $Der(\mathbf{k}\Gamma)$.

Characterization of a derivation

- Recall that $\mathcal{P}$ is a $\mathbf{k}$-basis of $\mathbf{k}\Gamma$. So for a $\mathbf{k}$-linear map $D : \mathbf{k}\Gamma \to \mathbf{k}\Gamma$ is determined by the coefficients $c_q^p \in \mathbf{k}$ in

$$D(p) = \sum_{q \in \mathcal{P}} c_q^p q, \quad p \in \mathcal{P}.$$

Convention: the sum over an empty set is defined to be zero.

- Theorem 1. $D : \mathbf{k}\Gamma \to \mathbf{k}\Gamma$ is a differential operator if and only if D is defined by

$$D(v) = \sum_{q \in \mathcal{P}, t(q)=v, h(q)\neq v} c_q^{t(q)} q - \sum_{q \in \mathcal{P}, h(q)=v, t(q)\neq v} c_q^{t(q)} q \quad \text{for } v \in \Gamma_0,$$

$$D(p) = \sum_{q \in \mathcal{P}\setminus V, h(q)=t(p), t(q)\neq t(p)} c_q^{t(q)} qp + \sum_{q \in \mathcal{P}\setminus V, t(q)=h(p), h(q)\neq h(p)} c_q^{t(q)} pq$$

$$+ \sum_{i=1}^{k} \sum_{q_i \in \mathcal{P}\setminus V, t(q_i)=t(p_i), h(q_i)=h(p_i)} c_{q_i}^{p_i} p_1 \cdots p_{i-1} q_i p_{i+1} \cdots p_k,$$

where $\mathcal{P}\setminus\Gamma_0 \ni p = p_1 \cdots p_k$ is the decomposition of p into arrows.

Standard basis of $Der(\mathbf{k}\Gamma)$

- For $r, s \in \mathcal{P} \backslash \Gamma_0$ with $r \in \Gamma_1$ and $h(s) = h(r), t(s) = t(r)$, define $D_{r,s} : \mathbf{k}\Gamma \to \mathbf{k}\Gamma$ by the conditions
  (i). $D_{r,s}(q) = \delta_{r,q} s, \quad q \in \Gamma_0 \cup \Gamma_1.$
  (ii). $D_{r,s}(q_1 q_2) = D_{r,s}(q_1) q_2 + q_1 D_{r,s}(q_2), \quad q_1, q_2 \in \mathcal{P}.$
- Then $D_{r,s}$ defines a differential operator on $\mathbf{k}\Gamma$.
- Theorem 2. Denote
  $\mathfrak{B}_1 := \{D_s \mid s \in \mathcal{P}, h(s) \neq t(s)\}.$
  $\mathfrak{B}_2 := \{D_{r,s} \mid r, s \in \mathcal{P}, \backslash \Gamma_0$ with $r \in \Gamma_1$ and $h(s) = h(r), t(s) = t(r)\}.$
  Then $\mathfrak{B} := \mathfrak{B}_1 \cup \mathfrak{B}_2$ is a basis of $Der(\mathbf{k}\Gamma)$.

Existence of derivations

- **Theorem 3.** $Der(\mathbf{k}\Gamma) \neq 0$ if and only if Γ contains an arrow.
- Proof. $(\Leftarrow)$. If Γ contains an arrow p, then by Theorem 2, $D_{p,p}$ is a non-zero differential operator on $\mathbf{k}\Gamma$.
  $(\Rightarrow)$. If Γ does not contain any arrow, then a differential operator D on $\mathbf{k}\Gamma$ is determined by $D(v)$, $v \in \Gamma_0$. By Theorem 1,

  $$D(v) = \sum_{q \in \mathcal{P}, t(q)=v, h(q)\neq v} c_q^{t(q)} q - \sum_{q \in \mathcal{P}, h(q)=v, t(q)\neq v} c_q^{t(q)} q.$$

  Since Γ does contain any arrow, this sum is over an empty set. So $D(v) = 0$, $\forall v \in \Gamma_0$ and hence $D = 0$.

Structure of the Lie algebra $Der(\mathbf{k}\Gamma)$

- We list some sub-structures of $Diff(\mathbf{k}\Gamma)$ and their relations as follows:
- $Indiff(\mathbf{k}\Gamma) := ad(\mathbf{k}\Gamma) = \{D_s \mid s \in \mathbf{k}\Gamma\}$.
- $ODiff(\mathbf{k}\Gamma) := Der(\mathbf{k}\Gamma)/InDiff(\mathbf{k}\Gamma)$.
- $\mathfrak{D}_1 := \mathbf{k}\mathfrak{B}_1 =$the subspace of $Indiff\,\mathbf{k}\Gamma$ generated by $\mathfrak{B}_1 := \{D_s \mid s \in \mathcal{P}, h(s) \neq t(s)\}$.
- $\mathfrak{D}_0 := \mathbf{k}\mathfrak{B}_0 =$the subspace of $\mathfrak{D}_1$ generated by $\mathfrak{B}_0 := \{D_s \mid s \in \mathcal{P}, h(s) = h(t)$, but $\ell(s) \geq 1\}$.
- $\mathfrak{D}_V := \mathbf{k}\mathfrak{B}_V =$the subspace of $\mathfrak{D}_1$ generated by $\mathfrak{B}_V := \{D_v : v \in \Gamma_0\}$.
- $\mathfrak{D}_2 := \mathbf{k}\mathfrak{B}_2 =$the subspace of $Diff(\mathbf{k}\Gamma)$ generated by $\mathfrak{B}_2 := \{D_{r,s} \mid r, s \in \mathcal{P}\backslash\Gamma_0, \ell(r) = 1, h(s) = h(r), t(s) = t(r)\}$.
- **Theorem 4.** We have the commutative diagram of exact sequences of Lie algebras.

$$\begin{tikzcd}
0 \arrow[r] & Indiff(\mathbf{k}\Gamma) \arrow[r] & Diff(\mathbf{k}\Gamma) \arrow[r] & ODiff(\mathbf{k}\Gamma) \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{D}_0 + \mathfrak{D}_V \arrow[r] \arrow[u] & \mathfrak{D}_2 \arrow[r] \arrow[u] & \mathfrak{D}_2/(\mathfrak{D}_0 + \mathfrak{D}_V) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

$ODiff(\mathbf{k}\Gamma)$ and the combinatorics of Γ

- In $0 \longrightarrow Indiff\,\mathbf{k}\Gamma \longrightarrow Diff\,\mathbf{k}\Gamma \longrightarrow ODiff(\mathbf{k}\Gamma) \longrightarrow 0$ $InDiff(\mathbf{k}\Gamma) \cong \mathbf{k}\Gamma/\mathbf{k}$ (except when Γ is a oriented cycle), so it pretty much recovers the algebra structure of $\mathbf{k}\Gamma$. One is wondering what extra information could $ODiff(\mathbf{k}\Gamma)$ provide.
- If Γ has an oriented cycle, then all the Lie algebras involved are infinite dimensional. So we consider quivers without oriented cycles.
- Let Γ be a planar quiver with a fixed embedding into $\mathbb{R}^2$. A primitive cycle is an unoriented cycle that contains no other unoriented cycles. Let Γ_p be the set of primitive cycles of Γ and let γ_p be its cardinality. It is one less than the number of connected components of $\mathbb{R}\backslash\Gamma$, so describes the topology of Γ.
- An almost oriented cycle is a pair (p, r) where $p \in \Gamma_1$ and $r \in \mathcal{P}$ with $h(r) = h(p), t(r) = t(p)$ such that, for the inverse arrow p^* of p, $p^* r$ is an oriented cycle. Let Γ_a be the set of almost oriented cycles of Γ and let γ_a be its cardinality.
- **Theorem 5.** Let Γ be a planar quiver with no oriented cycles. Then

$$\dim_{\mathbf{k}} ODiff(\mathbf{k}\Gamma) = \gamma_p + \gamma_a.$$

Structure of $ODiff(\mathbf{k}\Gamma)$

- Theorem 5 suggests a canonical basis of $ODiff(\mathbf{k}\Gamma) = Der(\mathbf{k}\Gamma)/InDiff(\mathbf{k}\Gamma)$.
- For each primitive cycle $\mathfrak{p}$ consisting of a list of arrows $(p_1, \cdots, p_k)$, define $D_{\mathfrak{p}} = \pm D_{p_1,p_1} \pm \cdots \pm D_{p_k,p_k}$, where a $\pm p_i$ is $+p_i$ if p_i is in the clockwise direction and is $-p_i$ otherwise.
- **Theorem 6.** The set

$$\{D_{\mathfrak{p}} \mid \mathfrak{p} \in \Gamma_p\} \sqcup \{D_{p,r} \mid (p, r) \in \Gamma_a\}$$

  is a basis of $ODiff(\mathbf{k}\Gamma)$.
- The Lie algebra structure of $ODiff(\mathbf{k}\Gamma)$ can be given in terms of this basis.

- **THANK YOU!**