Integro-Differential Algebras, Operators, and Polynomials

$$(\mathcal{F}, \partial) + \int$$

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joint work Markus Rosenkranz

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Motivation

Algebraic approaches and Symbolic Computation
for Boundary problems

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Green's Operator G via Green's Function g:

$$Gf(x) = \int_0^1 g(x,\xi)\, f(\xi)\, d\xi \qquad g(x,\xi) = \begin{cases} (x-1)\xi & \text{for } x \geq \xi \\ \xi(x-1) & \text{for } x \leq \xi \end{cases}$$

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Definition

$(\mathcal{F}, \partial, \int)$ is an integro-differential algebra if
$(\mathcal{F}, \partial)$ is a differential K-algebra and $\int$ is a K-linear section of $\partial = '$,
i.e. $(\int f)' = f$, such that the differential Baxter axiom

$$(\textstyle\int f')(\int g') + \int (fg)' = (\int f')g + f(\int g')$$

holds. cf. R-R '08, Guo-Keigher '08

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$$\partial\,(a_0, a_1, a_2, \dots) = (a_1, a_2, \dots) \quad \textstyle\int (a_0, a_1, \dots) = (0, a_0, a_1, \dots)$$

Sections and Multiplicative Projectors

Example: $C^\infty(\mathbb{R})$, $\int f' = f - f(a)$, and $f - \int f' = f(a)$

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Differential fields cannot have integral operators

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- Solve initial value problems with variation-of-constants formula

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$\mathcal{F}[\partial]$ differential operators over $(\mathcal{F}, \partial)$:

Via normal Forms $\sum f_i \partial^i$ with multiplication $\partial f = f\partial + f'$

Integro-Differential Operators, Preliminaries

"Integration gives one evaluation for free"

Need more characters ("evaluations") $\mathcal{F} \to K$ for boundary problems
Example: $C^\infty(\mathbb{R})$, point evaluations $f \mapsto f(b)$

Fix a set of characters $\Phi \subseteq \mathcal{F}^*$ including $\mathtt{E}$

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Free K-algebra generated by the symbol ∂ and the "functions" $f \in \mathcal{F}$ modulo the rewrite system (and linearity)

fg	$\to$	$f \bullet g$	∂f	$\to$	$f\partial + \partial \bullet f$

$\bullet$ denotes action on $\mathcal{F}$, $\partial \bullet f = f'$

Integro-Differential Operators, Construction

$\mathcal{F}_\Phi[\partial, \int]$ integro-differential operators over $(\mathcal{F}, \partial, \int)$ and Φ

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$\varphi\psi$	$\to$	ψ	$\partial\varphi$	$\to$	0
φf	$\to$	$(\varphi \bullet f)\,\varphi$	$\partial\int$	$\to$	1
$\int f \int$	$\to$	$(\int \bullet f)\int - \int(\int \bullet f)$			
$\int f\partial$	$\to$	$f - \int(\partial \bullet f) - (\mathbf{E} \bullet f)\,\mathbf{E}$			
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Proposition

The rewrite system is Noetherian and confluent
(forms a noncommutative Gröbner-Shirshov basis). (R-R '08)

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differential | integral | boundary operator

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differential integral boundary operator

Subalgebras

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Direct decomposition

$$\mathcal{F}_\Phi[\partial, \textstyle\int] = \mathcal{F}[\partial] \dotplus \mathcal{F}[\textstyle\int] \dotplus (\Phi)$$

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- compute Green's operator from a given fundamental system
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- lift factorization of differential operators to boundary problems

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Instance of general construction of polynomials in universal algebra
Free product of coefficient algebra and free algebra

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$$(4uu' \textstyle\int (x+3)u'^3)(u' \int u''^2) + \int x^6 u(0)^2 u''(0) uu''^5 \int (x^2+5x) u^3 u'^2 \int u,$$

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where each multi-index and n may be zero.
Not unique ("integration by parts"),

$$\textstyle\int fu' \quad \text{and} \quad fu - \int f'u - f(0)\,u(0)$$

represent the same polynomial

Canonical Forms for Integro-Differential Polynomials

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Confluence proof for the rewrite rules for integro-differential operators via a Gröbner-Shirshov basis computation in a suitable algebraic domain (Tec-R-R-Buchberger '10)

Conclusion and Outlook

- Integro-differential algebras: differential algebra + integral operator

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Thank you!