Invariants via Moving Frames: Computation and Applications

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Outline:

- Definitions and examples of invariants
- Applications:
  - congruence problem for curves;
  - symmetry reduction of variational problems;
- Structure theorems
- Computation via moving frames (classical, generalized, inductive and algebraic methods)

Group actions and invariants:

Group actions

An action of a group $\mathcal{G}$ on a set $\mathcal{Z}$ is a map $\Phi : \mathcal{G} \times \mathcal{Z} \longrightarrow \mathcal{Z}$ such that

i. $\Phi(e, \mathbf{z}) = \mathbf{z}, \quad \forall \mathbf{z} \in \mathcal{Z}$.

ii. $\Phi\,(g_1,\, \Phi(g_2, \mathbf{z})) = \Phi(g_1\, g_2, \mathbf{z})$, $\forall \mathbf{z} \in \mathcal{Z}$ and $\forall g_1, g_2 \in \mathcal{G}$.

Example: Let $M(n, \mathbb{K}) = \{n \times n$ matrices over a field $\mathbb{K}\}$.
A group $\mathcal{G}L(n, \mathbb{K}) = \{A \in M(n, \mathbb{K}) |\, \det(A) \neq 0\}$ acts on $\mathbb{K}^n$ by:

$$\Phi(A, \mathbf{z}) = A\mathbf{z},\ \forall A \in \mathcal{G}L(n, \mathbb{K}) \text{ and } \mathbf{z} \in \mathbb{K}^n.$$

Notation: $\mathcal{G} \curvearrowright \mathcal{Z}$ and $\Phi(g, \mathbf{z}) = g \cdot \mathbf{z}$.

We will consider

- $\mathcal{G}$ – smooth Lie group or algebraic group over a field $\mathbb{K}$

- $\mathcal{Z}$ – smooth manifold or algebraic variety

- Φ – smooth map or polynomial or rational map

A local action of a topological group $\mathcal{G}$ on a topological set $\mathcal{Z}$ is a map $\Phi : \Omega \longrightarrow \mathcal{Z}$ defined on some open subset $\Omega \subset G \times \mathcal{Z}$ containing $e \times \mathcal{Z}$, such that

i. $\Phi(e, \mathbf{z}) = \mathbf{z}, \quad \forall \mathbf{z} \in \mathcal{Z}$.

ii. $\Phi\,(g_1,\, \Phi(g_2, \mathbf{z})) \;=\; \Phi(g_1\, g_2, \mathbf{z}), \quad \forall g_1, g_2, \mathbf{z}$ such that $(g_2, \mathbf{z}) \in \Omega$ and $(g_1\, g_2, \mathbf{z}) \in \Omega$.

Invariants:

A function F on $\mathcal{Z}$ is invariant under $\mathcal{G} \curvearrowright \mathcal{Z}$ if

$$F(g \cdot \mathbf{z}) = F(\mathbf{z}), \quad \forall \mathbf{z} \in \mathcal{Z} \text{ and } \forall g \in \mathcal{G}.$$

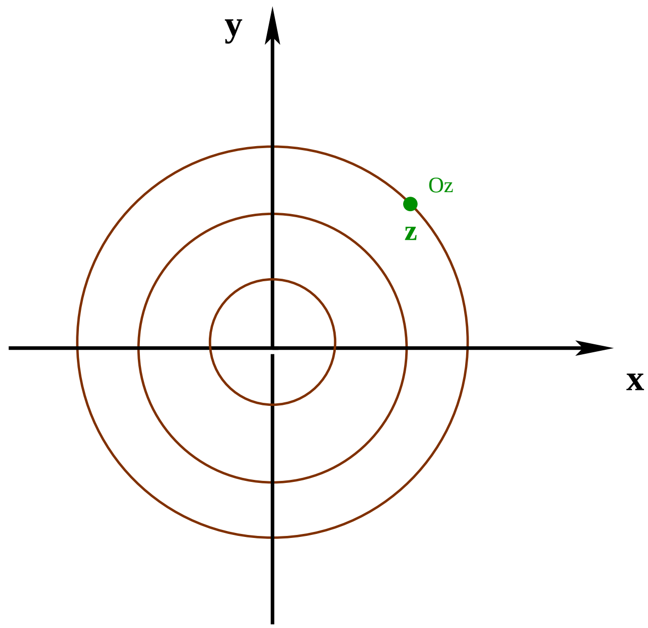
A function F, defined on an open subset $\mathcal{U}$ of a topological set $\mathcal{Z}$, is locally invariant under $\mathcal{G} \curvearrowright \mathcal{Z}$ if

$$F(g \cdot \mathbf{z}) = F(\mathbf{z}), \quad \forall (g, \mathbf{z}) \in \Omega.$$

for some open subset $\Omega \subset \mathcal{G} \times \mathcal{Z}$ such that $e \times \mathcal{U} \subset \Omega$.

Invariants under rotations on $\mathbb{R}^2$:

$SO(2,\mathbb{R}) \curvearrowright \mathbb{R}^2$ by rotations



Invariants

- Any smooth invariant on $\mathbb{R}^2 - \{(0,0)\}$ is functions of $r = \sqrt{x^2 + y^2}$.
- Any polynomial invariant on $\mathbb{R}^2$ is functions of $r^2 = x^2 + y^2$.

Orbits are level sets of r.

Invariants under rotations and translations on $\mathbb{R}^2$:

Action: $SE(2,\mathbb{R}) = SO(2,\mathbb{R}) \ltimes \mathbb{R}^2 \curvearrowright \mathbb{R}^2$ by rotations and translations.

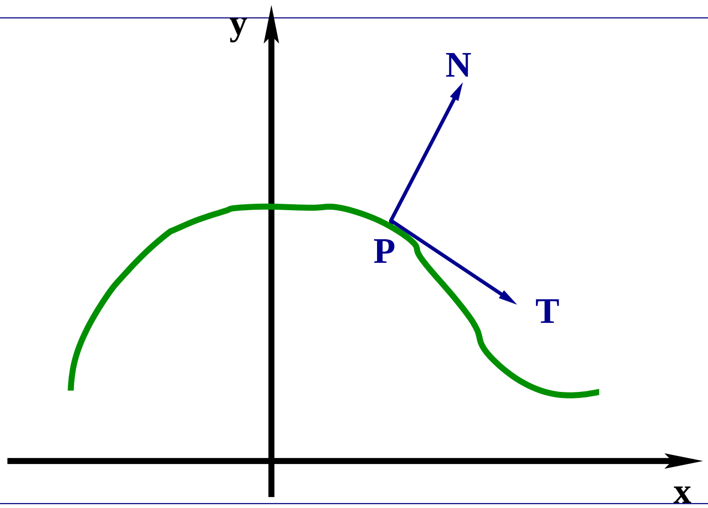
$\mathbb{R}^2$ is a single orbit.

Invariants: constant functions.

Differential invariants for planar curves $\gamma(t) = (x(t), y(t))$ under rotations and translations

$SE(2,\mathbb{R})$-action on $\mathbb{R}^2$ induces an action on $x(t), y(t), \dot{x}(t), \dot{y}(t), \ldots$ (jet bundle of curves in $\mathbb{R}^2$).

- Unit tangent: $T = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$, $|T| = 1 \Rightarrow$
  Infinitesimal arc-length: $ds = \sqrt{\dot{x}^2 + \dot{y}^2}\,dt$
- Unit normal: $N \perp T$, $|N| = 1$.
- The Frénet equation: $\frac{dT}{ds} = \kappa N$



- Generators of the differential algebra of invariants: κ and $\frac{d}{ds}$, where $\frac{d}{ds} = \frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}\frac{d}{dt}$ is an invariant differential operator.

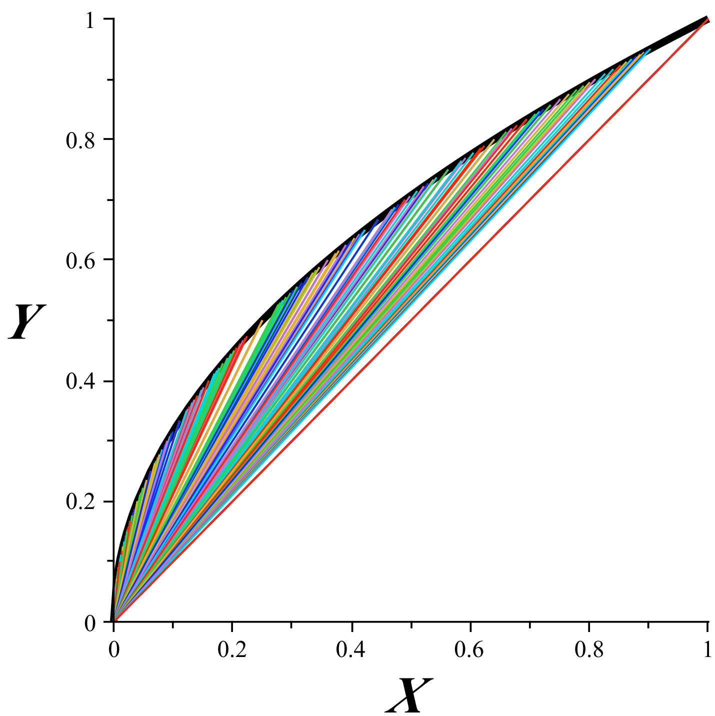
- Fundamental local diff. invariants:

$$\kappa, \kappa_s = \frac{d\kappa}{ds}, \kappa_{ss}, \ldots$$

An integral invariant for planar curves $\gamma(t) = (x(t), y(t))$, $t \in [a, b]$

Notation: $X(t) = x(t) - x(a)$, $Y(t) = y(t) - y(a)$,

$$\boxed{I^{[0,1]}(t) = \int_a^t Y(\tau)\, dX(\tau) - \frac{1}{2} X(t)\, Y(t)}$$



$I^{[0,1]}$ represnets the signed area between the curve and a secant. It is invariant under $\mathcal{SA}(2, \mathbb{R}) \supset SE(2, \mathbb{R})$ action.

An <u>discrete</u> invariants for quadratic forms

The standard action of $\mathcal{G}L(n,\mathbb{C})$ on $\mathbb{C}^n$ induces an action on the space V_d^n of homogeneous polynomials of degree d in n variables:

$$A \cdot P(\mathbf{x}) = P\left(A^{-1}\mathbf{x}\right), \forall A \in \mathcal{G}L(n,\mathbb{C}) \text{ and } \mathbf{x} \in \mathbb{C}^n.$$

There are well known canonical forms for $\mathcal{G}L(n,\mathbb{C}) \curvearrowright V_2^n$:

$$x_1^2 + \cdots + x_k^2, \text{ for } k = 0, \ldots n.$$

k is a discrete invariant for $\mathcal{G}L(n,\mathbb{C}) \curvearrowright V_2^n$.

Types of the invariants:

- local smooth;
- polynomial, rational, and algebraic;
- differential;
- integral;
- integro-differential;
- discrete;
- ...

Applications:

- Equivalence (congruence) problems for
  - sub-manifolds (in particular curves and surfaces)
  - for polynomials
  - differential equations
  - . . .
- Symmetry reduction of
  - differential equations
  - variational problems
  - algebraic equations
- Invariant geometric flows
- . . .

Equivalence problem for curves

Equivalence problem for curves in $\mathbb{R}^n$.

- Problem: Given an action of a group $\mathcal{G}$ on $\mathbb{R}^n$ and curves $\gamma_1\colon [a,b] \longrightarrow \mathbb{R}^n$ and $\gamma_2\colon [c,d] \longrightarrow \mathbb{R}^n$ decide whether there exists $g \in \mathcal{G}$ such that

$$Image(\gamma_1) = g \cdot Image(\gamma_2).$$

- If such $g \in \mathcal{G}$ exists then γ_1 and γ_2 are called $\mathcal{G}$-equivalent, or $\mathcal{G}$-congruent:

$$\gamma_1 \cong \gamma_2.$$

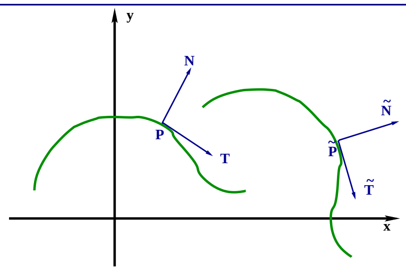
Transformations on $\mathbb{R}^2$ commonly appearing in computer image processing:

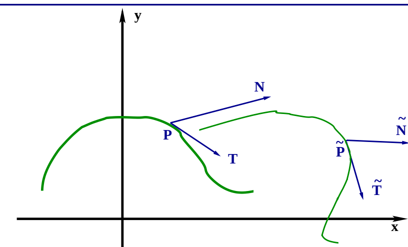
- Special Euclidean (orientation preserving rigid motions):
  $\boxed{X = \cos(\phi)x - \sin(\phi)y + a,\ Y = \sin(\phi)x + \cos(\phi)y + b.}$
- Euclidean (rigid motions):
  $\boxed{X = \cos(\phi)x - \sin(\phi)y + a,\ Y = \epsilon(\sin(\phi)x + \cos(\phi)y) + b}$
  $\epsilon = \pm 1$
- similarity
  $\boxed{X = \lambda(\cos(\phi)x - \sin(\phi)y) + a,\ Y = \epsilon\lambda(\sin(\phi)x + \cos(\phi)y) + b}$,
  $\epsilon = \pm 1, \lambda \neq 0.$
- equi-affine (area and orientation preserving):
  $\boxed{X = \alpha\, x + \beta\, y + a,\ Y = \gamma\, x + \delta\, y + b,}$ $\alpha\delta - \beta\gamma = 1$
- affine:
  $\boxed{X = \alpha\, x + \beta\, y + a,\ Y = \gamma\, x + \delta\, y + b}$ $\alpha\delta - \beta\,\gamma \neq 0$
- projective: $\boxed{X = \frac{\alpha\, x + \beta\, y + a}{\nu\, x + \mu\, y + c},\ Y = \frac{\gamma\, x + \delta\, y + b}{\nu\, x + \mu\, y + c},}$ $\det\begin{pmatrix} \alpha & \beta & a \\ \gamma & \delta & b \\ \nu & \mu & c \end{pmatrix} \neq 0$

Euclidean and equi-affine frame

Euclidean geometry in $\mathbb{R}^2$	Equi-affine geometry in $\mathbb{R}^2$
$SE(2,\mathbb{R}) = SO(2,\mathbb{R}) \ltimes \mathbb{R}^2$	$SA(2,\mathbb{R}) = SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$

Moving Frame:





$T = \left(\frac{dx}{ds}, \frac{dy}{ds}\right),\ N \perp T,\ \lvert N\rvert = 1$	$T = \left(\frac{dx}{d\alpha}, \frac{dy}{d\alpha}\right), \quad N = \frac{dT}{d\alpha}$

Infinitesimal arc-length:

$\lvert T\rvert = 1 \Rightarrow ds = \sqrt{1 + y_x^2}\, dx$	$\det \lvert TN\rvert = 1 \Rightarrow d\alpha = y_{xx}^{1/3}\, dx$

Fundamental differential invariants:

$\frac{dT}{ds} = \kappa N$	$\frac{dN}{d\alpha} = \mu T$
$\Downarrow$	$\Downarrow$
$\kappa_s = \frac{d\kappa}{ds}, \kappa_{ss}, \ldots$	$\mu_\alpha = \frac{d\mu}{d\alpha}, \mu_{\alpha\alpha}, \ldots.$

Differential invariants for planar curves

Let $\mathcal{G}$ be an r-dim'l Lie group acting on the plane. For almost all actions $\exists$

- a local differential invariant ξ ($\mathcal{G}$-curvature) of differential order $r-1$;
- an invariants differential form ϖ (infinitesimal $\mathcal{G}$-arclength) of differential order at most $r-2$ and the dual invariant differential operator D_{ϖ}.

s.t. any other local differential invariant on an open subset of the jet space $\mathcal{J}(\mathbb{R}^2, 1)$ is a smooth function of ξ, $D_{\varpi}\xi, D_{\varpi}^2\xi, \ldots$

Relations between invariants of a group and its subgroup*

- special Eucl.: $\kappa = \frac{(\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y})}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}$, $ds = \sqrt{\dot{x}^2 + \dot{y}^2}\,dt$, $\frac{d}{d\,s} = \frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}\frac{d}{d\,t}$

- equi-affine: $\mu = \frac{3\,\kappa\,(\kappa_{ss}+3\,\kappa^3) - 5\,\kappa_s^2}{9\,\kappa^{8/3}}$, $d\alpha = \kappa^{1/3} ds$, $\frac{d}{d\,\alpha} = \frac{1}{\kappa^{1/3}}\frac{d}{d\,s}$

- projective: $\eta = \frac{6\mu_{\alpha\alpha\alpha}\mu_\alpha - 7\,\mu_{\alpha\alpha}^2 - 9\mu_\alpha^2\,\mu}{6\mu_\alpha^{8/3}}$, $d\rho = \mu_\alpha^{1/3} d\alpha$, $\frac{d}{d\,\rho} = \frac{1}{\mu_\alpha^{1/3}}\frac{d}{d\,\alpha}$.

Definition: Curves for which $\mathcal{G}$-curvature or $\mathcal{G}$-arclength are undefined are called $\mathcal{G}$-exceptional.

*see (Kogan 2001, 2003) for a general method of deriving invariants of a group in terms of invariants of its subgroup

Congruence criteria for curves with specified initial point

- Theorem: *Two non $\mathcal{G}$-exceptional curves are $\mathcal{G}$-congruent iff their $\mathcal{G}$-curvatures as functions of $\mathcal{G}$-arclength coincide.*

For $\gamma_1(t),\ t \in [a,b] \longrightarrow \mathbb{R}^2$ and $\gamma_2(\tau),\ \tau \in [c,d] \longrightarrow \mathbb{R}^2$:

$$\exists g \in \mathcal{G} \quad \text{s. t.} \quad g \cdot \gamma_1(a) = \gamma_2(c) \text{ and } Image(\gamma_1) = g \cdot Image(\gamma_2)$$

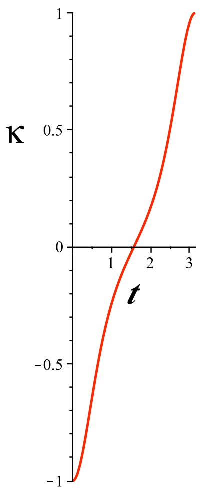
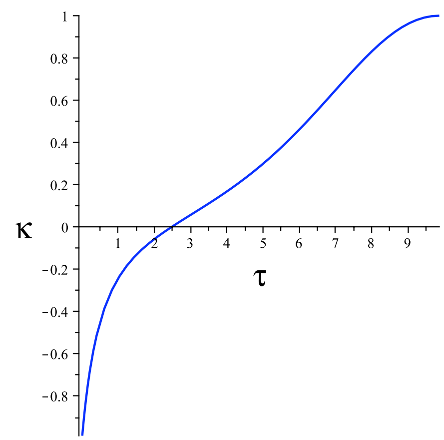
$$\Updownarrow$$

$$\xi|_{\gamma_1}(s_1) \quad = \quad \xi|_{\gamma_2}(s_2), \text{ where } s_1(t) = \int_a^t \varpi|_{\gamma_1} \text{ and } s_2(\tau) = \int_c^\tau \varpi|_{\gamma_2}$$

- Applicable only if:
  - initial point is specified
  - arc-length reparametrization is feasible in practice

$\mathcal{G}$-curvature under reparametrization

Euclidean example: $\kappa = \frac{(\ddot{y}\dot{x} - \ddot{x}\dot{y})}{(\dot{x}^2+\dot{y}^2)^{\frac{2}{3}}}$:

$\gamma(t) = (t, \cos t),\ t \in [0, \pi]$	$\tilde{\gamma}(\tau) = (\sqrt{\tau}, \cos\sqrt{\tau}),\ \tau \in [0, \pi^2]$
$\kappa\vert_{\gamma}(\phi(\tau)) = -\frac{\cos(t)}{(1+\sin^2(t))^{3/2}}$	$\kappa\vert_{\tilde{\gamma}}(\tau) = -\frac{\cos(\sqrt{\tau})}{(1+\sin^2(\sqrt{\tau}))^{3/2}}$
	

$\boxed{\kappa|_{\gamma}(\phi(\tau)) = \kappa|_{\bar{\gamma}}(\tau) \text{ where } t = \phi(\tau) = \sqrt{\tau}.}$

Differential signature for planar curves

(Calabi et al. (1998))

- Let ξ be $\mathcal{G}$-curvature, ϖ-infinitesimal $\mathcal{G}$-arclength and $\xi_\varpi = D_\varpi \xi$.

- Definition: The $\mathcal{G}$-signature of a non-exceptional curve $\gamma(t) = \Big(x(t), y(t)\Big)$, $t \in [a, b]$ is the image of a parametric curve $\Big(\xi|_\gamma(t), \xi_\varpi|_\gamma(t)\Big)$:

$$\mathcal{S}_\gamma(t) = \{\Big(\xi|_\gamma(t), \xi_\varpi|_\gamma(t)\Big) \,|\, t \in [a, b]\}.$$

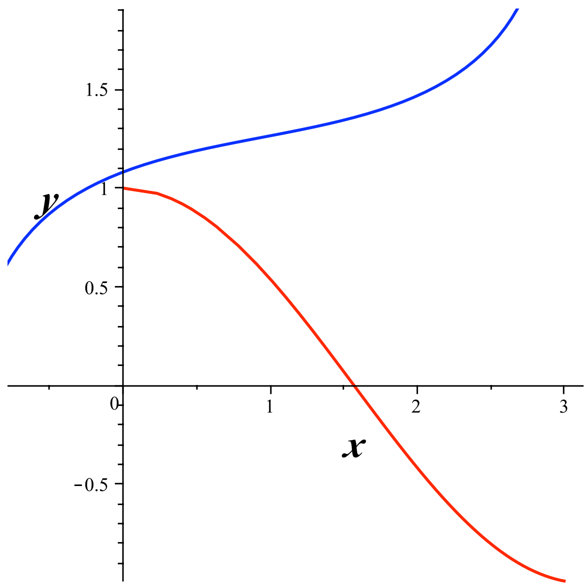
- $\mathcal{G}$-congruence criterion for non-exceptional curves

$$\begin{array}{ccl} \gamma_1 & \cong & \gamma_2 \\ & \Downarrow\Uparrow & \text{under certain conditions} \\ \mathcal{S}_{\gamma_1} & = & \mathcal{S}_{\gamma_2} \end{array}$$

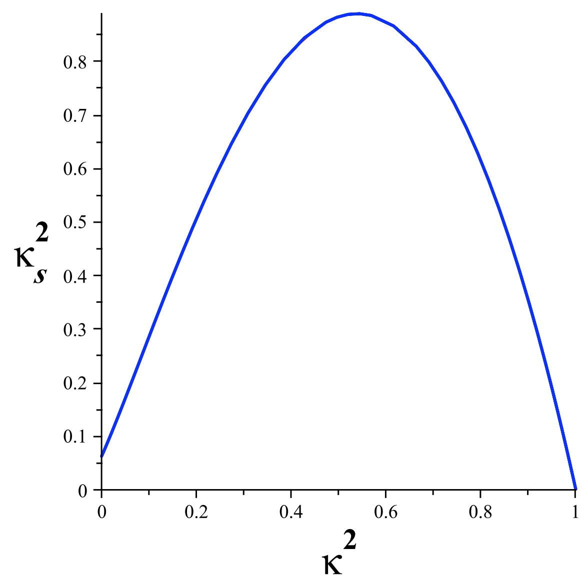
Example 1 of Euclidean differential signature:

$\gamma(t) = (\sqrt{t}, \cos\sqrt{t})$, $t \in [0, \pi^2]$	$\tilde{\gamma}(t) = (\frac{3}{5}t - \frac{4}{5}\cos t, \frac{4}{5}t + \frac{3}{5}\cos t)$, $t \in [0, \pi]$

Images of γ and $\tilde{\gamma}$ in $\mathbb{R}^2$



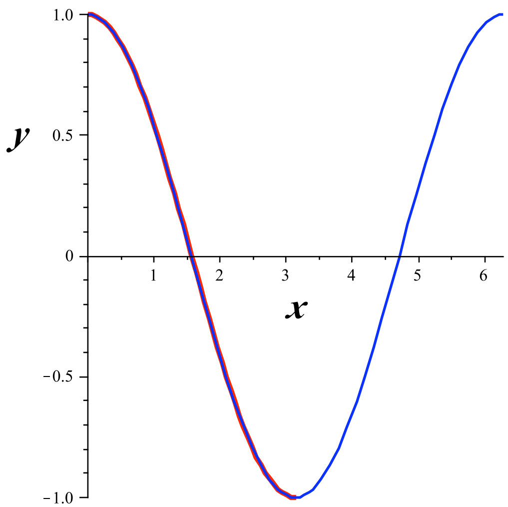
Signatures (κ^2, κ_s^2) for γ and $\tilde{\gamma}$



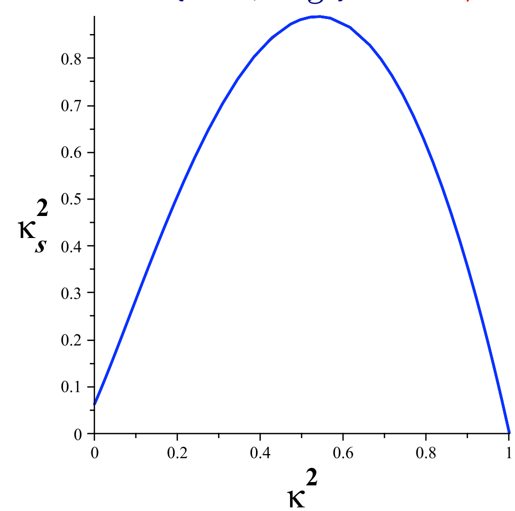
Example 2 of Euclidean differential signature:

$$\boxed{\gamma(t) = (t,\, \cos t),\ t \in [0, \pi]} \quad \boxed{\tilde{\gamma}(t) = (t,\, \cos t),\ t \in [0, 2\,\pi]}$$

Images of γ and $\tilde{\gamma}$ in $\mathbb{R}^2$



Signatures (κ^2, κ_s^2) for γ and $\tilde{\gamma}$



Images of signatures of γ and $\tilde{\gamma}$ coincide due to reflection symmetry of $\tilde{\gamma}$

Signature for γ is traced 2 times when $t \in [0, \pi]$ due to symmetry under rotations by π around the point $(\frac{\pi}{2}, 0)$.
Signature for $\tilde{\gamma}$ is traced 4 times when $t \in [0, 2\,\pi]$!

Local $\mathcal{G}$-congruence criterion for non-exceptional curves

γ_1 locally congruent to γ_2

$\Downarrow\Uparrow$ for smooth curves

$\mathcal{S}_{\gamma_1}$ and $\mathcal{S}_{\gamma_2}$ overlap

Advantages and disadvantages of differential signature

+ the construction extends to curves and higher dimensional submanifolds of $\mathbb{R}^n$ under majority of transformations.

+ independent of parametrization

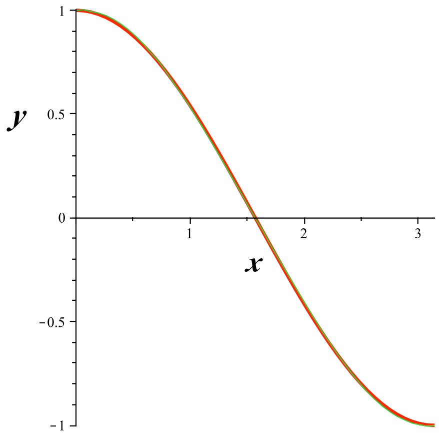
+ can be used for local comparison

+ can be used to detect symmetries

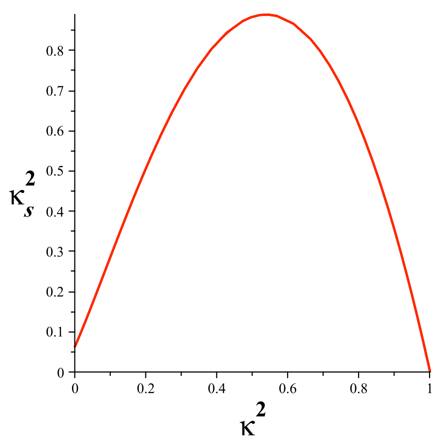
- depends on derivatives of high order (for planar curves of order = $\dim \mathcal{G}$) $\Longrightarrow$ very sensitive to high frequency perturbations

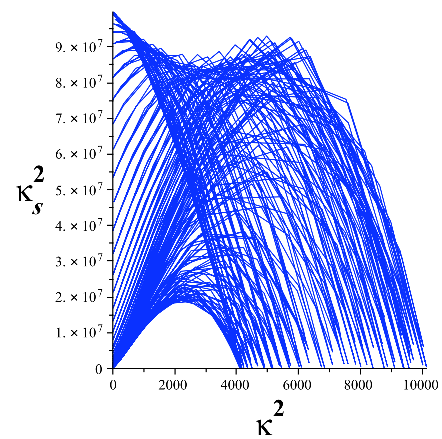
Sensitivity of differential signature to high frequency perturbation:

Images of $\gamma = (t,\, \cos t)$ and $\tilde\gamma = (t,\, \cos(t) + \frac{1}{100}\sin(100\,t),\, t \in [0,\pi]$



Signatures (κ^2, κ_s^2) for γ and $\tilde\gamma$





Integral variables for planar curves $\gamma(t) = (x(t), y(t))$, $t \in [a, b]$.

(Hann and Hickman (2002)

- $\mathcal{G}$-action on $\mathbb{R}^2$ induces an action on $x(0),\ y(0),\ x(t),\ y(t),$ and

$$x^{[i,j]}(t) = \int_a^t x(\tau)^i\, y(\tau)^j\, dx(\tau).$$

- Example: if $x \longrightarrow x + y$, and $y \longrightarrow y$ then

$$x^{[i,j]}(t) \longrightarrow \int_a^t \left[x(\tau) + y(\tau)\right]^i\, y(\tau)^j\, d\left[x(\tau) + y(\tau)\right]$$

- $y^{[i,j]}(t) = \int_a^t x(\tau)^i\, y(\tau)^j\, dy(\tau)$ can be expressed in terms of $x(a),\ y(a),\ x(t),\ y(t),\ x^{[k,l]}(t) = \int_a^t x(\tau)^k\, y(\tau)^l\, dx(\tau)$ via integration-by-parts.

- $i + j$ is called the order of integral variable $x^{[i,j]}$.

Integral invariants for planar curves*

- An affine action can be prolonged to an integral jet bundle of planar curves which is parametrized by $x(a)$, $y(a), x$, y, $x^{[i,j]}$, where $j > 0, i \geq 0$.

- Integral invariants are invariant functions on the integral jet bundle.

- Moving frame method can be applied to derive fundamental or generating sets of integral invariants.

- In (Feng, Kogan, Krim (2010)) we derived Euclidean and affine fundamental sets of integral invariants for curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ via inductive variation of the moving frame method.

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*Integral invariants defined here are not the same as moment invariants (Taubin and Cooper (1992))

Examples of integral invariants for planar curves

$$\gamma(t), \quad t \in [a,b]$$

- Notation: $X(t) = x(t) - x(a)$, $Y(t) = y(t) - y(a)$,
$$X^{[i,j]}(t) = \int_a^t X(\tau)^i \, Y(\tau)^j \, dX(\tau).$$
- Invariants:

0-th order $r = \sqrt{X^2 + Y^2}$ - E_2-invariant

1-st order $I^{[0,1]} = X^{[0,1]} - \frac{1}{2} X\,Y$ -($\mathcal{SA}_2 \supset SE_2$)-invariant.

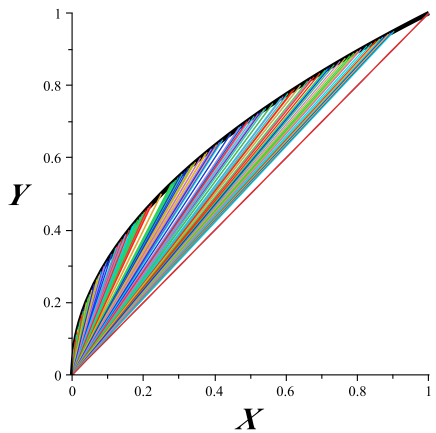
2-nd order $*$ $I^{[1,1]} = Y\,X^{[1,1]} - \frac{1}{2} X\,X^{[0,2]} - \frac{1}{6} X^2\,Y^2$-
$\mathcal{SA}_2$ and E_2-invariant

$*$ $I^{[0,2]} = Y\,X^{[0,2]} + 2X\,X^{[1,1]} - \frac{1}{3} X\,Y^3 - \frac{2}{3} X^3\,Y$
E_2-invariant

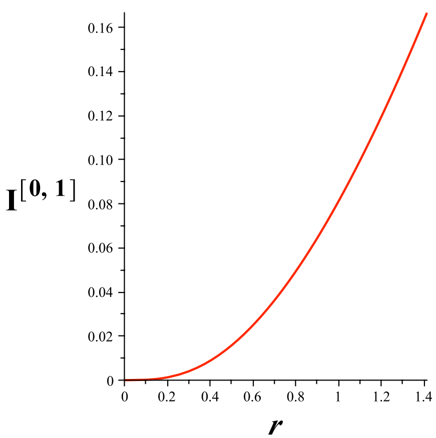
...

Geometric interpretation of

$$I^{[0,1]}(t) = X^{[0,1]} - \frac{1}{2} X\, Y = \int_a^t Y(\tau)\, dX(\tau) - \frac{1}{2} X(t)\, Y(t)$$



The signed area between the curve and a secant, originating at the initial point.



$(r, I^{[0,1]})$-signature is the graph of the length of a secant vs. the area between the curve and the secant. It is independent of parametrization.

Examples of integral signatures for planar curves*

- $SE(2)$-signature $\left(r,\ I^{[0,1]}\right)$

- $E(2)$- signatures $\left(r,\ (I^{[0,1]})^2\right)$ or $\left(r,\ I^{[1,1]}\right)$.

- similarity signature: $\left(\frac{(I^{[0,1]})^2}{r^4},\ \frac{I^{[1,1]}}{r^4}\right)$

- $\mathcal{SA}(2)$-signature $\left(I^{[0,1]},\ I^{[1,1]}\right)$

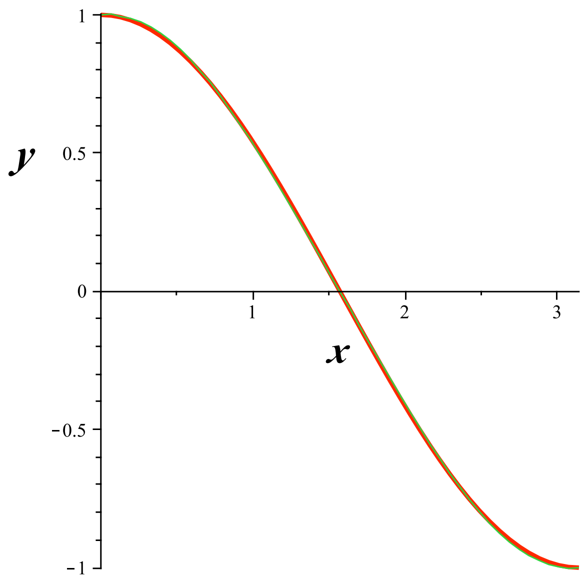
*see ((Feng, Kogan, Krim (2010))) for signatures of curves in $\mathbb{R}^3$

curves

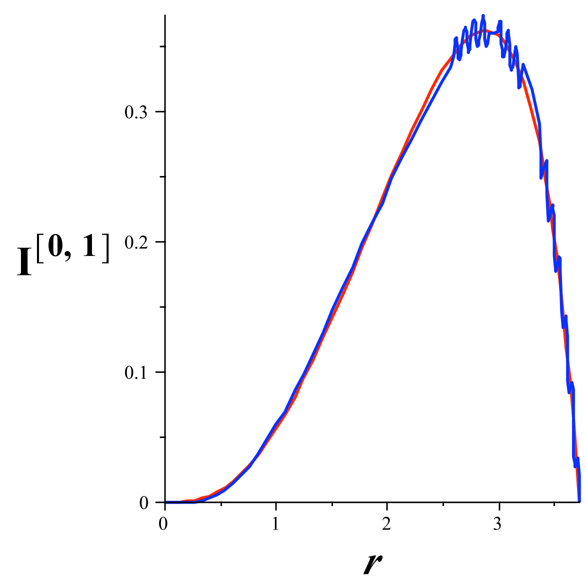
Reasonable behavior under high frequency perturbation:

$\gamma(t) = (t,\ \cos t)$, $t \in [0, \pi]$	$\tilde{\gamma}(t) = (t,\ \cos(t) + \frac{1}{100}\sin(100\,t))$, $t \in [0, \pi]$

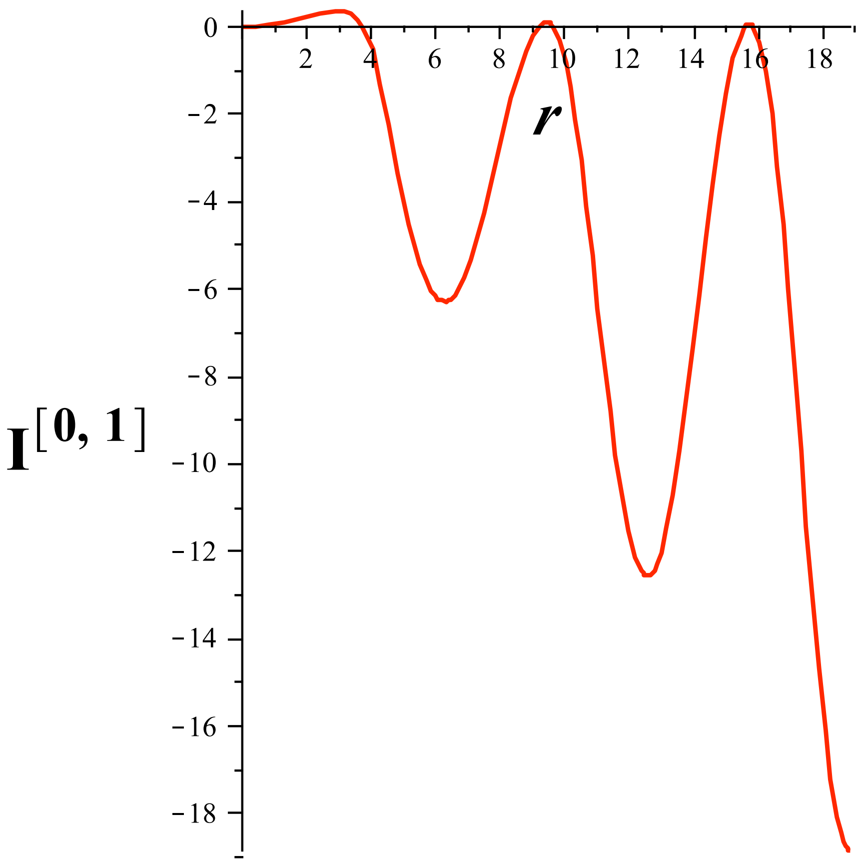
Images of γ and $\tilde{\gamma}$ in $\mathbb{R}^2$



$SE(2,\mathbb{R})$- signatures $(r,\ I_1)$ for γ and $\tilde{\gamma}$



Signature $(r,\ I_1)$ **for** $\gamma(t) = (t, \cos(t))$ **for** $t \in [0, 6\,\pi]$**:**



Equivalence theorem for curves with specified initial points:

$$\gamma_1 \quad \cong \quad \gamma_2$$

$$\Downarrow \quad \Uparrow \quad \text{conditions ?}$$

$$\text{integral signature}|_{\gamma_1} \quad = \quad \text{integral signature}|_{\gamma_2}$$

Remark:

- $\Downarrow$ follows from the definition of invariants
- $\Uparrow$ is proved for
  - $SE(2,\mathbb{R})$-signature $\left(r,\ I^{[0,1]}\right)$
  - $E(2,\mathbb{R})$- signature $\left(r,\ I^{[1,1]}\right)$.

Advantages and disadvantages of integral signature

+ extends to curves in $\mathbb{R}^n$ (see Feng, Kogan, Krim (2010) for curves in $\mathbb{R}^3$).

+ independent of parametrization

+ tolerant to data uncertainty and perturbations

$\mp$ requires an identified initial point

- possible, but problematic use for local comparison

- no straightforward generalization to rational action (i.e. projective actions), see Hann and Hickman (2002) for a numeric approach.)

General framework for solving an equivalence problem for an action of $\mathcal{G}$ on a set $\mathcal{Z}$

- find a finite set of invariants that separates generic orbits. i.e. orbits on an open dense subset $\mathcal{U} \subset \mathcal{Z}$.

- characterize orbits on $\mathcal{Z} - \mathcal{U}$ (possibly by another set of invariants).

A glimpse into the symmetry reduction

General framework for symmetry reduction

Definition: A group of transformations $\mathcal{G}$ on the space of independent and dependent variables is a Lie symmetry of a differential equation (or a variational problem) if each element of $\mathcal{G}$ maps a solution to a solution.

Theorem: (S. Lie (1897))

- (almost) any $\mathcal{G}$-symmetric system of differential equations can be written in terms of differential $\mathcal{G}$-invariants.

- (almost) any $\mathcal{G}$-symmetric variational problem can be written in terms of differential $\mathcal{G}$-invariants and $\mathcal{G}$-invariant differential forms.

Example: $SE(2,\mathbb{R})$-invariant variational problem for $y = u(x)$:

$$\mathcal{L}[u] = \int \frac{u_{xx}^2}{2\,(1+u_x^2)^{5/2}} dx \qquad \Longleftrightarrow \qquad \mathcal{L}[\kappa] = \int \tfrac{1}{2}\kappa^2\, ds$$

$$\Big\downarrow E = \frac{\partial}{\partial u} - \left(\frac{d}{dx}\right)\frac{\partial}{\partial u_x} + \left(\frac{d}{dx}\right)^2 \frac{\partial}{\partial u_{xx}} \cdots \qquad\qquad \Big\downarrow \qquad ?$$

$$\Delta = 0 \qquad\qquad \Longleftrightarrow \qquad \kappa_{ss} + \tfrac{1}{2}\kappa^3 = 0$$

$$\Delta = \frac{2\,u_4\,(1+u_1^2)^2 - 20\,u_1\,u_2\,u_3\,(1+u_1^2) + 30\,{u_2}^3\,{u_1}^2 - 5\,{u_2}^3}{2\,(1+{u_1}^2)^{\frac{9}{2}}}.$$

$(\,u_1 = u_x, \ldots, u_4 = u_{xxxx})$

$\mathcal{G}$-invariant Euler-Lagrange operator for planar curves $y = u(x)$:

$$
\begin{array}{ccc}
\int L(x,u,u_1,\ldots,u_n)\,dx & \Leftrightarrow & \int \mathcal{L}\left(\xi, D_\varpi \xi, \ldots, D_\varpi^m \xi\right)\,\varpi \\
\downarrow & & \downarrow \\
E(L) = \sum_i \left(-\dfrac{d}{d\,x}\right)^i \dfrac{\partial\,L}{\partial\,u_i} = 0 & \Leftrightarrow & \left[A^* \mathcal{E}\left(\mathcal{L}\right) - B^* \mathcal{H}\left(\mathcal{L}\right)\right] = 0,
\end{array}
$$

where

$$
\mathcal{E}\left(\mathcal{L}\right) = \sum_{i=0}^{n} (-D_\varpi)^i \frac{\partial\,\mathcal{L}}{\partial\,\xi_i}, \quad \mathcal{H}\left(\mathcal{L}\right) = \sum_{i>j\geq 0}^{n} \xi_{i-j}\left(-D_\varpi\right)^j \frac{\partial\,\mathcal{L}}{\partial\,\xi_i} - \mathcal{L}.
$$

- A^* and B^* – $\mathcal{G}$-invariant diff. operators, computable by differentiation and linear algebra.
- *general formula for any number of independent variables and unknown functions is obtained in Kogan and Olver(2003)*
- Completely algorithmic – iVB package (IK) in MAPLE.

Structure theorems

Structure theorems of algebraic invariant theory:

- Hilbert theorem (1890): If an algebraic reductive group $\mathcal{G}$ acts regularly on an affine variety $\mathcal{Z}$ then the ring of polynomial invariants $\mathbb{K}[\mathcal{Z}]^{\mathcal{G}}$ is finitely generated.
$$\mathbb{K}[\mathcal{Z}]^{G} = \mathbb{K}[u_1, \ldots, u_d] \backslash R,$$
where R is a finitely generated ideal of syzygies.

- If an algebraic group $\mathcal{G}$ acts rationally on an affine variety $\mathcal{Z}$ of dimension m then the field of rational invariants $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$ is finitely generated.
If $\dim \mathcal{Z} = m$ and $\max_{\mathbf{z}} \dim \mathcal{O}_{\mathbf{z}} = s$, then the transcendence degree of $\mathbb{K}(\mathcal{Z})^{\mathcal{G}} : \mathbb{K}$ is $m - s$.

- Rosenlicht theorem (1956): Rational invariants separate orbits on an open dense subset of $\mathcal{Z}$. Any separating subset of rational invariants is generating.

- Problems:
  - Find (minimal) generating set of $\mathbb{K}[\mathcal{Z}]^{\mathcal{G}}$ and $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$.
  - Describe the structure of $\mathbb{K}[\mathcal{Z}]^{\mathcal{G}}$ and $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$ (find syzygy ideal, transcendence basis, ...).

Theorem of smooth invariant theory:

- Definition: Let $\mathcal{G}$ be a smooth Lie group acting on a smooth manifold $\mathcal{Z}$. A collection of local invariants on an open subset $\mathcal{U} \subset \mathcal{Z}$ forms a fundamental set if they are functionally independent, and any local invariant on $\mathcal{U}$ can be expressed as a smooth function of the invariants from this set.

- Frobenious integrability theorem $\Rightarrow$ If $\dim \mathcal{Z} = m$ and all orbits have the same dimension s, then for each point $\mathbf{z} \in \mathcal{Z}$ there exists a fundamental set of $m - s$ local smooth invariants defined on an open neighborhood $\mathcal{U}_{\mathbf{z}}$.

- Problem:
  - Find a fundamental set of invariants.

Structure theorem of differential invariant theory:

Let $\mathcal{G}$ be a Lie group acting on an n-dim'l manifold $\mathcal{Z}$. For $1 \leq p < n$ $\exists$!prolongation of $\mathcal{G}$-action to the jet bundle $\mathcal{J}(\mathcal{Z}, p)$ of p-dim'l submanifolds of $\mathcal{Z}$.

Tresse theorem (1894): Local smooth invariants on $\mathcal{J}(\mathcal{Z}, p)$ have a structure of finitely generated differential algebra*:

- $\exists\{\mathcal{I}^1, \ldots, \mathcal{I}^\nu\}$ - invariant function on $\mathcal{J}(\mathcal{Z}, p)$
- $\exists\mathcal{D}_1, \ldots, \mathcal{D}_p$ - invariant differential operators

such that any invariant $\mathcal{I}$ on $\mathcal{J}(\mathcal{Z}, p)$ can be expressed as

$$\mathcal{I} = F\left(\ldots, \mathcal{D}_J(\mathcal{I}^l), \ldots\right)$$

*in general it is a non-free algebra with non-commutative derivations

Problem:

- Find (minimal) set of generators
- Finite (minimal) set of generating syzygies $H\left(\ldots, \mathcal{D}_J(\mathcal{I}^l), \ldots\right) \equiv 0$

Structure theorems of integral invariant theory ???

or may be

Structure theorems of integro-differential invariant theory ???

Invariants via moving farmes

- Classical moving frames (Frénet (1847), Serret (1851), Darboux (1887), Cartan (1935))

- Generalization of moving frame construction to arbitrary Lie group actions on manifolds (Fels and Olver (1999))

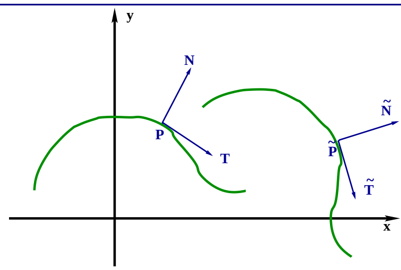
- Inductive and recursive variations (Kogan(2001, 2003))

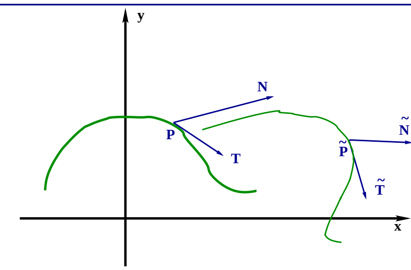
- Algebraic formulation (Hubert, Kogan(2007))

Euclidean and affine moving frames for curves

Euclidean geometry in $\mathbb{R}^2$	Equi-affine geometry in $\mathbb{R}^2$
$SE(2) = SO(2) \ltimes \mathbb{R}^2$	$SA(2) = SL(2) \ltimes \mathbb{R}^2$

Moving Frame:





$T = \left(\frac{dx}{ds}, \frac{dy}{ds}\right), \ N \perp T, \ \lvert N \rvert = 1$	$T = \left(\frac{dx}{d\alpha}, \frac{dy}{d\alpha}\right), \quad N = \frac{dT}{d\alpha}$

Infinitesimal arc-length:

$\lvert T \rvert = 1 \Rightarrow ds = \sqrt{1 + y_x^2}\, dx$	$\det \lvert T\,N \rvert = 1 \Rightarrow d\alpha = y_{xx}^{1/3}\, dx$

Fundamental differential invariants:

$\frac{dT}{ds} = \kappa N$	$\frac{dN}{d\alpha} = \mu T$
$\Downarrow$	$\Downarrow$
$\kappa_s = \frac{d\kappa}{ds}, \kappa_{ss}, \ldots$	$\mu_\alpha = \frac{d\mu}{d\alpha}, \mu_{\alpha\alpha}, \ldots.$

Observe that in the affine and in the Euclidean case:

- Moving frame defines a map from the jets of curve to $\mathcal{G}$, i. e. $([T, N], (x, y)) \in \mathcal{G}$.
- Invariants can be obtained from the pull-backs of a basis of invariant differential forms on $\mathcal{G}$ by ρ.

Generalizations to submanifolds of homogeneous spaces (Cartan (1935), Griffiths (1974), Green(1978), Chern (1985))

Definition. (Fels and Olver (1999)) Given $\mathcal{G} \curvearrowright \mathcal{Z}$, a (local) moving frame is an equivariant smooth (local) map ρ: $\mathcal{Z} \to \mathcal{G}$.

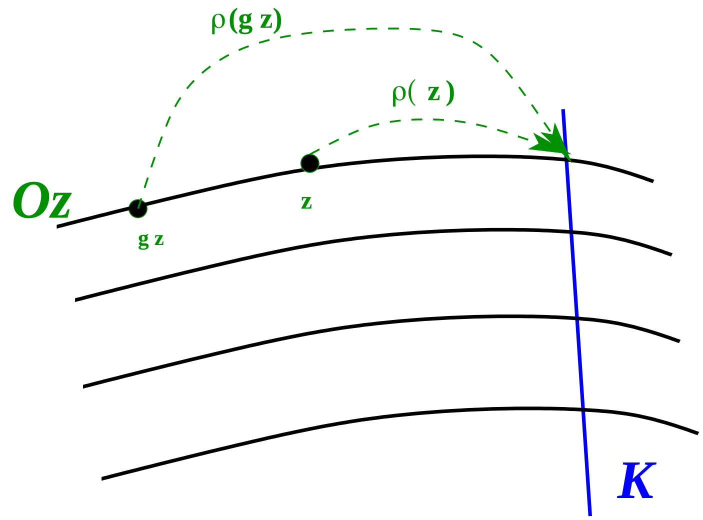
$$\begin{tikzcd}
\mathcal{G} \arrow[r, "R_{g^{-1}}"] & \mathcal{G} \\
\mathcal{Z} \arrow[u, "\rho"] \arrow[r, "g"'] & \mathcal{Z} \arrow[u, "\rho"']
\end{tikzcd}$$

Theorem. (Fels and Olver (1999))

$\exists$ loc. moving frame

$\Updownarrow$

$\mathcal{G}$ action is locally free* and $\exists$ local cross-section $\mathcal{K}$ on $\mathcal{Z}$: $T|_{\mathbf{z}}\mathcal{K} \bigoplus T|_{\mathbf{z}}\mathcal{O}_{\mathbf{z}} = T|_{\mathbf{z}}\mathcal{Z}$, $\forall \mathbf{z} \in \mathcal{K}$.



$\rho : \mathcal{Z} \longrightarrow \mathcal{G}$ is defined by the condition $\rho(\mathbf{z}) \cdot \mathbf{z} \in \mathcal{K}$

$$\boxed{\rho(g \cdot \mathbf{z})(g \cdot \mathbf{z}) = \rho(\mathbf{z}) \cdot \mathbf{z},\ \text{freeness} \Longrightarrow \rho(g \cdot \mathbf{z}) = \rho(\mathbf{z})g^{-1}}$$

$\Downarrow$

ρ is a $\mathcal{G}$-equivariant map.

* The dimension of each orbit $= \dim \mathcal{G}$.

computation

Implicit invariantization ι:

Let $z^1, \ldots, z^m$ be loc. coordinates on $\mathcal{Z}$ and $\mathcal{K}$ be a loc. cross-section.

Functions: $\forall f \in \mathcal{F}(\mathcal{Z}) \quad \exists!$ loc. inv. $\iota f \in \mathcal{F}(\mathcal{Z})$ s. t. $\iota f|_{\mathcal{K}} = f|_{\mathcal{K}}$.

$\{\iota(z^1), \ldots, \iota(z^m)\} \supset$ fundamental set of inv.

If the $\mathcal{G}$-action is locally free then

- **differential forms:** $\forall \Omega \in \Lambda^k \quad \exists!$ loc. inv. $\iota\Omega \in \Lambda^k$. s. t. $\iota\Omega|_{\mathcal{K}} = \Omega|_{\mathcal{K}}$.
  $\varpi = \iota dz^1, \ldots, \varpi_n = \iota dz^m$ is the dual basis of invariant differential 1-forms
- **vector fields:** $\forall$ vector field V on $\mathcal{Z} \quad \exists!$ loc. inv. vector field ιV s. t. $\iota V|_{\mathcal{K}} = V|_{\mathcal{K}}$.

  $\mathcal{D}_1 = \iota\left(\frac{\partial}{\partial z^1}\right), \ldots, \mathcal{D}_n = \iota\left(\frac{\partial}{\partial z^m}\right)$ is a basis of invariant differential operators (non-commutative in general)

computation

Explicit invariantization steps:

1. Write down a system of equations that describes $g \in \mathcal{G}$ which brings an arbitrary point $\mathbf{z} \in \mathcal{Z}$ to the cross-section;

2. Solve the system for the group parameters ($g = \rho(\mathbf{z})$);

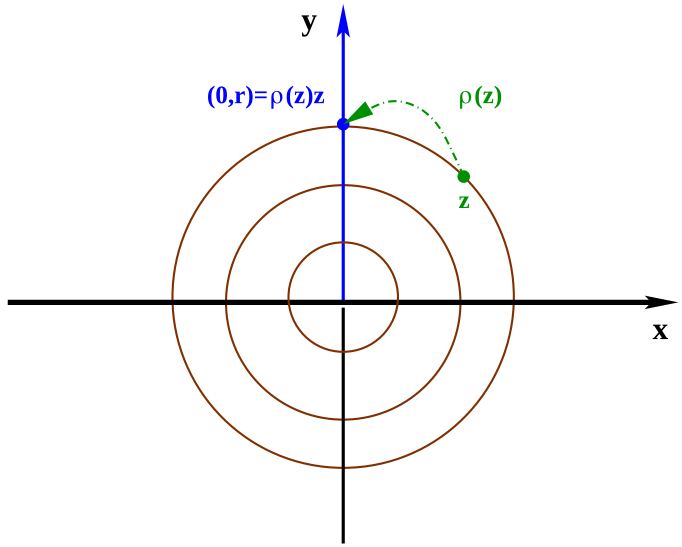
3. Replace g with $\rho(\mathbf{z})$ in the pull-back of a function (or a form) by the action of $g \in \mathcal{G}$.

Constructive idea in the algebraic setting is to replace steps 2 and 3 with elimination of the group parameters (Hubert, Kogan (2007)) .

Example: $SO(\mathbb{R},2) \curvearrowright \mathbb{R}^2 - \{(0,0)\}$:

Action:

$$X = \cos(\phi)x - \sin(\phi)y,$$
$$Y = \sin(\phi)x + \cos(\phi)y.$$



Cross-section:

$$\mathcal{K} = \{(x,y)|x = 0,\ y > 0\}$$

1. Equations: $\cos(\phi)x - \sin(\phi)y = 0, \quad Y = \sin(\phi)x + \cos(\phi)y > 0.$
2. Solution: $\cos\phi = \frac{y}{\sqrt{x^2+y^2}}$, $\sin\phi = \frac{x}{\sqrt{x^2+y^2}}$
3. Substitution:
   - into $Y \Rightarrow r = \sqrt{x^2+y^2}$ - invariant function;
   - into $dX \Rightarrow \varpi_1 = \frac{1}{\sqrt{x_1^2+y^2}}(y\,dx - x\,dy)$
   - into $dY \Rightarrow \varpi_2 = \frac{1}{\sqrt{x^2+y^2}}(x\,dx + y\,dy)$

$SE(2,\mathbb{R}) = SO(2,\mathbb{R}) \ltimes \mathbb{R}^2 \curvearrowright$ **on plane curves:**

$$\boxed{X = \cos(\phi)x - \sin(\phi)y + a, \quad Y = \sin(\phi)x + \cos(\phi)y + b}$$

$$Y_X = \frac{\sin(\phi) + \cos(\phi)y_x}{\cos(\phi) - \sin(\phi)y_x}, \quad Y_{XX} = \frac{y_{xx}}{(\cos(\phi) - \sin(\phi)y_x)^3},$$

$$Y_{XXX} = \frac{(\cos(\phi)-\sin(\phi)y_x)y_{xxx}+3\sin(\phi)y_{xx}^2}{(\cos(\phi)-\sin(\phi)y_x)^5}.$$

cross-section: $\mathcal{K} = \{x = 0,\ y = 0,\ y_x = 0\}$

$$\Downarrow$$

solve $X = 0,\ Y = 0,\ Y_X = 0$: for $a, b, \phi \Rightarrow$ moving frame:

$$\cos\phi = \frac{1}{\sqrt{y_x^2+1}}, \ \sin\phi = -\frac{y_x}{\sqrt{y_x^2+1}}, \quad a = -\frac{x + y_x y}{\sqrt{y_x^2+1}}, \quad b = \frac{y_x x - y}{\sqrt{y_x^2+1}}.$$

Substitute: $\cos\phi = \frac{1}{\sqrt{y_x^2+1}}$, $\sin\phi = -\frac{y_x}{\sqrt{y_x^2+1}}$ into

$$Y_{XX} = \frac{y_{xx}}{(\cos(\phi) - \sin(\phi) y_x)^3} \quad \Rightarrow \quad I_2 = \kappa = \frac{y_{xx}}{(1+y_x^2)^{3/2}}$$

$$Y_{XXX} \quad \Rightarrow \quad I_3 = \kappa_s = \frac{y_{xxx}(1+y_x^2) - 3\, y_x\, y_{xx}^2}{(1+y_x^2)^{5/2}}$$

$$Y_{XXXX} \quad \Rightarrow \quad I_4 = \kappa_{ss} + 3\,\kappa^3$$

$$d\,X = \cos(\phi) dx - \sin(\phi) dy \quad \Rightarrow \quad \varpi = \frac{dx + y_x dy}{\sqrt{1+y_x^2}} = \sqrt{1+y_x^2}\, dx + \frac{y_x}{\sqrt{1+y_x^2}}\theta,$$

where $\theta = dy - y_x\, dx$.

Recursive and inductive variations of a moving frame construction.

(Kogan 2000, 2003)

- Recursive:
  - does not require freeness, but requires a slice - a cross-section with a constant isotropy group;
  - on a jet bundle allows to construct moving frames and invariants order-by-order.

- Inductive:
  - requires splitting of the group into a product of two subgoups $\mathcal{G} = A\,B$ s. t. $A \cap B$ is discrete;
  - invariants and moving frames for A (or B) can be used to construct invariants and a moving frame for $\mathcal{G}$.

$$\Downarrow$$

Relations among the invariants of $\mathcal{G}$ and its subgroups.

Ex.: from the Euclidean to the affine action on the planar curves.

$SA(2,\mathbb{R}) = SL(2,\mathbb{R}) \ltimes \mathbb{R}^2 = B \cdot A$, where $A = SE(2,\mathbb{R})$ and

$$B = \left\{ \begin{pmatrix} \tau & \lambda \\ 0 & \frac{1}{\tau} \end{pmatrix} \right\}$$

Notation: $y_1 = y_x,\ y_2 = y_{xx},\ \ldots$

$\mathcal{K}_A = \{\mathbf{z} \in \mathcal{J}^k | x = 0, y = 0, y_1 = 0\}$ is stable under the B-action.

$\mathcal{K}_B = \{\mathbf{z} \in \mathcal{K}_A | y_2 = 0, y_3 = 1\} \subset \mathcal{K}_A$ is a cross-section to the $\mathcal{SA}(2,\mathbb{R})$-action on the jets of curves.

$$\Downarrow$$

a moving frame for B on $\mathcal{K}_A^4$

$$\Downarrow$$

$$\boxed{\mu = \frac{\kappa(\kappa_{ss} + 3\kappa^3) - \frac{5}{3}\kappa_s^2}{\kappa^{8/3}}, \qquad d\alpha = \kappa^{1/3} ds, \quad \frac{d}{d\alpha} = \frac{1}{\kappa^{1/3}} \frac{d}{ds}}$$

Example: from the affine to the projective action on the planar curves.

$PGL(3,\mathbb{R}) = B \cdot A$, where $A = SL(2,\mathbb{R})$ and $B = \left\{ \begin{pmatrix} 1 & ab & 0 \\ 0 & a & 0 \\ b & c & \frac{1}{a} \end{pmatrix} \right\}$.

$\mathcal{K}_A = \{\mathbf{z} \in J^k | x = 0,\ y = 0,\ y_x = 0,\ y_{xx} = 1, y_{xxx} = 0\}$ is stable under the B-action.

$\mathcal{K}_B = \{\mathbf{z} \in \mathcal{K}_A | y_4 = 0, y_5 = 1, y_6 = 0\} \subset \mathcal{K}_A$ is a cross-section to the $PGL(3,\mathbb{R})$-action on the jets of curves.

$$\Downarrow$$

moving frame for B on $\mathcal{K}_A$

$$\Downarrow$$

$$\boxed{\eta = \frac{-7\mu_{\alpha\alpha}^2 + 6\mu_\alpha\mu_{\alpha\alpha\alpha} - 3\mu\mu_\alpha^2}{6\mu_\alpha^{8/3}}, \quad d\varrho = \mu_\alpha^{1/3} d\alpha, \quad \frac{d}{d\varrho} = \frac{1}{\mu_\alpha^{1/3}} \frac{d}{d\alpha}}$$

computation

Algebraic formulation of the moving frame method.

(Hubert, Kogan 2007)

- applicable to rational actions of algebraic groups
- replaces non-constructive step of solving for group parameters with constructive elimination algorithms
- produces a generating set of rational invariants
- produces a set of algebraic invariants with replacement property, (corresponds to invariantization of coordinate functions in the smooth construction).

Ideals and varieties

Let V be an affine variety, then $\mathbb{K}[V]$ denotes the ring of regular functions on V and $\mathbb{K}(V)$ denotes the field of rational functions on V. For $U \subset V$, $\overline{U}$ denotes Zariski closure of U.

An alg. group $\mathcal{G}$ acts rationally on a variety $\mathcal{Z}$ over a field $\mathbb{K}$, $char\mathbb{K} = 0$.

- Source and target space: $\mathcal{Z} \times \mathcal{Z}$,
- Graph of the action: $\mathcal{O} = \overline{\{(\mathbf{z}, \mathbf{Z}) \subset \mathcal{Z} \times \mathcal{Z} | \exists g \in \mathcal{G} : \mathbf{Z} = g \cdot \mathbf{z}\}} \leftrightarrow$ ideal: $O \subset \mathbb{K}[\mathcal{Z} \times \mathcal{Z}]$ extension: $\boxed{O^e \subset \mathbb{K}(Z)[Z]}$
- Orbit: $\mathcal{O}_{\mathbf{z}} = \overline{\{\mathbf{Z} \in \mathcal{Z} | \exists g \in \mathcal{G} : \mathbf{Z} = g \cdot \mathbf{z}\}} \leftrightarrow$ ideal: $O_{\mathbf{z}} \subset \mathbb{K}[\mathcal{Z}]$

- Cross-section of degree d: an irreducible variety $\mathcal{K} \subset \mathcal{Z}$ s.t. $\mathcal{O}_{\mathbf{z}} \cap \mathcal{K}$ consists of d simple points $\forall \mathbf{z}$ in a dense subset of $\mathcal{Z}$ (transversality cond.)

  Cross-section ideal: K is prime, s.t. $codim K = \max_{\mathbf{z}} \dim \mathcal{O}_{\mathbf{z}} = s$ and $\boxed{I^e = O^e + K \subset \mathbb{K}(\mathcal{Z})[\mathcal{Z}]}$ is radical zero-dimensional (transversality cond.)

- Graph-section: $\mathcal{I} = \{(\mathbf{z}, \mathbf{Z}) \subset \mathcal{Z} \times \mathcal{K} | \exists g \in \mathcal{G} : \mathbf{Z} = g \cdot \mathbf{z}\} \leftrightarrow$ ideal: $I = O + K \subset \mathbb{K}[\mathcal{Z} \times \mathcal{Z}]$

Theorem: Coeff. of a reduced Gröbner basis of either O^e or I^e generate $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$.

Previous work. Rosenlicht (1956): $\forall$ subset set of $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$ that separates orbits generates $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$; coeffs. of Chow form of O^e have this property.

Popov, Vinberg (1989): if coeff. of a generating set of O^e are in $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$, then they generate $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$; $\exists$ such generating set.

Beth, Müller-Quade (1999): rewriting algorithm for linear actions.

Hubert, Kogan (2007) contribution: simple algorithm to compute rational and replacement invariants; $\dim I^e = 0 \Rightarrow$ computational advantage; rewriting algorithms.

Example: $SO(2,\mathbb{R}) \curvearrowright \mathbb{R}^2$.

- group: $G = (\lambda_1^2 + \lambda_2^2 - 1) \subset \mathbb{R}[\lambda_1, \lambda_2]$, $\quad (\lambda_1 = \cos\phi, \lambda_2 = \sin\phi)$
- action: $J = A + G$, where

$$A = (Z_1 - \lambda_1 z_1 - \lambda_2 z_2, \quad Z_2 - \lambda_2 z_1 + \lambda_1 z_2)$$

- graph: $O = J \cap \mathbb{R}[z, Z] = \left\langle Z_1^2 + Z_2^2 - z_1^2 - z_2^2 \right\rangle$.

$$O^e = \left\langle Z_1^2 + Z_2^2 - (z_1^2 + z_2^2) \right\rangle \subset \mathbb{R}(z)[Z].$$

- cross-section: $K = (Z_1)$
- $I^e = O^e + K = \left\langle Z_1, Z_2^2 - (z_1^2 + z_2^2) \right\rangle$
- $\mathbb{R}(\mathcal{Z})^G = \mathbb{R}(z_1^2 + z_2^2)$
- $\overline{\mathbb{R}(\mathcal{Z})^G}$ zeros $\xi^{(\pm)} = (\xi_1^{(\pm)}, \xi_2^{(\pm)}) = (0, \pm\sqrt{z_1^2 + z_2^2})$ of I^e are replacement invariants. (e.g. $z_1^2 + z_2^2 = \left[\xi_1^{(\pm)}\right]^2 + \left[\xi_2^{(\pm)}\right]^2$).

Replacement invariants

$\boxed{I^e = (O^e + K) \subset \mathbb{R}(z)[Z] \text{ radical, zero-dimensional.}}$

Theorem:

- coefficients of a reduced Gröbner Q basis of I^e generate $\mathbb{R}(z)^G$.
- $I^G = I^e \cap \mathbb{R}(z)^G[Z] =< Q >$ is prime
- if c.-s. $\mathcal{K}$ intersects generic orbit at d points then I^G has d zeros of n-tuples $\xi^{(i)} = (\xi_1^{(i)}, \ldots, \xi_n^{(i)}),\ i = 1..d\,,\ \xi_j^{(i)} \in \overline{\mathbb{K}(\mathcal{Z})^{\mathcal{G}}}$.
- Each $\xi^{(i)}$ has replacement property: $F(z_1, \ldots, z_n) \in \mathbb{R}(z)^G \Rightarrow F(z_1, \ldots, z_n) = F(\xi_1^{(i)}, \ldots, \xi_n^{(i)})$

Example: $SE_2(\mathbb{R}) \curvearrowright \mathbb{R}^4$ **(second jet bundle of plane curves).**

- the group and the action $J = G + A$, where:
  $G = (\lambda_1^2 + \lambda_2^2 - 1) \subset \mathbb{R}[\lambda_1, \lambda_2, \lambda_3, \lambda_4], \quad (\lambda_1 = \cos\phi, \lambda_2 = \sin\phi)$
  $$A = \begin{pmatrix} Z_1 - \lambda_1 z_1 - \lambda_2 z_2 + \lambda_3, \quad Z_2 - \lambda_2 z_1 + \lambda_1 z_2 + \lambda_4, \\ Z_3 - \dfrac{\lambda_2 + \lambda_1 z_3}{\lambda_1 - \lambda_2 z_2}, \quad Z_4 - \dfrac{z_4}{(\lambda_1 - \lambda_2 z_2)^3}. \end{pmatrix}$$

- graph: $O = \left\langle \left(1 + z_3^2\right)^3 Z_4^2 - \left(1 + Z_3^2\right)^3 z_4^2 \right\rangle = (G + A) \cap \mathbb{R}[z, Z].$

  $$O^e = \left\langle Z_4^2 - \frac{z_4^2}{(1+z_3^2)^3} Z_3^2 - \frac{z_4^2}{(1+z_3^2)^3} \right\rangle \subset \mathbb{R}(z)[Z].$$

- cross-section: $K = (Z_1, Z_2, Z_3)$

- $I^e = \left\langle Z_1, Z_2, Z_3, Z_4^2 - \frac{z_4^2}{(1+z_3^2)^3} \right\rangle$

- ring of rational invariants: $\mathbb{R}(z)^G = \mathbb{R}\left(\frac{z_4^2}{(1+z_3^2)^3}\right)$

- 2 replacement invariants: $\xi^{(\pm)} = (\xi_1^{(\pm)}, \xi_2^{(\pm)}, \xi_3^{(\pm)}, \xi_4^{(\pm)}) = \left(0, 0, 0, \pm\frac{z_4}{(1+z_3^2)^{3/2}}\right)$

- Replacement illustration: $\frac{z_4}{(1+z_3^2)^{3/2}} = \frac{\xi_4^{(\pm)}}{\left(1+\xi_3^{(\pm)}\right)^{3/2}}$.

THANK YOU!