Generic Galois groups for q-difference equations

Lucia Di Vizio

(joint work with Charlotte Hardouin)

Beijing, October 27th, 2010

$\mathbb{C}$ field of complex numbers,
$q \in \mathbb{C} \smallsetminus \{0,1\}$, $\sigma_q : f(x) \mapsto f(qx)$

q-difference system

$Y(qx) = A(x)Y(x)$ with $A(x) \in Gl_\nu(\mathbb{C}(x))$

q-difference module $\mathcal{M} = (M, \Sigma_q)$ over $\mathbb{C}(x)$

M	$\mathbb{C}(x)$-vector space of dimension ν
$\Sigma_q : M \to M$	σ_q-semilinear bijection

Rmk.
Horizontal vectors for $\Sigma_q \leftrightarrow$ solutions of a q-difference system

If $|q| \neq 1$:

$A(x) \in Gl_\nu(\mathbb{C}(x)) \Rightarrow \exists U \in Gl_\nu(\mathcal{M}er(\mathbb{C}^*))$ s.t. $U(qx) = A(x)U(x)$

$\rightsquigarrow$ $Y(qx) = A(x)Y(x)$ has a fundamental solution in $\mathcal{E}_q(x)(U) \subset \mathcal{M}er(\mathbb{C}^*)$, with $\mathcal{M}er(\mathbb{C}^*)^{\sigma_q} = \mathcal{E}_q$

$\rightsquigarrow$ algebraic group over the algebraic closure over $\mathcal{E}_q$

$\rightsquigarrow$ differential group over the differential closure over $\mathcal{E}_q$ (Hardouin-Singer)

Problem. Large field of definition
Groups difficult to calculate and characterize

$\mathcal{M}_{\mathbb{C}(x)} = (M_{\mathbb{C}(x)}, \Sigma_q) =$ q-difference module/$\mathbb{C}(x)$

$Constr(\mathcal{M}_{\mathbb{C}(x)}) =$ family of q-diff. modules/$\mathbb{C}(x)$ closed w.r.t. algebraic constructions

$\rightsquigarrow Gl(M_{\mathbb{C}(x)})$ naturally acts on $Constr(\mathcal{M}_{\mathbb{C}(x)})$

Generic Galois group

$Gal(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)}) = \{\varphi \in Gl_\nu(M_{\mathbb{C}(x)}) : \varphi$ stabilises every sub-q-difference module in every object of $Constr(\mathcal{M}_{\mathbb{C}(x)})\}$

Theorem.

$\dim_{\mathbb{C}(x)} Gal(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$ is equal to the transcendence degree of the Picard-Vessiot extension over the algebraic closure of the field $\mathcal{E}_q$ of q-elliptic functions.

RMK The weak Picard-Vessiot extension over $\mathcal{E}_q$ is actually enough (*cf.* Chatdzidakis-Hardouin-Singer).

$Constr^{\partial}(\mathcal{M}_{\mathbb{C}(x)})$ = family of q-diff. modules/$\mathbb{C}(x)$ closed w.r.t. constructions of differential algebra

i.e. algebraic constructions plus the prolongation functor F

$F(\mathcal{M}_{\mathbb{C}(x)})$= extension of $\mathcal{M}_{\mathbb{C}(x)}$ by itself

s.t. if $\underline{e}$ is a basis of $\mathcal{M}_{\mathbb{C}(x)}$ with $\Sigma_q \underline{e} = \underline{e}A(x)$, then there exists a basis $(\underline{e}, \underline{e}')$ of $F(\mathcal{M}_{\mathbb{C}(x)})$ such that $\Sigma_q(\underline{e}, \underline{e}') = (\underline{e}, \underline{e}') \begin{pmatrix} A & \partial(A) \\ 0 & A \end{pmatrix}$

$\rightsquigarrow$ $Gl(M_{\mathbb{C}(x)})$ naturally acts on $Constr^{\partial}(\mathcal{M}_{\mathbb{C}(x)})$

Generic differential Galois group

$Gal^{\partial}(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)}) = \{\varphi \in Gl_{\nu}(M_{\mathbb{C}(x)}) \, : \, \varphi$ stabilises every sub-q-difference module in every object of $Constr^{\partial}(\mathcal{M}_{\mathbb{C}(x)})\}$

Theorem.

$\mathrm{diff.dim}_{\mathbb{C}(x)} Gal^{\partial}(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$ is equal to the hypertranscendence degree of the differential Picard-Vessiot extension over the differential closure of the field $\mathcal{E}_q$ of q-elliptic functions.

RMK The weak Picard-Vessiot extension over $\mathcal{E}_q$ is actually enough.

To summarize :

We give up

the Galois correspondence.

We keep

the information on the algebraic and differential relations between the solutions.

We gain

- a smaller field of definition ;
- the intrinsic construction of the generic Galois group : no need of Picard-Vessiot theory or of construction of solutions ;
- an arithmetic description of the Galois group.

$\mathcal{M}_{\mathbb{C}(x)}$ q-difference module/$\mathbb{C}(x)$, with $q \in \mathbb{C} \smallsetminus \{0,1\}$
$\Rightarrow \exists K \subset \mathbb{C}$, finitely generated/$\mathbb{Q}$ and $\mathcal{M}_{K(x)}$ such that

- $\mathcal{M}_{K(x)} \otimes_{K(x)} \mathbb{C}(x) \cong \mathcal{M}_{\mathbb{C}(x)}$
- $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes_{K(x)} \mathbb{C}(x) \cong Gal(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$
- $Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)}) \otimes_{K(x)} \mathbb{C}(x) \cong Gal^{\partial}(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$

We can always work on a finitely generated extension K of $\mathbb{Q}$.

Three cases :

- q root of unity
  (for $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ *cf.* Hendriks, 1996)
- q transcendant
- q algebraic not a root of unity
  (for $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$, with K number field *cf.* DV, 2002)

q root of unity of order κ

$\Rightarrow$ there is only one curvature to take into account : Σ_q^κ

Theorem

- $Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the Zariski closure of Σ_q^κ.
- $Gal^\partial(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the Kolchin closure of Σ_q^κ.

$\exists k \subset K$ such that $K/k(q)$ finite and $k(q)/k$ transcendental

For simplicity, we state the theorem for $K = k(q)$:

ϕ_v= irreducible factor of a cyclotomic polynomial in $k[q]$

Theorem

$Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest <u>algebraic</u> subgroup of $Gl_\nu(M_{K(x)})$ whose reduction modulo ϕ_v contains $\Sigma_q^{\kappa_v} : M_{\mathcal{A}} \otimes \mathcal{O}_K/\phi_v \rightarrow M_{\mathcal{A}} \otimes \mathcal{O}_K/\phi_v$, for almost all v.

$\exists k \subset K$ such that $K/k(q)$ finite and $k(q)/k$ transcendental

For simplicity, we state the theorem for $K = k(q)$:

ϕ_v= irreducible factor of a cyclotomic polynomial in $k[q]$

Theorem

$Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest differential subgroup of $Gl_{\nu}(M_{K(x)})$ whose reduction modulo ϕ_v contains $\Sigma_q^{\kappa_v} : M_{\mathcal{A}} \otimes \mathcal{O}_K/\phi_v \rightarrow M_{\mathcal{A}} \otimes \mathcal{O}_K/\phi_v$, for almost all v.

q algebraic, not a root of unity
$Q=$ algebraic closure of $\mathbb{Q}$ in K, with ring of integers $\mathcal{O}_Q$

For simplicity we take $\mathbb{Q} = Q$
$\kappa_p=$ order as a root of unity of q mod p
$p^{\ell_p}=$ integer power of p s.t. $p^{-\ell_p}(1-q^{\kappa_p}) \in \mathbb{Z}_p^\times$

Theorem

$Gal(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest algebraic subgroup of $Gl_\nu(M_{K(x)})$ whose reduction modulo p^{ℓ_p} contains $\Sigma_q^{\kappa_p} : M_{\mathcal{A}} \otimes \mathcal{O}_{\mathbb{Q}}/p^{\ell_p} \to M_{\mathcal{A}} \otimes \mathcal{O}_{\mathbb{Q}}/p^{\ell_p}$ for almost all places p.

q algebraic, not a root of unity
$Q=$ algebraic closure of $\mathbb{Q}$ in K, with ring of integers $\mathcal{O}_Q$

For simplicity we take $\mathbb{Q} = Q$
$\kappa_p=$ order as a root of unity of q mod p
$p^{\ell_p}=$ integer power of p s.t. $p^{-\ell_p}(1-q^{\kappa_p}) \in \mathbb{Z}_p^{\times}$

Theorem

$Gal^{\partial}(\mathcal{M}_{K(x)}, \eta_{K(x)})$ is the smallest <u>differential</u> subgroup of $Gl_{\nu}(M_{K(x)})$ whose reduction modulo p^{ℓ_p} contains $\Sigma_q^{\kappa_p} : M_{\mathcal{A}} \otimes \mathcal{O}_{\mathbb{Q}}/p^{\ell_p} \rightarrow M_{\mathcal{A}} \otimes \mathcal{O}_{\mathbb{Q}}/p^{\ell_p}$ for almost all places p.

Corollary

$Gal(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$ and $Gal^{\partial}(\mathcal{M}_{\mathbb{C}(x)}, \eta_{\mathbb{C}(x)})$ can always be characterized by curvature means.

To keep in mind for the applications :

$Y(qx) = A(x)Y(x)$, with $A(x) \in Gl_\nu(\mathbb{C}(x))$

$Y(q^k x) = A_k(x)Y(x)$, $k \in \mathbb{Z}$, where

$$\begin{array}{l} A_k(x) = A(q^{k-1}x)\cdots A(qx)A(x)\,,\ k > 0\,, \\ A_0 = id\,, \\ A_{-k}(x) = A_k(q^{-k}x)^{-1}\,,\ k > 0\,. \end{array}$$

The generic (differential) Galois group of $(\mathbb{C}(x)^\nu, X \mapsto A(x)^{-1}\sigma_q(X))$ is the smallest algebraic (differential) Galois group containing

- if q is transcendent, the specialization of $A_{\kappa_\xi}(x)$ at $q = \xi$, for almost all primitive root of unity ξ in an algebraic closure of K, of order κ_ξ.
- if q is algebraic, the reduction of $A_{\kappa_p}(x)$ modulo p^{ℓ_p} for almost all primes p.