Hypergeometric Series Solutions of Linear Operator Equations

Qing-Hu Hou

Center for Combinatorics
Nankai University
P.R. China

Joint work with Yan-Ping Mu, Tianjin University of Technology

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- Background

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- Solving linear operator equations

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- Differential/Difference equations

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- Differential/Difference equations
- Recurrence relations

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- Abramov, Paule and Petkovšek visited formal power series solutions and basic hypergeometric series solutions for q-difference equations.

$$y(x) = \sum_{k=0}^{\infty} c(q^k) x^k.$$

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Given a linear differential/difference equation

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$$\Rightarrow p(x) = \sum_{k=0}^{n} \frac{(-n)_k (n)_k}{(1/2)_k k!} \left(\frac{1-x}{2}\right)^k = {}_2F_1\left(\begin{matrix} -n, n \\ 1/2 \end{matrix} \middle| \frac{1-x}{2}\right).$$

Solving linear operator equations

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Ve solve $x_1, \ldots, x_k$ for explicit integer k. Recall that

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We do not obtain any equation on x_1.

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$$A_2 = n^2 - 4, \quad B_2 = -3(x_1 + x_2), \quad \text{and} \quad x_1 x_2 = 3x_1^2 - 2.$$

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$$b_k(x) = (x + 1)^k \quad \text{or} \quad b_k(x) = (x - 1)^k.$$

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holds if and only if

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- Verify
- solve out A_k and B_k

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$$L(p(x)) = (1 - x^2)p''(x) - xp'(x) + n^2 p(x).$$

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$$A_k = n^2 - k^2 \quad \text{and} \quad B_k = k - 2k^2.$$

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Vhen $\sum_{k=0}^{\infty} c_k b_k(x)$ is a finite summation, it is a real solution.

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$$y(x) = \sum_{k=0}^{\infty} t_k = t_0 \cdot {}_2F_1\left(\begin{matrix} -n, n \\ 1/2 \end{matrix} \middle| \frac{1-x}{2} \right)$$

Differential/Difference equations

cobi polynomials

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$_k$: $(x-1)^k$ or $(x+1)^k$.

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}\,\middle|\,\frac{1-x}{2}\right) \\ &= (-1)^n\frac{(\beta+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \beta+1\end{matrix}\,\middle|\,\frac{1+x}{2}\right). \end{aligned}$$

ahn polynomials

$$(p(x)) = B(x)y(x+1) - (n(n+\alpha+\beta+1) + B(x) + D(x))y(x) + D(x)y(x-1),$$

where $B(x) = (x+\alpha+1)(x-N)$ and $D(x) = x(x-\beta-N-1)$.

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$$(p(x)) = B(x)y(x{+}1) - (n(n{+}\alpha{+}\beta{+}1) + B(x) + D(x))y(x) + D(x)y(x{-}1),$$

where $B(x) = (x + \alpha + 1)(x - N)$ and $D(x) = x(x - \beta - N - 1)$.

$_k$:

$$(x{+}\alpha{+}1)_k\}, \quad \{(-1)^k(-x{+}N{+}\beta{+}1)_k\}, \quad \{(x{-}N)_k\}, \quad \{(-1)^k(-x)_k\}.$$

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$_k$:

$(x+\alpha+1)_k\}, \quad \{(-1)^k(-x+N+\beta+1)_k\}, \quad \{(x-N)_k\}, \quad \{(-1)^k(-x)_k\}.$

$$Q_n(x) = c_n \cdot {}_3F_2\left(\left.\begin{matrix} -n, n+\alpha+\beta+1, x+\alpha+1 \\ \alpha+1, \alpha+\beta+N+2 \end{matrix}\,\right|\, 1\right).$$

on-uniform lattice

$= x(s)$ and L acts on s.

$$(p(s)) = B(s)p(s+1) - (n(n+\alpha+\beta+1) + B(s) + D(s))p(s) + D(s)p(s-1),$$

(s) and $D(s)$ are rational functions.

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$$b_k(s) = (x(s) - x_1)(x(s) - x_2) \cdots (x(s) - x_k)$$

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$$b_k(s) = (x(s) - x_1)(x(s) - x_2) \cdots (x(s) - x_k)$$

or Racah polynomials, we find $x(s) = s(s + \gamma + \delta + 1)$ and $_k = x(s_k)$:

$$_k = k+\alpha-\gamma-\delta-1, \quad s_k = k-\delta-1, \quad s_k = k-1, \quad \text{or} \quad s_k = k+\beta-\gamma-1.$$

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$$_n(x(s)) = {}_4F_3\left(\begin{matrix} -n, n+\alpha+\beta+1, -s+\alpha-\gamma-\delta, s+\alpha+1 \\ \alpha+1, \alpha-\delta+1, \alpha+\beta-\gamma+1 \end{matrix} \middle| 1 \right).$$

Recurrence relations

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$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x).$$

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$$L(p(n)) = \alpha_n r(n) p(n+1) + (\beta_n - x) p(n) + \frac{\gamma_n}{r(n-1)} p(n-1).$$

hoose $r(n)$

a) $r(n) = p(n)/q(n)$, $q(n)$ is a factor of the numerator of α_n and $p(n-1)$ is a factor of the numerator of γ_n.

b) The numerator of $\frac{\gamma_n}{r(n-1)}$ is divisible by n.

c) $L(1)$ is a constant independent of n.

xample

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$$L(p(n)) = \frac{u - 1}{2}(n + 1)p(n + 1) + (n - x)p(n) - \frac{u + 1}{2}np(n - 1),$$

here $u^2 = 1 - 4\alpha$.

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here $u^2 = 1 - 4\alpha$.

$_k(n) = (-1)^k(-n)_k$ and

$$P_n(x) = a_0 \left(\frac{u-1}{2}\right)^n n!\, {}_2F_1 \left(\begin{matrix} -n, (-2x+u-1)/2u \\ 1 \end{matrix} \middle| \frac{2u}{u-1} \right).$$

he Al-Salam-Chihara polynomials

$$xQ_n(x) = Q_{n+1}(x) + (a+b)q^n Q_n(x) + (1-q^n)(1-abq^{n-1})Q_{n-1}(x).$$

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$$L(p(t)) = \frac{t-ab}{2at}p(t/q) + \left(\frac{a+b}{2t} - x\right)p(t) + \frac{a(t-1)}{2t}p(tq).$$

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$_k(t) = (t-1)(t-q^{-1})\cdots(t-q^{-k+1})$ and

$$Q_n(x) = a_0\frac{(ab;q)_n}{a^n}{}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0\end{matrix}\middle| q;q\right), \quad x = \cos\theta.$$

anks

Thanks for attending