Symbolic Integration

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Topics

- Iterated Antiderivative Extensions.
- Picard-Vessiot Extensions with Certain Unipotent Algebraic Groups as Galois Groups.

Introduction and Preliminaries

All fields considered in this talk are of characteristic zero.

We consider only ordinary differential fields (one derivation).

NNC Extensions

Let F be a differential field.

Definition

A differential field extension $E \supset F$ is a *No New Constants* (NNC) extension of F if the constants of E are the same as the constants of F.

Antiderivative Extension

Definition

Let $E \supset F$ be a NNC extension. An element $u \in E$ is an *antiderivative* (of an element) of F if $u' \in F$. A differential field extension $E \supset F$ is an *antiderivative extension* of F if for $i = 1, 2, \cdots, n$, there exists $u_i \in E$ such that $u_i' \in F$ and $E = F(u_1, u_2, \cdots, u_n)$.

Iterated Antiderivative Extension

Definition

A No New Constant extension E of F is called a *Iterated Antiderivative Extension* of F if $E = F(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ and for each i, $\mathbf{x}_i' \in F(\mathbf{x}_1, \cdots, \mathbf{x}_{i-1})$, that is, $\mathbf{x}_i$ is an antiderivative of an element of $F(\mathbf{x}_1, \cdots, \mathbf{x}_{i-1})$.

Basic Theorems

Theorem

Let $E \supset F$ be a NNC extension and let $\mathbf{x} \in E$ with $\mathbf{x}' \in F$. Then either $\mathbf{x}$ is transcendental over F or $\mathbf{x} \in F$.

Theorem

Let $E \supset F$ be a differential field extension. Suppose that there is an $\mathbf{x} \in E - F$, $\mathbf{x}' \in F$ and that $F(\mathbf{x})$ contains a new constant. Then there is an element $y \in F$ such that $y' = \mathbf{x}'$.

Kaplansky, Magid, Rosenlicht-Singer.

Algebraic Dependence of Antiderivatives

Theorem

Let $E \supset F$ be a NNC differential field extension and for $i = 1, 2, \cdots, n$, let $\mathbf{x}_i \in E$ be antiderivatives of F. Then either $\mathbf{x}_i$'s are algebraically independent over F or there is a tuple $(c_1, \cdots, c_n) \in C^n - \{0\}$ such that $\sum_{i=1}^{n} c_i \mathbf{x}_i \in F$.

Ostrowski, Kolchin, Ax, Rosenlicht,...

Structure of Antiderivative Extensions

Corollary

Let $E = F(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_t)$ be an antiderivative extension of F and let K be a differential subfield of E containing F. Then K is an antiderivative extension of F.

Iterated Antiderivative Extension

Definition

We recall that a No New Constant extension E of F is called a *Iterated Antiderivative Extension* of F if $E = F(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ and for each i, $\mathbf{x}_i' \in F(\mathbf{x}_1, \cdots, \mathbf{x}_{i-1})$, that is, $\mathbf{x}_i$ is an antiderivative of an element of $F(\mathbf{x}_1, \cdots, \mathbf{x}_{i-1})$.

Structure Theorem

Theorem

Let F be a differential field with an algebraically closed field of constants C. Let E be an iterated antiderivative extension of F and let K be a differential subfield of E such that $K \supseteq F$. Then K is an iterated antiderivative extension of F.

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Remark

The structure theorem is not true in general if we consider a liouvillian extensions instead of an iterated antiderivative extensions.

Liouvillian Extensions

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Example 1, Rosenlicht, Singer

Consider

$$E := \mathbb{C}(z, e^{z^2}, \int e^{z^2}) \supset K := \mathbb{C}(z, \frac{\int e^{z^2}}{e^{z^2}}) \supset \mathbb{C}(z).$$

Then E is liouvillian over $\mathbb{C}(z)$ but K is not Liouvillian over $\mathbb{C}(z)$.

Example 2, Rosenlicht, Singer

$$E := F(\sqrt{1-z^2}, \sin^{-1} z)$$

is a liouvillian extension (generalized elementary extension) of $\mathbb{C}(z)$ but $K := F(\sqrt{1-z^2}\sin^{-1} z)$ is not liouvillian (generalized elementary) over $\mathbb{C}(z)$.

Algebraically Independent Antiderivatives

Theorem

Let F be a differential that has elements $f_1, f_2, \cdots, f_n \in F$ such that for any $c_1, c_2, \cdots, c_n \in C$ and for any $f \in F$ if $\sum_{i=1}^{n} c_i f_i = f'$ then $c_i = 0$ for all i.

Algebraically Independent Antiderivatives

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Let $E = F(\mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{y}_1, \cdots, \mathbf{y}_m)$ be the field of rational functions with $n + m$ variables. For $i = 1, \cdots, m$, let $P_i, Q_i, R_i \in F[\mathbf{x}_1, \cdots, \mathbf{x}_n]$, $(P_i, Q_i) = (P_i, R_i) = (Q_i, R_i) = 1$ be polynomials satisfying the following condition:

> *R_i is an irreducible polynomial, $R_i \nmid R_j$ if $i \neq j$ and $R_i \nmid Q_j$ for any $1 \leq i, j \leq m$.*

Algebraically Independent Antiderivatives

Theorem

Let F be a differential that has elements $f_1, f_2, \cdots, f_n \in F$ such that for any $c_1, c_2, \cdots, c_n \in C$ and for any $f \in F$ if $\sum_{i=1}^{n} c_i f_i = f'$ then $c_i = 0$ for all i.
Let $E = F(\mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{y}_1, \cdots, \mathbf{y}_m)$ be the field of rational functions with $n + m$ variables. For $i = 1, \cdots, m$, let $P_i, Q_i, R_i \in F[\mathbf{x}_1, \cdots, \mathbf{x}_n]$, $(P_i, Q_i) = (P_i, R_i) = (Q_i, R_i) = 1$ be polynomials satisfying the following condition:

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Extend derivation of F to E by setting $\mathbf{x}_i' = f_i$ and $\mathbf{y}_i' = \frac{P_i}{R_i Q_i}$. Then E is a no new constants extension of F.

Algebraically Independent Antiderivatives

Corollary

Let $y \in E$ be an antiderivative of F. Then $y = \sum_{i=1}^{m} \alpha_i x_i + f$, where $\alpha_i \in C$ and $f \in F$.

Algebraically Independent Antiderivatives

Let

$$E = \mathbb{C}(z)\Big(\ln z, \ln(z+1), \int \frac{1}{\ln(z+1)}, \int \frac{(z+1)^2}{z\ln(z)}\Big).$$

Algebraically Independent Antiderivatives

Let

$$E = \mathbb{C}(z)\Big(\ln z, \ln(z+1), \int \frac{1}{\ln(z+1)}, \int \frac{(z+1)^2}{z\ln(z)}\Big).$$

If $y \in E$ and $y' \in \mathbb{C}(z)$ then $y' = \frac{c_1}{x} + \frac{c_2}{x+1} + f'$, where $c_1, c_2 \in \mathbb{C}$ and $f \in \mathbb{C}(z)$

- Iterated antiderivative extensions need not be Picard-Vessiot Extensions: Consider $\mathbb{C}(z, \ln z)$ over $\mathbb{C}$.

Part II

Picard-Vessiot Extensions with Certain Unipotent Algebraic Groups as Galois Groups.

Extensions with unipotent algebraic groups as Galois Groups

Let F be a differential field with a field of constants C.

Let $\mathcal{U}(n+1, C)$ denote a full unipotent subgroup of the general linear group $GL(n+1, C)$.

We ask the following question: Under what conditions on F does there exist a P-V extension (of F), whose Galois group is Isomorphic to $\mathcal{U}(n+1, C)$

Extensions with unipotent algebraic groups as Galois Groups

Condition on F

(NS) Let F be a differential field F that satisfies the following condition: there are elements $f_1, f_2, \cdots, f_n \in F$ such that for any $c_1, c_2, \cdots, c_n \in C$ and for any $f \in F$ if $\sum_{i=1}^{n} c_i f_i = f'$ then $c_i = 0$ for all i.

Kovacic, Bialynicki-Birula

Extensions with unipotent algebraic groups as Galois Groups

Let $E = F(g)$, where $g := \begin{pmatrix} 1 & \mathbf{x}_{1,1} & \mathbf{x}_{2,1} & \cdots & \mathbf{x}_{n,1} \\ 0 & 1 & \mathbf{x}_{1,2} & \cdots & \mathbf{x}_{n-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \mathbf{x}_{1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$. Extend the derivation of F to E by setting $g' = Ag$, where

$$A := \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 \\ 0 & 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & f_n \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then E is a Picard-Vessiot extension of F with a differential Galois group naturally isomorphic to $\mathcal{U}(n+1, C)$.

Extensions with unipotent algebraic groups as Galois Groups

We can also compute a linear differential operator over F for E.

$$L(Y) := \frac{w(Y, 1, \mathbf{x}_{1,1}, \mathbf{x}_{2,1}, \cdots, \mathbf{x}_{n,1})}{w(1, \mathbf{x}_{1,1}, \mathbf{x}_{2,1}, \cdots, \mathbf{x}_{n,1})}$$

$L^{-1}\{0\} = \mathrm{span}_C \{1, \mathbf{x}_{1,1}, \mathbf{x}_{2,1}, \cdots, \mathbf{x}_{n,1}\}$

Computing a differential equation

Let $E \supseteq F$ be differential field extensions and assume that the field of constants C of F is algebraically closed. Then E is a Picard-Vessiot Extension of F if and only if

1. $E = F\langle V\rangle$, where $V \subset E$ is a finite dimensional $C-$ vector space
2. There is a group G of differential automorphisms of E with $G(V) \subseteq V$ and $E^G = F$.
3. $E \supseteq F$ is a NNC extension.

In particular if the above conditions hold and if $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ is a $C-$basis of V then E is a Picard-Vessiot Extension of F for the differential operator

$$L(Y) := \frac{w(Y, \mathbf{x}_1, \cdots, \mathbf{x}_n)}{w(\mathbf{x}_1, \cdots, \mathbf{x}_n)}.$$

and $L^{-1}\{0\} = V$.

Computing a differential equation for $\mathcal{U}(3, C)$

Let $f_1, f_2 \in F$ be elements satisfying the condition NS.
Outline of the proof:

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Outline of the proof:

Let $E = F(g)$, where $g = \begin{pmatrix} 1 & \mathbf{x}_1 & \mathbf{y} \\ 0 & 1 & \mathbf{x}_2 \\ 0 & 0 & 1 \end{pmatrix}$. Extend the derivation on F to E by setting

$$\begin{pmatrix} 1 & \mathbf{x}_1 & \mathbf{y} \\ 0 & 1 & \mathbf{x}_2 \\ 0 & 0 & 1 \end{pmatrix}' = \begin{pmatrix} 0 & f_1 & 0 \\ 0 & 0 & f_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{x}_1 & \mathbf{y} \\ 0 & 1 & \mathbf{x}_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Computing a differential equation for $\mathcal{U}(3, C)$

Note that $\mathbf{x}_1' = f_1$, $\mathbf{x}_2' = f_2$ and $\mathbf{y}' = f_1\mathbf{x}_2$.

1. The differential field E is a NNC extension of F. (Suppose not. Then by an earlier theorem, there should be a $y \in F(\mathbf{x}_1, \mathbf{x}_2)$ such that $y' = f_1\mathbf{x}_2 + f$ and one can show that there is no such $y \in F(\mathbf{x}_1, \mathbf{x}_2)$.)

Computing a differential equation for $\mathcal{U}(3, C)$

Let G be the differential Galois group of E fixing F and let $\sigma \in G$.

2 Note that $\sigma(\mathbf{x}_1) = \mathbf{x}_1 + \alpha_\sigma$, $\sigma(\mathbf{x}_2) = \mathbf{x}_2 + \beta_\sigma$, where $\alpha_\sigma, \beta_\sigma \in C$. and that

$$\sigma(\mathbf{y})' = \sigma(\mathbf{y}') = \sigma(\mathbf{x}_1'\mathbf{x}_2) = \mathbf{x}_1'\sigma(\mathbf{x}_2) = \mathbf{y}' + \beta_\sigma \mathbf{x}_1' = (\mathbf{y} + \beta_\sigma \mathbf{x}_1)'.$$

Therefore $\sigma(\mathbf{y}) = \mathbf{y} + \beta_\sigma \mathbf{x}_1 + \gamma_\sigma$ for some $\gamma_\sigma \in C$.

Let $V :=\mathrm{span}_C\{1, \mathbf{x}_1, \mathbf{y}\}$ and note that $GV \subseteq V$ and that $F\langle V\rangle = E$.

Computing a differential equation for $\mathcal{U}(3, C)$

Moreover, G maps to $\mathcal{U}(3, C)$ via

$$\sigma \mapsto \begin{pmatrix} 1 & \alpha_\sigma & \gamma_\sigma \\ 0 & 1 & \beta_\sigma \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to the the ordered basis $\{1, \mathbf{x}_1, \mathbf{y}\}$. And conversely, every element in $\mathcal{U}(3, C)$ naturally induces a map on V, which in turn induces a differential automorphism of the differential field E fixing F.

Computing a differential equation for $\mathcal{U}(3, C)$

3 Show that if $u \in E - F$ then the automorphism induced by

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

will not fix u.

Computing a differential equation for $\mathcal{U}(3, C)$

Since $GV = V$ and $E^G = F$, the differential equation

$$L(y) = w(Y, 1, \mathbf{x}_1, \mathbf{y})/w(1, \mathbf{x}_1, \mathbf{y})$$

has coefficients in F. Clearly $L^{-1}\{0\} = V$.

Examples

A computation shows

$$L(Y) = Y''' - \left(\frac{f_2'}{f_2} + \frac{2f_1'}{f_1} \right) Y'' + \left(\frac{f_1' f_2'}{f_1 f_2} + 2\left(\frac{f_1'}{f_1} \right)^2 - \frac{f_1''}{f_1} \right) Y'$$

$F = \mathbb{C}(z), z' = 1$

- Let $c_1, c_2 \in \mathbb{C}$ be distinct complex numbers and let $f_i = \frac{1}{x+c_i}$. Then

$$Y''' - \frac{3z + 2c_2 + c_1}{(z+c_1)(z+c_2)} Y'' + \frac{1}{(z+c_1)(z+c_2)} Y' = 0$$

has differential Galois group $\mathcal{U}(3, C)$.

Solution Space: $\mathrm{span}_C\{1, \ln(z+c_1), \int \frac{\ln(z+c_2)}{z+c_1}\}$.

$F = C(z), z' = 1, \; G \cong \mathcal{U}(4, C)$

let $f_i = \frac{1}{x+c_i}$, where c_i are distinct complex numbers for $i = 1, 2, 3, 4$.

- The differential equation

$$\frac{d^4}{dz^4} + \frac{6z^2 + (3c_1 + 4c_2 + 5c_3)z + c_2c_1 + 2c_3c_1 + 3c_3c_2}{(z+c_1)(z+c_2)(z+c_3)}\frac{d^3}{dz^3} + \frac{7z + c_1 + 2c_2 + 4c_3}{(z+c_1)(z+c_2)(z+c_3)}\frac{d^2}{dz^2} + \frac{1}{(z+c_1)(z+c_2)(z+c_3)}\frac{d}{dz}.$$

Solution Space:

$$\text{span}_C\left\{1, \ln(z+c_1), \int \frac{\ln(z+c_2)}{z+c_1}, \int \frac{\int \frac{\ln(z+c_3)}{z+c_2}}{z+c_1}\right\}.$$