Methods of Solving Flag Partial Differential Equations

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Plan of Talk

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A *partial differential equation of flag type* is the linear differential equation of the form:

$$(d_1 + f_1 d_2 + f_2 d_3 + \cdots + f_{n-1} d_n)(u) = 0,$$

where $d_1, d_2, \ldots, d_n$ are certain commuting locally nilpotent differential operators on the polynomial algebra $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and $f_1, \ldots, f_{n-1}$ are polynomials satisfying

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Flag partial differential equations naturally appear in geometry, physics and the representation theory of Lie algebras (groups). Many variable-coefficient (generalized) Laplace equations, wave equations, Klein-Gordon equations, Helmholtz equations are of this type. Solving such equations is also important in finding invariant solutions of nonlinear partial differential equations.

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Lemma 1. *Let $\mathcal{B}$ be a commutative associative algebra and let $\mathcal{A}$ be a free $\mathcal{B}$-module generated by a filtrated subspace $V = \bigcup_{r=0}^{\infty} V_r$ (i.e., $V_r \subset V_{r+1}$). Let T_1 be a linear operator on $\mathcal{B} \oplus \mathcal{A}$ with a right inverse T_1^- such that*

$$T_1(\mathcal{B}, \mathcal{A}),\ T_1^-(\mathcal{B}, \mathcal{A}) \subset (\mathcal{B}, \mathcal{A}),$$

$$T_1(\eta_1\eta_2) = T_1(\eta_1)\eta_2,\quad T_1^-(\eta_1\eta_2) = T_1^-(\eta_1)\eta_2$$

for $\eta_1 \in \mathcal{B},\ \eta_2 \in V$.

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Then we have

$$\begin{aligned} & \{g \in \mathcal{A} \mid (T_1 + T_2)(g) = 0\} \\ = \; & Span\{\sum_{i=0}^{\infty}(-T_1^- T_2)^i(hg) \mid g \in V,\; h \in \mathcal{B};\; T_1(h) = 0\}, \end{aligned}$$

where the summation is finite under our assumption. Moreover, the operator $\sum_{i=0}^{\infty}(-T_1^- T_2)^i T_1^-$ *is a right inverse of* $T_1 + T_2$.

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We remark that the above operator T_1 and T_2 may not commute.

Take the notion

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Define

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \ \text{ for } \ \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n.$$

Moreover, we denote

$$\epsilon_i = (0, ..., 0, \overset{i}{1}, 0, ..., 0) \in \mathbb{N}^n.$$

For each $i \in \overline{1, n}$, we define the linear operator $\int_{(x_i)}$ on $\mathcal{A}$ by:

$$\int_{(x_i)} (x^\alpha) = \frac{x^{\alpha + \epsilon_i}}{\alpha_i + 1} \ \text{ for } \ \alpha \in \mathbb{N}^n.$$

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Furthermore, we let

$$\int_{(x_i)}^{(0)} = 1, \quad \int_{(x_i)}^{(m)} = \overbrace{\int_{(x_i)} \cdots \int_{(x_i)}}^{m}$$

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$$\partial^{\alpha} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}, \quad \int^{(\alpha)} = \int_{(x_1)}^{(\alpha_1)} \int_{(x_2)}^{(\alpha_2)} \cdots \int_{(x_n)}^{(\alpha_n)} \quad \text{for} \quad \alpha \in \mathbb{N}^n.$$

Obviously, $\int^{(\alpha)}$ is a right inverse of ∂^{α} for $\alpha \in \mathbb{N}^n$. We remark that $\int^{(\alpha)} \partial^{\alpha} \neq 1$ if $\alpha \neq 0$ due to $\partial^{\alpha}(1) = 0$.

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Consider the wave equation in Riemannian space with a nontrivial conformal group:

$$u_{tt} - u_{x_1x_1} - \sum_{i,j=2}^{n} g_{i,j}(x_1 - t)u_{x_ix_j} = 0, \qquad (*)$$

where we assume that $g_{i,j}(z)$ are one-variable polynomials. Change variables:

$$z_0 = x_1 + t, \qquad z_1 = x_1 - t.$$

Then

$$\partial_t^2 = (\partial_{z_0} - \partial_{z_1})^2, \;\; \partial_{x_1}^2 = (\partial_{z_0} + \partial_{z_1})^2.$$

So the equation $(*)$ changes to:

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$$\mathcal{B} = \mathbb{F}[z_0, z_1], \quad V = \mathbb{F}[x_2, ..., x_n], \quad V_r = \{f \in V \mid \deg f \leq r\}.$$

Then the conditions in Lemma 1 hold. Thus we have:

Theorem 2. *The space of all polynomial solutions for the equation* $(*)$ *is:*

$$Span\{\sum_{m=0}^{\infty}(-2)^{-m}(\sum_{i,j=2}^{n}\int_{(z_0)}\int_{(z_1)}g_{i,j}(z_1)\partial_{x_i}\partial_{x_j})^m(f_0g_0+f_1g_1)$$
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Let $m_1, m_2, ..., m_n$ be positive integers. According to Lemma 1, the set

$$\{\sum_{k_2,...,k_n=0}^{\infty}(-1)^{k_2+\cdots+k_n}\binom{k_2+\cdots+k_k}{k_2,...,k_n}\int_{(x_1)}^{((k_2+\cdots+k_n)m_1)}(x_1^{\ell_1})$$
$$\times\partial_{x_2}^{k_2 m_2}(x_2^{\ell_2})\cdots\partial_{x_n}^{k_n m_n}(x_n^{\ell_n}) \mid \ell_1\in\overline{0,m_1-1},\ \ell_2,...,\ell_n\in\mathbb{N}\}$$

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Lemma 1 indeed gives an algorithm of finding polynomial solutions for more general equations.

Let

$$f_i \in \mathbb{R}[x_1, ..., x_i] \qquad \text{for} \quad i \in \overline{1, n-1}.$$

Consider the equation:

$$(\partial_{x_1}^{m_1} + f_1 \partial_{x_2}^{m_2} + \cdots + f_{n-1} \partial_{x_n}^{m_n})(u) = 0.$$

Denote

$$d_1 = \partial_{x_1}^{m_1}, \quad d_r = \partial_{x_1}^{m_1} + f_1 \partial_{x_2}^{m_2} + \cdots + f_{r-1} \partial_{x_r}^{m_r} \qquad \text{for} \quad r \in \overline{2, n}.$$

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We will apply Lemma 1 with $T_1 = d_r,\ T_2 = \sum_{i=r}^{n-1} f_i \partial_{x_{i+1}}^{m_{i+1}}$ inductively. Take a right inverse $d_1^- = \int_{(x_1)}^{(m_1)}$. Suppose that we have found a right inverse d_s^- of d_s for some $s \in \overline{1, n-1}$ such that

$$x_i d_s^- = d_s^- x_i,\ \ \partial_{x_i} d_s^- = d_s^- \partial_{x_i} \qquad \text{for}\ \ i \in \overline{s+1, n} \qquad (**)$$

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$$d_{s+1}^{-}=\sum_{i=0}^{\infty}(-d_s^{-}f_s)^i d_s^{-}\partial_{x_{s+1}}^{im_{s+1}}$$

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By induction, we have found a right inverse d_s^{-} of d_s such that $(**)$ holds for each $s\in\overline{1,n}$.

We set

$$\mathcal{S}_r = \{g \in \mathbb{R}[x_1, ..., x_r] \mid d_r(g) = 0\} \qquad \text{for} \;\; r \in \overline{1, k}.$$

Then

$$\mathcal{S}_1 = \sum_{i=0}^{m_1 - 1} \mathbb{R} x_1^i.$$

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Given $h \in \mathcal{S}_r$ and $\ell \in \mathbb{N}$, we define

$$\sigma_{r+1,\ell}(h) = \sum_{i=0}^{\infty} (-d_r^- f_r)^i (h) \partial_{x_{r+1}}^{im_{r+1}} (x_{r+1}^{\ell}),$$

which is actually a finite summation. Lemma 1 says

$$\mathcal{S}_{r+1} = \sum_{\ell=0}^{\infty} \sigma_{r+1,\ell}(\mathcal{S}_r).$$

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forms a basis of the polynomial solution space $\mathcal{S}_n$ of the partial differential equation:

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Evolution Equations

First we want to solve the following evolution partial differential equation:

$$u_t = (\partial_{x_1} + x_1^{m_1}\partial_{x_2} + x_2^{m_2}\partial_{x_3} + \cdots + x_{n-1}^{m_{n-1}}\partial_{x_n})(u)$$

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Theoretically, the solution is

$$u = e^{t(\partial_{x_1}+\sum_{r=1}^{n-1} x_r^{m_r}\partial_{x_{r+1}})}(f).$$

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For convenience, we denote $m_0 = 1$ and $x_0 = 1$. Set

$$D_i = t\sum_{r=0}^{i-1} x_i^{m_i}\partial_{x_{i+1}} \qquad \text{for} \quad i \in \overline{1,n}.$$

Denote

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Thus

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In our special case, the Campbell-Hausdorff formula becomes

$$\ln e^A e^B = A + B + \sum_{r=1}^{\infty} a_r (\mathrm{ad} A)^r (B), \qquad a_r \in \mathbb{R},$$

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$$\vartheta(x) = \frac{1-e^{-x}}{x} = \int_{-1}^{0} e^{yx} dy = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i!} x^{i-1}.$$

After a long calculation, we obtain

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Set

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By induction, we get

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Our final solution is

$$u = f(x_1 + \eta_1(t), x_2 + \eta_2(t), ..., x_n + \eta_n(t)).$$

Indeed we have solve more general equations associated with weighted root trees in graph theory.

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Indeed we have solve more general equations associated with weighted root trees in graph theory.

Given a continuous function $f(x_1, x_2, ..., x_n)$ on the region:

$$-a_i \leq x_i \leq a_i, \qquad 0 < a_i \in \mathbb{R}, \qquad \text{for } \ i \in \overline{1,n}.$$

We want to solve the differential equation:

$$u_t = (\partial_{x_1}^{m_1} + x_1\partial_{x_2}^{m_2} + x_2\partial_{x_3}^{m_3} + \cdots + x_{n-1}\partial_{x_n}^{m_n})(u)$$

subject to the initial condition:

$$u(0, x_1, ..., x_n) = f(x_1, x_2, ..., x_n) \qquad \text{for } \ x_i \in [-a_i, a_i].$$

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Denote

$$D(t) = t(\partial_{x_1}^{m_1} + x_1\partial_{x_2}^{m_2} + x_2\partial_{x_3}^{m_3} + \cdots + x_{n-1}\partial_{x_n}^{m_n}).$$

Define

$$\xi_1(t, \partial_{x_1}, ..., \partial_{x_n}) = \int_0^t (\partial_{x_1} + \int_0^{y_1} (\partial_{x_2} + ... + \int_0^{y_{n-2}} (\partial_{x_{n-1}}$$
$$+ y_{n-1}\partial_{x_n}^{m_n})^{m_{n-1}} dy_{n-1} ...)^{m_2} dy_2)^{m_1} dy_1,$$

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Dual arguments show

$$e^{D(t)} = e^{\xi_n(t,\partial_{x_1},\ldots,\partial_{x_n})} e^{\xi_{n-1}(t,\partial_{x_1},\ldots,\partial_{x_n})} \ldots e^{\xi_1(t,\partial_{x_1},\ldots,\partial_{x_n})}.$$

For convenience, we denote

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Set

$$e^{2\pi(\vec{k}^\dagger\cdot\vec{x})\sqrt{-1}} = e^{\sum_{r=1}^n 2\pi k_r^\dagger x_r\sqrt{-1}}.$$

Define

$$\phi_{\vec{k}}(t, x_1, ..., x_n) = \frac{1}{2}[(\prod_{i=1}^n e^{\xi_i(t, 2\pi k_1^\dagger\sqrt{-1}, ..., 2\pi k_n^\dagger\sqrt{-1})})e^{2\pi(\vec{k}^\dagger\cdot\vec{x})\sqrt{-1}}$$

$$+(\prod_{i=1}^n e^{\xi_i(t, -2\pi k_1^\dagger\sqrt{-1}, ..., -2\pi k_n^\dagger\sqrt{-1})})e^{-2\pi(\vec{k}^\dagger\cdot\vec{x})\sqrt{-1}}]$$

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Then

$$\phi_{\vec{k}}(0,x_1,...,x_n)=\cos 2\pi(\vec{k}^\dagger\cdot\vec{x}),$$

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We define $0 \prec \vec{k}$ if its first nonzero coordinate is a positive integer.

The solution of our second problem is

$$u = \sum_{0 \preceq \vec{k} \in \mathbb{Z}^n} (b_{\vec{k}} \phi_{\vec{k}}(t, x_1, ..., x_n) + c_{\vec{k}} \psi_{\vec{k}}(t, x_1, ..., x_n))$$

with

$$b_{\vec{k}} = \frac{1}{2^{n-1} a_1 a_2 \cdots a_n} \int_{-a_1}^{a_1} \cdots \int_{-a_n}^{a_n} f(x_1, ..., x_n) \cos 2\pi (\vec{k}^{\dagger} \cdot \vec{x}) dx_n \cdots dx_1$$

and

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Constant-Coefficient PDEs

Let m and $n > 1$ be positive integers and let

$$f_i(\partial_{x_2},...,\partial_{x_n}) \in \mathbb{R}[\partial_{x_2},...,\partial_{x_n}] \qquad \text{for} \quad i \in \overline{1,m}.$$

We want to solve the equation:

$$(\partial_{x_1}^m - \sum_{r=1}^{m} \partial_{x_1}^{m-i} f_i(\partial_{x_2},...,\partial_{x_n}))(u) = 0$$

with $x_1 \in \mathbb{R}$ and $x_r \in [-a_r, a_r]$ for $r \in \overline{2,n}$, subject to the condition

$$\partial_{x_1}^s(u)(0,x_2,...,x_n) = g_s(x_2,...,x_n) \qquad \text{for} \quad s \in \overline{0,m-1},$$

where $a_2,...,a_n$ are positive real numbers and $g_0,...,g_{m-1}$ are continuous functions.

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$$\frac{1}{r!}\sum_{i_1,\ldots,i_m=0}^{\infty}\binom{i_1+\cdots+i_m}{i_1,\ldots,i_m}\int_{(x_1)}^{(\sum_{s=1}^m si_s)}(x_1^r)$$
$$\times\Big(\prod_{p=1}^{m} f_p(\partial_{x_2},\ldots,\partial_{x_n})^{i_p}\Big)\big(e^{2\pi(\vec{k}^\dagger\cdot\vec{x})\sqrt{-1}}\big)$$
$$=\sum_{i_1,\ldots,i_m=0}^{\infty}\binom{i_1+\cdots+i_m}{i_1,\ldots,i_m}\frac{x_1^{r+\sum_{s=1}^m si_s}}{(r+\sum_{s=1}^m si_s)!}$$
$$\times\left[\prod_{p=1}^{m} f_p(2k_2^\dagger\pi\sqrt{-1},\ldots,2k_n^\dagger\pi\sqrt{-1})^{i_p}\right]e^{2\pi(\vec{k}^\dagger\cdot\vec{x})\sqrt{-1}}$$

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$$\times\left[\prod_{p=1}^m f_p(2k_2^\dagger\pi\sqrt{-1},\ldots,2k_n^\dagger\pi\sqrt{-1})^{i_p}\right]e^{2\pi(\vec{k}^\dagger\cdot\vec{x})\sqrt{-1}}$$

is a complex solution of the equation for any $\vec{k} \in \mathbb{Z}^{n-1}$.

We write

$$\sum_{i_1,\ldots,i_m=0}^{\infty}\binom{i_1+\cdots+i_m}{i_1,\ldots,i_m} \times \frac{x_1^r\prod_{p=1}^m(x_1^p f_p(2k_2^\dagger\pi\sqrt{-1},\ldots,2k_n^\dagger\pi\sqrt{-1}))^{i_p}}{(r+\sum_{s=1}^m si_s)!} = \phi_r(x_1,\vec{k})+\psi_r(x_1,\vec{k})\sqrt{-1},$$

where $\phi_r(x_1,\vec{k})$ and $\psi_r(x_1,\vec{k})$ are real functions. Moreover,

$$\partial_{x_1}^s(\phi_r)(0,\vec{k})=\delta_{r,s},\;\; \partial_{x_1}^s(\psi_r)(0,\vec{k})=0 \qquad \text{for}\;\; s\in\overline{0,r}.$$

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The solution of our initial-value problem is:

$$u = \sum_{r=0}^{m-1} \sum_{\vec{0}\preceq\vec{k}\in\mathbb{Z}^{n-1}} [b_r(\vec{k})(\phi_r(x_1,\vec{k}^\dagger)\cos 2\pi(\vec{k}^\dagger\cdot\vec{x})$$
$$-\psi_r(x_1,\vec{k}^\dagger)\sin 2\pi(\vec{k}^\dagger\cdot\vec{x}))$$
$$+c_r(\vec{k})(\phi_r(x_1,\vec{k}^\dagger)\sin 2\pi(\vec{k}^\dagger\cdot\vec{x})$$
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with

$$b_r(\vec{k}) = \frac{1}{2^{n-2}a_2\cdots a_n}\int_{-a_2}^{a_2}\cdots\int_{-a_n}^{a_n} g_r(x_2,...,x_n) \times \cos 2\pi(\vec{k}^{\dagger}\cdot\vec{x})\,dx_n\cdots dx_2 - \sum_{s=0}^{r-1}(b_s(\vec{k})\partial_{x_1}^{r}(\phi_s)(0,\vec{k}) + c_s(\vec{k})\partial_{x_1}^{r}(\psi_s)(0,\vec{k}))$$

and

$$c_r(\vec{k}) = \frac{1}{2^{n-2}a_2\cdots a_n}\int_{-a_2}^{a_2}\cdots\int_{-a_n}^{a_n} g_r(x_2,...,x_n) \times \sin 2\pi(\vec{k}^{\dagger}\cdot\vec{x})\,dx_n\cdots dx_2 - \sum_{s=0}^{r-1}(c_s(\vec{k})\partial_{x_1}^{r}(\phi_s)(0,\vec{k}) - b_s(\vec{k})\partial_{x_1}^{r}(\psi_s)(0,\vec{k})).$$

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Remark . If we take $f_i = b_i$ with $i \in \overline{1, m}$ to be constant functions, we get m fundamental solutions

$$\varphi_r(x) = \sum_{i_1,\ldots,i_m=0}^{\infty} \binom{i_1+\cdots+i_m}{i_1,\ldots,i_m} \frac{x^r \prod_{p=1}^m (b_p x^p)^{i_p}}{(r+\sum_{s=1}^m s i_s)!}, \qquad r \in \overline{0, m-1}$$

of the constant-coefficient ordinary differential equation

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Given the initial conditions:

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we define $a_0 = c_0$ and

$$a_r = c_r - \sum_{s=0}^{r-1} \sum_{i_1,\dots,i_{r-s}\in\mathbb{N};\ \sum_{p=1}^{r} p i_p = r-s} \binom{r-s}{i_1,\dots,i_{r-s}} a_s b_1^{i_1} \cdots b_{r-s}^{i_{r-s}}$$

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From the above results, it seems that the following functions

$$\mathcal{Y}_r(y_1,...,y_m) = \sum_{i_1,...,i_m=0}^{\infty} \binom{i_1+\cdots+i_m}{i_1,...,i_m} \frac{y_1^{i_1}y_2^{i_2}\cdots y_m^{i_m}}{(r+\sum_{s=1}^m si_s)!} \qquad \text{for} \quad r\in\mathbb{N}$$

are important natural functions. Indeed,

$$\mathcal{Y}_1(x)=e^x,\;\; \mathcal{Y}_0(0,-x)=\cos x,\;\; \mathcal{Y}_1(0,-x)=\frac{\sin x}{x},$$

$$\varphi_r(x)=x^r\mathcal{Y}_r(b_1x,b_2x^2,...,b_mx^m)$$

and

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Thank You!