Differential universes of control systems on time scales

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Outline

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- Calculus on time scales
- Control systems on time scales
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Universes

Theory of universes was developed by Joseph Johnson in

J. Johnson, A generalized global differential calculus I, *Cahiers Top. et Geom. Diff.* XXVII(1986)

Function universes were first applied to control theory in

Z. Bartosiewicz, J. Johnson, Systems on universe spaces. *Acta Applicandae Mathematicae*, 34 (1994)

Function universe

Let $\mathcal{F}_n$ be a family of real-valued functions defined on open subsets of $\mathbb{R}^n$ and let $\mathcal{F}$ be the disjoint union of all $\mathcal{F}_n$ for $n \in \mathbb{N}$.

Let X be an arbitrary set. A set $\mathcal{U}$ of real-valued partially defined functions on X is called a *(function) $\mathcal{F}$-universe on X* (or just *universe* if $\mathcal{F}$ and X are fixed), if

- $\mathcal{U}$ contains the global 0 function,
- $\mathcal{U}$ is closed with respect to amalgamation (i.e. glueing up functions that agree on the common domain)
- $\mathcal{U}$ is closed with respect to substitutions to functions from $\mathcal{F}$.

If $\mathcal{F}$ consists of all analytic functions of finitely many variables, then $\mathcal{U}$ is an *analytic (C^ω) universe*.

Morphisms and derivations of function universes

A *morphism* of two $\mathcal{F}$-universes $\mathcal{U}_1$ on X_1 and $\mathcal{U}_2$ on X_2 is a map $\tau : \mathcal{U}_1 \to \mathcal{U}_2$ that transfers the global 0 function on X_1 to the global 0 function on X_2 and commutes with substitutions and amalgamation. A bijective morphism is an *isomorphism*.

Let $\mathcal{U}$ be an analytic universe and $\sigma : \mathcal{U} \to \mathcal{U}$ be a morphism, $\sigma \neq \mathrm{id}$. A map $\Delta : \mathcal{U} \to \mathcal{U}$ is a *σ-derivation* of $\mathcal{U}$ if there is $\mu > 0$ such that $\sigma = \mathrm{id} + \mu\Delta$ (thus $\Delta = (\sigma - \mathrm{id})/\mu$).
We extend this definition to $\sigma = \mathrm{id}$ and $\mu = 0$ adding the standard requirement (the chain rule)

$$\Delta(F(\varphi_1, \ldots, \varphi_n)) = \sum_{k=1}^{n} \frac{\partial F}{\partial x_k}(\varphi_1, \ldots, \varphi_n)\Delta(\varphi_k).$$

Let $\varphi^\sigma := \sigma(\varphi)$ and $\varphi^\Delta := \Delta(\varphi)$.

Differential universes

A *(skew) differential universe* is a universe $\mathcal{U}$ together with a σ-derivation Δ (for some σ). Differential universes $(\mathcal{U}_1, \Delta_1)$ and $(\mathcal{U}_2, \Delta_2)$, corresponding to the same μ, are *isomorphic*, if there is an isomorphism $\tau : \mathcal{U}_1 \to \mathcal{U}_2$ such that $\tau \circ \Delta_1 = \Delta_2 \circ \tau$.

Proposition

Let $F \in \mathcal{F}_n$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{U}$. If F is of class C^1 then

$$F(\varphi_1, \ldots, \varphi_n)^\Delta = \int_0^1 \sum_{k=1}^n \frac{\partial F}{\partial x_k}(\varphi_1 + s\mu\varphi_1^\Delta, \ldots, \varphi_n + s\mu\varphi_n^\Delta)\varphi_k^\Delta ds$$

for the σ-derivation Δ corresponding to μ.

Corollary

If Δ is a σ-derivation then for $\varphi, \psi \in \mathcal{U}$

$$(\varphi\psi)^\Delta = \varphi^\sigma\psi^\Delta + \varphi^\Delta\psi = \varphi\psi^\Delta + \varphi^\Delta\psi^\sigma = \varphi\psi^\Delta + \varphi^\Delta\psi + \mu\varphi^\Delta\psi^\Delta.$$

Calculus on time scales

Calculus on time scales was developed by Stefan Hilger in his Ph.D. thesis. It unifies differential calculus and calculus of finite differences.

Main references:

- S. Hilger, Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph.D. thesis, Universität Würzburg, 1988
- M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, Birkhauser, Boston 2001

Time scales

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers.
Examples: $\mathbb{R}$, $h\mathbb{Z} = \{nh : n \in \mathbb{Z}\}$ $(h > 0)$, $q^{\mathbb{N}} = \{q^n : n \in \mathbb{N}\}$ $(q > 1)$.
For a time scale $\mathbb{T}$ we define:
the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by
$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, if $\sup \mathbb{T} = +\infty$, and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, if $\sup \mathbb{T}$ is finite;
the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$.

Delta derivative

If $\mathbb{T}$ has an isolated maximum M, then we set $\mathbb{T}^\kappa := \mathbb{T} \setminus \{M\}$. Otherwise $\mathbb{T}^\kappa := \mathbb{T}$.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. *Delta derivative* of f at t, denoted by $f^\Delta(t)$, is the real number with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ such that

$$|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$.
We say that f is *delta differentiable* on $\mathbb{T}$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Examples

For $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = f'(t)$.
For $\mathbb{T} = h\mathbb{Z}$, $f^\Delta(t) = \frac{f(t+h)-f(t)}{h}$.
For $\mathbb{T} = q^{\mathbb{N}}$, $f^\Delta(t) = \frac{f(tq)-f(t)}{t(q-1)}$.

Basic properties:

$(af + bg)^{\Delta} = af^{\Delta} + bg^{\Delta}$ for $a, b \in \mathbb{R}$
$(fg)^{\Delta} = f^{\Delta}g^{\sigma} + fg^{\Delta}$, where $g^{\sigma} = g \circ \sigma$.

Example. For $f(t) = t^2$, $f^{\Delta}(t) = t + \sigma(t)$.

Chain rule

Let $g : \mathbb{T} \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$. If g is delta differentiable and f is differentiable, then $F = f \circ g$ is delta differentiable and

$$F^{\Delta}(t) = \int_0^1 f'(g(t) + \theta\mu(t)g^{\Delta}(t))d\theta \cdot g^{\Delta}(t).$$

If $f : \mathbb{T} \to \mathbb{R}$, then $f^{[k]}$ denotes the delta derivative of order k.

Control systems on time scales

Let $\mathbb{T}$ be a homogeneous time scale, i.e. $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mu\mathbb{Z}$ for $\mu > 0$.
Consider the control system with output

$$\Sigma : \; x^{\Delta} = f(x, u), \; y = h(x),$$

where $x = x(t) \in \mathbb{R}^n$ is the state, $u = u(t) \in \mathbb{R}^m$ is the control (input) and $y = y(t) \in \mathbb{R}^p$ is the output (observed variable). The maps f and h are analytic.
A triple (x, u, y) defined on some $(a, b) \cap \mathbb{T}$ and satisfying Σ is called a *trajectory of Σ*. Its projection onto (u, y) is an *external trajectory of Σ*.

Dynamic equivalence

Two analytic control systems with output, on a time scale $\mathbb{T}$,

$$\Sigma: \; x^{\Delta} = f(x,u), \; y = h(x), \quad \tilde{\Sigma}: \; \tilde{x}^{\Delta} = \tilde{f}(\tilde{x},\tilde{u}), \; \tilde{y} = \tilde{h}(\tilde{x})$$

are *externally dynamically equivalent*, if there exist dynamic transformations

$$y = \phi(\tilde{y},\dots,\tilde{y}^{[k]},\tilde{u},\dots,\tilde{u}^{[k]}), \; u = \psi(\tilde{y},\dots,\tilde{y}^{[k]},\tilde{u},\dots,\tilde{u}^{[k]})$$

and

$$\tilde{y} = \tilde{\phi}(y,\dots,y^{[k]},u,\dots,u^{[k]}), \; \tilde{u} = \tilde{\psi}(y,\dots,y^{[k]},u,\dots,u^{[k]})$$

that transform external trajectories $(\tilde{y},\tilde{u})$ of $\tilde{\Sigma}$ onto external trajectories (y,u) of Σ and vice versa, and are mutually inverse on trajectories.

Differential universe of the system

Let $\mathcal{U}$ denote the C^ω-universe of all analytic partially defined functions depending on finitely many variables from the set $\{x_i, i = 1, \ldots, n,\ u_j^{[k]}, j = 1, \ldots, m; k \geq 0\}$. The map

$$\sigma_\Sigma(\varphi)(x, u^{[0]}, u^{[1]}, \ldots) := \varphi(x + \mu f(x, u^{[0]}), u^{[0]} + \mu u^{[1]}, \ldots)$$

is a morphism of $\mathcal{U}$ and the map Δ_Σ given by

$$\Delta_\Sigma(\varphi)(x, u^{[0]}, u^{[1]}, \ldots) :=$$
$$\int_0^1 \left[\frac{\partial \varphi}{\partial x}(x + s\mu f(x, u^{[0]}), u^{[0]} + s\mu u^{[1]}, \ldots) f(x, u^{[0]}) \right.$$
$$\left. + \sum_{k=0}^{\infty} \frac{\partial \varphi}{\partial u^{[k]}}(x + s\mu f(x, u^{[0]}), u^{[0]} + s\mu u^{[1]}, \ldots) u^{[k+1]} \right] ds$$

is a σ_Σ-derivation of $\mathcal{U}$.

Differential universe of the system

Let $\mathcal{U}_\Sigma$ be the smallest C^ω-universe contained in $\mathcal{U}$, containing h_j, $j = 1, \ldots, r$, (the components of h) and u_i, $i = 1, \ldots, m$, (the components of u) and invariant with respect to the derivation Δ_Σ. Then $(\mathcal{U}_\Sigma, \Delta_\Sigma)$ is called the *differential universe of the system Σ*.

The system Σ is *uniformly observable* if $x_i \in \mathcal{U}_\Sigma$ for $i = 1, \dots, n$. This property means that locally we can express each x_i as a composition of some analytic analytic function with a finite number of Δ_Σ derivatives of the output function h and the control variable u.

Example. Consider the system

$$\Sigma : \quad x^\Delta = f(x,u) = u, \quad y = h(x) = \sin x.$$

Then $g(x,u) := (\Delta_\Sigma h)(x,u) =$

$$= \int_0^1 \cos(x + \mu s u) \cdot u ds = \begin{cases} u \cos x & \text{if } \mu = 0 \\ u \frac{\sin(x+\mu u) - \sin x}{\mu} & \text{if } \mu > 0. \end{cases}$$

Locally, around any $x_0 \in \mathbb{R}$ we can compute x as an analytic function of h, g and u. Amalgamation gives a global x function. Thus x belongs to the differential universe of the system Σ, which means that Σ is uniformly observable.

Characterization of dynamical equivalence

Under some technical assumptions about the systems, the following can be shown.

Theorem

Two uniformly observable systems Σ and $\tilde{\Sigma}$ are externally dynamically equivalent if and only if the differential universes $\mathcal{U}_{\Sigma}$ and $\mathcal{U}_{\tilde{\Sigma}}$ are isomorphic.

Z. Bartosiewicz, E. Pawłuszewicz, External Dynamical Equivalence of Analytic Control Systems, in: *Mathematical Control Theory and Finance*, Springer-Verlag, Berlin 2008
B. Jakubczyk, Remarks on equivalence and linearization of nonlinear systems, in: Proceedings of the 2nd IFAC NOLCOS Symposium, 1992, Bordeaux, France,