Classical Yang-Baxter Equation and Its Extensions

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Outline

1. What is classical Yang-Baxter equation (CYBE)?
2. Extensions of CYBE: Lie algebras
3. Extensions of CYBE: general algebras

What is classical Yang-Baxter equation (CYBE)?

Definition

Let $\mathfrak{g}$ *be a Lie algebra and* $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$. r *is called a solution of* **classical Yang-Baxter equation (CYBE)** *in* $\mathfrak{g}$ *if*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \text{ in } U(\mathfrak{g}), \tag{1}$$

where $U(\mathfrak{g})$ *is the universal enveloping algebra of* $\mathfrak{g}$ *and*

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1; r_{13} = \sum_i a_i \otimes 1 \otimes b_i; r_{23} = \sum_i 1 \otimes a_i \otimes b_i. \tag{2}$$

r *is said to be* **skew-symmetric** *if*

$$r = \sum_i (a_i \otimes b_i - b_i \otimes a_i). \tag{3}$$

We also denote $r^{21} = \sum_i b_i \otimes a_i$.

What is classical Yang-Baxter equation (CYBE)?

◦ Background and application:

1. Arose in the study of inverse scattering theory.
2. Schouten bracket in differential geometry.
3. "Classical limit" of quantum Yang-Baxter equation.
4. Classical integrable systems (Lax pair approach).
5. Lie bialgebras (coboundary Lie bialgebras).
6. Symplectic geometry (invertible solutions).
7. ...

∘ Interpretation in terms of matrices (linear maps)

⋆ Classical r-matrix

Set $r = \sum_{i,j} r_{ij} e_i \otimes e_j$, where $\{e_1, \cdots, e_j\}$ is a basis of the Lie algebra $\mathfrak{g}$. Then the matrix

$$r = (r_{ij}) = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \cdots & \cdots & \cdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix}, \tag{6}$$

is called a **classical r-matrix**.

Natural question: if a linear transformation (or generally, a linear map) R is given by the classical r-matrix under a basis, what should r satisfy?

What is classical Yang-Baxter equation (CYBE)?

$\star$ Semenov-Tian-Shansky's approach: Operator form of CYBE (Rota-Baxter operator)

M.A. Semenov-Tian-Shansky, What is a classical R-matrix? Funct. Anal. Appl. **17** *(1983) 259-272.*

A linear map $R : \mathfrak{g} \to \mathfrak{g}$ satisfies

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]), \forall x, y \in \mathfrak{g}. \qquad (7)$$

It is equivalent to the tensor form (1) of CYBE under the following two conditions:

1. there exists a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$.
2. r is skew-symmetric.

What is classical Yang-Baxter equation (CYBE)?

On the other hand, it is exactly the **Rota-Baxter operator (of weight zero)** in the context of Lie algebras:

$$R(x)R(y) = R(R(x)y + xR(y)), \forall x \in A, \tag{8}$$

where A is an associative algebra and $R : A \to A$ is a linear map.

Rota-Baxter operators arose from probability and combinatorics and have connections with many fields.

(See *L. Guo, WHAT is a Rota-Baxter algebra, Notice of Amer. Math. Soc.* **56** *(2009) 1436-1437*)

What is classical Yang-Baxter equation (CYBE)?

$\star$ Kupershmidt's approach: $\mathcal{O}$-operators

B.A. Kupershmidt, What a classical r-matrix really is, J. Nonlinear Math. Phys. **6** *(1999), 448-488.*

When r is skew-symmetric, the tensor form (1) of CYBE is equivalent to a linear map $r:\mathfrak{g}^*\rightarrow\mathfrak{g}$ satisfying

$$[r(x),r(y)]=r(\mathrm{ad}^*r(x)(y)-\mathrm{ad}^*r(y)(x)),\forall x,y\in\mathfrak{g}^*, \qquad (9)$$

where $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$ and ad^* is the dual representation of adjoint representation (coadjoint representation).

Definition

*Let $\mathfrak{g}$ be a Lie algebra and $\rho:\mathfrak{g}\rightarrow gl(V)$ be a representation of $\mathfrak{g}$. A linear map $T:V\rightarrow\mathfrak{g}$ is called an $\mathcal{O}$-**operator** if T satisfies*

$$[T(u),T(v)]=T(\rho(T(u))v-\rho(T(v))u),\forall u,v\in V. \qquad (10)$$

Kupershmidt introduced the notion of $\mathcal{O}$-operator as a natural generalization of CYBE!

What is classical Yang-Baxter equation (CYBE)?

$\star$ "Duality" between Rota-Baxter operators and CYBE

R is a Rota-Baxter operator of weight zero
$\Longleftrightarrow$ an $\mathcal{O}$-operator associated to ad

When r is skew-symmetric, we know that
CYBE $\Longleftrightarrow$ an $\mathcal{O}$-operator associated to ad^*
(From CYBE to $\mathcal{O}$-operators)

◦ From $\mathcal{O}$-operators to CYBE

C. Bai, A unified algebraic approach to the classical Yang-Baxter equation, J. Phys. A **40** *(2007) 11073-11082.*

Notation: let $\rho : \mathfrak{g} \to gl(V)$ be a representation of the Lie algebra $\mathfrak{g}$. On the vector space $\mathfrak{g} \oplus V$, there is a natural Lie algebra structure (denoted by $\mathfrak{g} \ltimes_\rho V$) given as follows:

$$[x_1 + v_1, x_2 + v_2] = [x_1, x_2] + \rho(x_1)v_2 - \rho(x_2)v_1, \qquad (11)$$

for any $x_1, x_2 \in \mathfrak{g}, v_1, v_2 \in V$.

Proposition

Let $\mathfrak{g}$ be a Lie algebra. Let $\rho : \mathfrak{g} \to gl(V)$ be a representation of $\mathfrak{g}$ and $\rho^* : \mathfrak{g} \to gl(V^*)$ be the dual representation. Let $T : V \to \mathfrak{g}$ be a linear map which is identified as an element in $\mathfrak{g} \otimes V^* \subset (\mathfrak{g} \ltimes_{\rho^*} V^*) \otimes (\mathfrak{g} \ltimes_{\rho^*} V^*)$. Then $r = T - T^{21}$ is a skew-symmetric solution of CYBE in $\mathfrak{g} \ltimes_{\rho^*} V^*$ if and only if T is an $\mathcal{O}$-operator.

What is classical Yang-Baxter equation (CYBE)?

◦ Left-symmetric algebras: the algebra structures behind CYBE ($\mathcal{O}$-operators approach)

Definition

Let A be a vector space equipped with a bilinear product $(x, y) \to xy$. A is called a **left-symmetric algebra** *if*

$$(xy)z - x(yz) = (yx)z - y(xz), \forall x, y, z \in A. \tag{12}$$

Two basic properties:

1. The commutator
$$[x, y] = xy - yx, \ \ \forall x, y \in A, \tag{13}$$
defines a Lie algebra $\mathfrak{g}(A)$, which is called the sub-adjacent Lie algebra of A and A is also called the compatible left-symmetric algebra structure on the Lie algebra $\mathfrak{g}(A)$.
2. $L : \mathfrak{g}(A) \to gl(\mathfrak{g}(A))$ with $x \to L_x$ gives a regular representation of the Lie algebra $\mathfrak{g}(A)$.

What is classical Yang-Baxter equation (CYBE)?

⋆ From $\mathcal{O}$-operators to left-symmetric algebras

Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to gl(V)$ be a representation. Let $T : V \to \mathfrak{g}$ be an $\mathcal{O}$-operator associated to ρ, then

$$u * v = \rho(T(u))v, \quad \forall u, v \in V \tag{14}$$

defines a left-symmetric algebra on V.

⋆ Sufficient and necessary condition:

Proposition

Let $\mathfrak{g}$ be a Lie algebra. There is a compatible left-symmetric algebra structure on $\mathfrak{g}$ if and only if there exists an invertible $\mathcal{O}$-operator of $\mathfrak{g}$.

"$\Longleftarrow$" The left-symmetric algebra structure is given by

$$x \circ y = T(\rho(x)T^{-1}(y)), \forall x, y \in \mathfrak{g}. \tag{15}$$

"$\Longrightarrow$" $\mathrm{id} : \mathfrak{g}(A) \to \mathfrak{g}(A)$ is an $\mathcal{O}$-operator of $\mathfrak{g}(A)$ associated to the representation $(L_\circ, A)$.

What is classical Yang-Baxter equation (CYBE)?

⋆ From left-symmetric algebras to CYBE

Proposition

Let A *be a left-symmetric algebra. Then*

$$r = \sum_{i=1}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i) \tag{16}$$

is a solution of the classical Yang-Baxter equation in the Lie algebra $\mathfrak{g}(A) \ltimes_{L^*} \mathfrak{g}(A)^*$*, where* $\{e_1, ..., e_n\}$ *is a basis of* A *and* $\{e_1^*, ..., e_n^*\}$ *is the dual basis.*

◦ Motivations and some examples

⋆ Semenov-Tian-Shansky's **modified classical Yang-Baxter equation (MCYBE)**

Let $\mathfrak{g}$ be a Lie algebra. A linear map $R:\mathfrak{g}\to\mathfrak{g}$ is a solution of the MCYBE if R satisfies

$$[R(x),R(y)]-R([R(x),y]+[x,R(y)])=-[x,y],\forall x,y\in\mathfrak{g}. \quad (17)$$

⋆ Bordemann's generalization of MCYBE

M. Bordemann, Generalized Lax pairs, the modified classical Yang-Baxter equation, and affine geometry of Lie groups, Comm. Math. Phys. **135** *(1990) 201-216.*

Let $\rho : \mathfrak{g} \to gl(V)$ be a representation of a Lie algebra $\mathfrak{g}$. Set

$$x \cdot v =: \rho(x)v, \quad \forall x \in \mathfrak{g}, v \in V. \tag{18}$$

Let $\beta : V \to \mathfrak{g}$ be a linear map satisfies

$$\beta(u) \cdot v + \beta(v) \cdot u = 0, \quad \forall u, v \in V; \tag{19}$$

$$\beta(x \cdot v) = [x, \beta(v)], \quad \forall x \in \mathfrak{g}, v \in V. \tag{20}$$

A linear map $r : V \to \mathfrak{g}$ satisfies MCYBE if r satisfies

$$[r(u), r(v)] = r(r(u) \cdot v - r(v) \cdot u) - [\beta(u), \beta(v)], \quad \forall u, v \in V. \tag{21}$$

1. When $\beta = 0$, r is the $\mathcal{O}$-operator;
2. When $\rho = \mathrm{ad}$ and $\beta = \mathrm{id}$, r reduces to the S.-T.-S.'s MCYBE.

Extensions of CYBE: Lie algebras

⋆ **Rota-Baxter operator of any weight**

Let $\mathfrak{g}$ be a Lie algebra. A linear map $R : \mathfrak{g} \to \mathfrak{g}$ is called a Rota-Baxter operator of weight λ if R satisfies

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)] + \lambda[x, y]), \forall x, y \in \mathfrak{g}. \quad (22)$$

⋆ Questions:

1. Whether it is possible to extend the notion of $\mathcal{O}$-operator to the non-zero weight?
2. If (1) holds, whether it is possible to deal with it and the Bordemann's generalization by a unified way?
3. Whether there are the tensor forms related to the above operator forms?
4. How to deal with the non-skew-symmetric cases?

∘ Extended $\mathcal{O}$-operators and extended CYBE

⋆ Extensions from representations to $\mathfrak{g}$-Lie algebras

Definition

1. *Let* $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$, *or simply* $\mathfrak{g}$, *denote a Lie algebra* $\mathfrak{g}$ *with Lie bracket* $[\,,\,]_{\mathfrak{g}}$.
2. *For a Lie algebra* $\mathfrak{b}$, *let* $\mathrm{Der}_{\mathbf{k}}\mathfrak{b}$ *denote the Lie algebra of derivations of* $\mathfrak{b}$.
3. *Let* $\mathfrak{a}$ *be a Lie algebra. An* $\mathfrak{a}$-**Lie algebra** *is a triple* $(\mathfrak{b}, [\,,\,]_{\mathfrak{b}}, \pi)$ *consisting of a Lie algebra* $(\mathfrak{b}, [\,,\,]_{\mathfrak{b}})$ *and a Lie algebra homomorphism* $\pi : \mathfrak{a} \to \mathrm{Der}_{\mathbf{k}}\mathfrak{b}$. *To simplify the notation, we also let* $(\mathfrak{b}, \pi)$ *or simply* $\mathfrak{b}$ *denote* $(\mathfrak{b}, [\,,\,]_{\mathfrak{b}}, \pi)$.
4. *Let* $\mathfrak{a}$ *be a Lie algebra and let* $(\mathfrak{g}, \pi)$ *be an* $\mathfrak{a}$*-Lie algebra. Let* $a \cdot b$ *denote* $\pi(a)b$ *for* $a \in \mathfrak{a}$ *and* $b \in \mathfrak{g}$.

Proposition

Let $\mathfrak{a}$ be a Lie algebra and let $(\mathfrak{b}, \pi)$ be an $\mathfrak{a}$-Lie algebra. Then there exists a unique Lie algebra structure on the vector space direct sum $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ retaining the old brackets in $\mathfrak{a}$ and $\mathfrak{b}$ and satisfying $[x, a] = \pi(x)a$ for $x \in \mathfrak{a}$ and $a \in \mathfrak{b}$. That is,

$$[x+a, y+b] = [x, y]+\pi(x)b-\pi(y)a+[a, b], \forall x, y \in \mathfrak{a}, a, b \in \mathfrak{b}. \tag{23}$$

Extensions of CYBE: Lie algebras

$\star$ Extensions of $\mathcal{O}$-operators

Definition

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{k}$ be a $\mathfrak{g}$-Lie algebra. Let $\alpha, \beta : \mathfrak{k} \to \mathfrak{g}$ be two linear maps. Suppose that

$$\kappa\beta(x) \cdot y + \kappa\beta(y) \cdot x = 0, \forall x, y \in \mathfrak{k} \tag{24}$$

$$\kappa\beta(\xi \cdot x) = \kappa[\xi, \beta(x)], \forall \xi \in \mathfrak{g}, x \in \mathfrak{k}, \tag{25}$$

$$\mu\beta([x, y]) \cdot z = \mu[\beta(x) \cdot y, z], \forall x, y, z \in \mathfrak{k}, \tag{26}$$

The pair (α, β) or simply α is called an **extended $\mathcal{O}$-operator of weight λ with extension β of mass (κ, μ)** *if*

$$[\alpha(x), \alpha(y)]_{\mathfrak{g}} - \alpha(\alpha(x) \cdot y - \alpha(y) \cdot x + \lambda[x, y]_{\mathfrak{k}})$$

$$= \kappa[\beta(x), \beta(y)]_{\mathfrak{g}} + \mu\beta([x, y]_{\mathfrak{k}}), \forall x, y \in \mathfrak{k}. \tag{27}$$

Definition

When (V, ρ) is a $\mathfrak{g}$-module, we regard (V, ρ) as a $\mathfrak{g}$-Lie algebra with the trivial bracket. Then λ, μ are irrelevant. We then call the pair (α, β) **an extended $\mathcal{O}$-operator with extension β of mass κ**.

1. If (V, ρ) is a $\mathfrak{g}$-module (or $\lambda = \mu = 0$), and in addition $\kappa = -1$, we obtain the Bordemann's MCYBE;
2. When $\beta = 0$, we obtain an **$\mathcal{O}$-operator of weight** $\lambda \in \mathbf{k}$, i.e.,

$$[\alpha(x), \alpha(y)]_{\mathfrak{g}} = \alpha\big(\alpha(x) \cdot y - \alpha(y) \cdot x + \lambda[x, y]_{\mathfrak{k}}\big), \quad \forall x, y \in \mathfrak{k}. \tag{28}$$

If in addition, $(\mathfrak{k}, \pi) = (\mathfrak{g}, \mathrm{ad})$, we obtain the Rota-Baxter operator of weight λ.

⋆ Extensions of CYBE

Definition

Let $\mathfrak{g}$ be a Lie algebra. Fix $\epsilon \in \mathbb{R}$. The equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \epsilon[(r_{13} + r_{31}), (r_{23} + r_{32})] \quad (29)$$

is called the **extended classical Yang-Baxter equation (ECYBE) of mass ϵ.**

When $\epsilon = 0$ or r is skew-symmetric, then the ECYBE of mass ϵ is the same as the CYBE.

Extensions of CYBE: Lie algebras

$\star$ From extended CYBE to extended $\mathcal{O}$-operators

C. Bai, L. Guo and X. Ni, Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and PostLie algebras, Comm. Math. Phys. **297** *(2010) 553-596.*

C. Bai, L. Guo and X. Ni, Generalizations of the classical Yang-Baxter equation and $\mathcal{O}$-operators, preprint 2010.

Notations: Let $\mathfrak{g}$ be a Lie algebra and let $r \in \mathfrak{g} \otimes \mathfrak{g}$. Set

$$r_{\pm} = (r \pm r^{21})/2. \tag{30}$$

On the other hand, $r \in \mathfrak{g} \otimes \mathfrak{g}$ is said to be **invariant** if r satisfies

$$(\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))r = 0, \forall x \in \mathfrak{g}. \tag{31}$$

Theorem

Let $\mathfrak{g}$ be a Lie algebra and let $r \in \mathfrak{g} \otimes \mathfrak{g}$. Define $r_{\pm}$ by Eq. (30) which are identified as linear maps from $\mathfrak{g}^*$ to $\mathfrak{g}$. Suppose that r_+ is invariant. Then r is a solution of ECYBE of mass $\frac{\kappa+1}{4}$:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \frac{\kappa+1}{4}[(r_{13}+r_{31}), (r_{23}+r_{32})] \quad (32)$$

if and only if r_- is an extended $\mathcal{O}$-operator with extension r_+ of mass κ, i.e., the following equation holds:

$$[r_-(a^*), r_-(b^*)] - r_-(\mathrm{ad}^*(r_-(a^*))b^* - \mathrm{ad}^*(r_-(b^*))a^*)$$

$$= \kappa[r_+(a^*), r_+(b^*)], \quad \forall a^*, b^* \in \mathfrak{g}^*. \quad (33)$$

Special cases:

(1) *Y. Kosmann-Schwarzbach and F. Magri, Poisson-Lie groups and complete integrability, I. Drinfeld bialgebras, dual extensions and their canonical representations, Ann. Inst. Henri Poincaré, Phys. Théor. A* **49** *(1988) 433-460.*

Suppose that r is not skew-symmetric. If the symmetric part r_+ is invariant, then r is a solution of the CYBE if and only if r_- is an extended $\mathcal{O}$-operator with extension r_+ of mass -1.

(2) Let $\mathfrak{g}$ be a real Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Then

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \frac{1}{2}[r_{13} + r_{31}, r_{23} + r_{32}] \qquad (32)$$

is called the **type II CYBE**.

Suppose r_+ is invariant.

r is a solution of the type II CYBE.

$\Leftrightarrow$ r_- is an extended $\mathcal{O}$-operator with extension r_+ of mass 1.

$\Leftrightarrow$ $r_- \pm i r_+$ are solutions of the CYBE in $\mathfrak{g} \oplus \sqrt{-1}\mathfrak{g}$.

Extensions of CYBE: Lie algebras

$\star$ From extended $\mathcal{O}$-operators to extended CYBE

Let $\mathfrak{g}$ be a Lie algebra and let (V, ρ) be a $\mathfrak{g}$-module.

1. Let $\alpha, \beta : V \to \mathfrak{g}$ be linear maps. Then α is an extended $\mathcal{O}$-operator with extension β of mass k if and only if $(\alpha - \alpha^{21} \pm (\beta + \beta^{21})$ is a solution of ECYBE of mass $\frac{\kappa+1}{4}$ in $\mathfrak{g} \ltimes_{\rho^*} V^*$.
2. Let $\alpha : V \to \mathfrak{g}$ be a linear map. Then α is an $\mathcal{O}$-operator of weight zero if and only if $\alpha - \alpha^{21}$ is a skew-symmetric solution of CYBE in $\mathfrak{g} \ltimes_{\rho^*} V^*$.
3. Let $R : \mathfrak{g} \to \mathfrak{g}$ be a linear map. R satisfies Semenov-Tian-Shansky's MCYBE if and only if $R - R^{21} \pm (\mathrm{id} + \mathrm{id}^{21})$ is a solution of CYBE in $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$.
4. Let $P : \mathfrak{g} \to \mathfrak{g}$ be a linear map. Then P is a Rota-Baxter operator of weight $\lambda \neq 0$ if and only if both $\frac{2}{\lambda}(P - P^{21}) + 2\mathrm{id}$ and $\frac{2}{\lambda}(P - P^{21}) - 2\mathrm{id}^{21}$ are solutions of CYBE in $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$.

⋆ Applications in integrable systems

Definition

A **nonabelian generalized Lax pair** *for a Hamiltonian system* $(P, w, \mathcal{H})$ *is a quintuple* $(\mathfrak{g}, \rho, \mathfrak{a}, L, M)$ *satisfying the following conditions:*

1. $\mathfrak{g}$ *is a (finite-dimensional) Lie algebra;*
2. $(\mathfrak{a}, \rho)$ *is a (finite-dimensional)* $\mathfrak{g}$*-Lie algebra with the Lie algebra homomorphism* $\rho : \mathfrak{g} \to \mathrm{Der}_{\mathbb{R}}(\mathfrak{a})$*;*
3. $L : P \to \mathfrak{a}$ *is a smooth map,*
4. $M : P \to \mathfrak{g}$ *is a smooth map such that*

$$dL(p)X_{\mathcal{H}}(p) = -\rho(M(p))L(p), \quad \forall p \in P. \tag{35}$$

1. When the Lie bracket on $\mathfrak{a}$ happens to be trivial, the $\mathfrak{g}$-Lie algebra $(\mathfrak{a}, \rho)$ becomes a representation of $\mathfrak{g}$ and the nonabelian generalized Lax pair becomes the **generalized Lax pair** in the sense of Bordemann.
2. For $\mathfrak{a} = \mathfrak{g}$ and $\rho = \mathrm{ad}$, Eq. (35) is the usual **Lax equation**. Moreover, the Lax pair can be realized as a nonabelian generalized Lax pair in two different ways, by either taking ρ to be ad and $\mathfrak{a}$ to be the Lie algebra $\mathfrak{g}$, or taking ρ to be ad and $\mathfrak{a}$ to be the underlying vector space of $\mathfrak{g}$ equipped with the trivial Lie bracket.

Remark

Nonabelian generalized r-matrix ansatz gives a natural motivation to extended $\mathcal{O}$-operators.

⋆ New algebras behind: PostLie algebras

B. Vallette, Homology of generalized partition posets, J. Pure Appl. Algebra **208** *(2007) 699-725.*

Definition

A **(left) PostLie algebra** *is a* $\mathbb{R}$*-vector space* L *with two bilinear operations* $\circ$ *and* $[,]$ *which satisfy the relations:*

$$[x,y] = -[y,x], \tag{36}$$

$$[[x,y],z] + [[z,x],y] + [[y,z],x] = 0, \tag{37}$$

$$z\circ(y\circ x) - y\circ(z\circ x) + (y\circ z)\circ x - (z\circ y)\circ x + [y,z]\circ x = 0, \tag{38}$$

$$z\circ[x,y] - [z\circ x, y] - [x, z\circ y] = 0, \tag{39}$$

for all $x, y \in L$.

Eq. (36) and Eq. (37) mean that L is a Lie algebra for the bracket $[,]$, and we denote it by $(\mathfrak{G}(L), [,])$. Moreover, we say that $(L, [,], \circ)$ **is a PostLie algebra structure on** $(\mathfrak{G}(L), [,])$. On the other hand, it is straightforward to check that L is also a Lie algebra for the operation:

$$\{x, y\} \equiv x \circ y - y \circ x + [x, y], \ \ \forall x, y \in L. \tag{40}$$

We shall denote it by $(\mathcal{G}(L), \{,\})$ and say that $(\mathcal{G}(L), \{,\})$ **has a compatible PostLie algebra structure given by** $(L, [,], \circ)$.

Proposition

Let $\mathfrak{g}$ be a Lie algebra. Then there is a compatible PostLie algebra structure on $\mathfrak{g}$ if and only if there exists a $\mathfrak{g}$-Lie algebra $(\mathfrak{k}, \pi)$ and an invertible $\mathcal{O}$-operator $r : \mathfrak{k} \to \mathfrak{g}$ of weight 1.

Application: There is a typical example of nonabelian generalized Lax pair constructed from PostLie algebras!

◦ Lie bialgebras and generalized CYBE

V. Chari and A. Pressley, A guide to quantum groups, Cambridge University Press, Cambridge (1994).

⋆ Lie bialgebras:

Definition

Let $\mathfrak{g}$ be a Lie algebra. A **Lie bialgebra** *structure on $\mathfrak{g}$ is an antisymmetric linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$ and δ is a 1-cocycle of $\mathfrak{g}$ associated to $\mathrm{ad} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$:*

$$\delta([x,y]) = (\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))\delta(y) - (\mathrm{ad}(y) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(y))\delta(x), \tag{41}$$

for any $x, y \in \mathfrak{g}$. We denote it by $(\mathfrak{g}, \delta)$.

⋆ Coboundary Lie bialgebras:

Definition

A Lie bialgebra $(\mathfrak{g},\delta)$ *is called* **coboundary** *if* δ *is a 1-coboundary of* $\mathfrak{g}$ *associated to* $\mathrm{ad}\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}$, *that is, there exists an* $r\in\mathfrak{g}\otimes\mathfrak{g}$ *such that*

$$\delta(x)=(\mathrm{ad}(x)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(x))r,\quad \forall x\in\mathfrak{g}. \tag{42}$$

Theorem

Let $\mathfrak{g}$ *be a Lie algebra and* $r\in\mathfrak{g}\otimes\mathfrak{g}$. *Then the map* $\delta:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ *defined by Eq. (42) induces a Lie bialgebra structure on* $\mathfrak{g}$ *if and only if the following two conditions are satisfied (for any* $x\in\mathfrak{g}$*):*

1. $(\mathrm{ad}(x)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(x))(r+r^{21})=0$;
2. $(\mathrm{ad}(x)\otimes\mathrm{id}\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(x)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{ad}(x))([r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}])=0.$

⋆ Generalized CYBE:

Definition

Let $\mathfrak{g}$ be a Lie algebra and let $r \in \mathfrak{g} \otimes \mathfrak{g}$. r is said to be a solution of **generalized classical Yang-Baxter equation (GCYBE)** *if r satisfies*

$$(\mathrm{ad}(x) \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ad}(x))$$

$$([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]) = 0. \tag{43}$$

Let $\mathfrak{g}$ be a Lie algebra.

1. If the symmetric part of $r \in \mathfrak{g} \otimes \mathfrak{g}$ is invariant, then r is a solution of GCYBE if r satisfies ECYBE.
2. Let $(\mathfrak{k}, \pi)$ be a $\mathfrak{g}$-Lie algebra. Let $\alpha, \beta : \mathfrak{k} \to \mathfrak{g}$ be two linear maps such that α is an extended $\mathcal{O}$-operator of weight λ with extension β of mass (κ, μ). Then $\alpha - \alpha^{21} \in (\mathfrak{g} \ltimes_{\pi^*} \mathfrak{k}^*) \otimes (\mathfrak{g} \ltimes_{\pi^*} \mathfrak{k}^*)$ is a skew-symmetric solution of GCYBE if and only if the following equations hold:

$$\lambda\pi(\alpha([u,v]_{\mathfrak{k}}))w + \lambda\pi(\alpha([w,u]_{\mathfrak{k}}))v + \lambda\pi(\alpha([v,w]_{\mathfrak{k}}))u = 0, \tag{44}$$

$$\lambda[x, \alpha([u,v]_{\mathfrak{k}})]_{\mathfrak{g}} = \lambda\alpha([\pi(x)u, v]_{\mathfrak{k}}) + \lambda\alpha([u, \pi(x)v]_{\mathfrak{k}}), \tag{45}$$

for any $x \in \mathfrak{g}, u, v, w \in \mathfrak{k}$. In particular, if $\lambda = 0$, i.e., α is an extended $\mathcal{O}$-operator of weight 0 with extension β of mass (κ, μ), then $\alpha - \alpha^{21} \in (\mathfrak{g} \ltimes_{\pi^*} \mathfrak{k}^*) \otimes (\mathfrak{g} \ltimes_{\pi^*} \mathfrak{k}^*)$ is a skew-symmetric solution of GCYBE.

1. Let $(\mathfrak{k},\pi)$ be a $\mathfrak{g}$-Lie algebra. Let $\alpha:\mathfrak{k}\to\mathfrak{g}$ an $\mathcal{O}$-operator of weight λ. Then $\alpha-\alpha^{21}\in(\mathfrak{g}\ltimes_{\pi^*}\mathfrak{k}^*)\otimes(\mathfrak{g}\ltimes_{\pi^*}\mathfrak{k}^*)$ is a skew-symmetric solution of GCYBE if and only if Eq. (45) and Eq. (46) hold.
2. Let $\rho:\mathfrak{g}\to\mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. Let $\alpha,\beta:\mathfrak{k}\to\mathfrak{g}$ be two linear maps such that α is an extended $\mathcal{O}$-operator with extension β of mass κ. Then $\alpha-\alpha^{21}\in(\mathfrak{g}\ltimes_{\rho^*}V^*)\otimes(\mathfrak{g}\ltimes_{\rho^*}V^*)$ is a skew-symmetric solution of GCYBE.

◦ **Main Idea**:

Let A be an algebra. Then (L, R) is the natural bimodule. Find the dual bimodule of (L, R). Then with a suitable symmetry, the $\mathcal{O}$-operator of A associated to the dual bimodule gives the equation of CYBE for A. Finally find the corresponding tensor form.

◦ **Application**: A solution of CYBE provides a construction (coboundary cases) for a bialgebra structure!

Extensions of CYBE: general algebras

$\star$ Associative algebra $(A, *)$

V.N. Zhelyabin, Jordan bialgebras and their connection with Lie bialgebras, Algebra and Logic **36** *(1997) 1-15.*

M. Aguiar, On the associative analog of Lie bialgebras, J. Algebra **244** *(2001) 492-532.*

C. Bai, Double construction of Frobenius algebras, Connes cocycles and their duality, J. Noncommutative Geometry **4** *(2010) 475-530.*

• **Associative Yang-Baxter equation** (skew-symmetric solution)

$$r_{12} * r_{13} + r_{13} * r_{23} - r_{23} * r_{12} = 0. \tag{46}$$

Associative D-bialgebra (Zhelyabin)
balanced infinitesimal bialgebra (Aguiar)
antisymmetric infinitesimal bialgebra (Bai)

⋆ Dendriform algebra $(A, \succ, \prec)$

- D-**equation** (symmetric solution)

$$r_{12} * r_{13} = r_{13} \prec r_{23} + r_{23} \succ r_{12}, \tag{47}$$

where $x * y = x \prec y + x \succ y$.

Dendriform D-bialgebra

⋆ Quadri-algebra $(A, \searrow, \nearrow, \nwarrow, \swarrow)$

- Q-**equation** (skew-symmetric solution)

$$r_{13} \succ r_{23} = r_{23} \nearrow r_{12} + r_{23} \nwarrow r_{12} + r_{12} \swarrow r_{13}, \tag{48}$$

$$r_{13} \prec r_{23} = -r_{23} \nearrow r_{12} - r_{12} \searrow r_{13} - r_{12} \swarrow r_{13}, \tag{49}$$

where $x \succ y = x \nearrow y + x \searrow y, \;\; x \prec y = x \nwarrow y + x \swarrow y$.

Quadri-bialgebra

Associative analogues of the study on (extended) $\mathcal{O}$-operators, (extended) CYBE and related algebras:

C. Bai, L. Guo and X. Ni, $\mathcal{O}$-operators on associative algebras and associative Yang-Baxter equations, arXiv:0910.3261.

C. Bai, L. Guo and X. Ni, $\mathcal{O}$-operators on associative algebras and dendriform algebras, arXiv: 1003.2432.

$\star$ Left-symmetric Lie algebra $(A, \circ)$

C. Bai, Left-symmetric bialgebras and an analogy of the classical Yang-Baxter equation, Comm. Contemp. Math. 10 (2008) 221-260.

- S-**equation** (symmetric solution)

$$-r_{12} \circ r_{13} + r_{12} \circ r_{23} + [r_{13}, r_{23}] = 0. \tag{50}$$

left-symmetric bialgebra

⋆ Jordan algebra $(A, \circ)$

V.N. Zhelyabin, On a class of Jordan D-bialgebras, St. Petersburg Math. J. 11 (2000) 589-609.

• **Jordan Yang-Baxter equation** (skew-symmetric solution)

$$r_{12} \circ r_{13} + r_{13} \circ r_{23} - r_{12} \circ r_{23} = 0. \tag{51}$$

Jordan D-bialgebra

⋆ Pre-Jordan algebra $(A, \cdot)$

D. Hou, X. Ni, C. Bai, Pre-Jordan algebras, appear in Mathematica Scandinavica.

D. Hou, C. Bai, Jordan D-bialgebras and pre-Jordan bialgebras, preprint.

• JP**-equation** (symmetric solution)

$$r_{13} \circ r_{23} - r_{12} \cdot r_{23} - r_{12} \cdot r_{13} = 0, \tag{52}$$

where $x \circ y = x \cdot y + y \cdot x$.

pre-Jordan bialgebra

Prospect

1. We have been trying to give an operadic interpretation on the study of classical Yang-Baxter equation, its analogues and extensions, bialgebra structures and related structures.
   C. Bai, L. Guo, X. Ni, in preparation.
2. It is natural to consider the possible "quantized" structures.
   No idea yet!

Thank You!