- $I_6 = \langle 8x_1^2x_3x_4 - 7x_1x_3^3 + 8x_3x_4, x_1x_2^2 - 6x_1x_3^2 - 7x_4, -5x_2^4 + 2x_1x_2x_4\rangle$;
- $I_7 = \langle 5x_1^2x_2^2 - 3x_1x_2x_3^2 + x_1^2x_2, 3x_2x_3^3 + 2x_3^2x_4^2 + x_3x_4^3, 5x_1^4 - x_1x_2x_3^2 - 7x_2x_4^2\rangle$.
- $I_8 = \langle -6x_4^3 - x_1x_3 + 7x_4^2, -x_2^4 + 4x_1x_2x_3 + 4x_3x_4, 2x_1x_3^2x_4 - 3x_1x_3x_4^2 + 7x_2^2x_4^2\rangle$.
- $I_9 = \langle -x_2^3x_3 + 6x_2^2x_4^2 - 4x_1x_2x_4, 2x_1^3x_4 - 4x_2x_3x_4 + 2x_3, 6x_1x_3^2 + 4x_1x_2^2x_4 + 3x_1x_4^2\rangle$.
- $I_{10} = \langle 3x_1x_4^2 + 7x_2^3 + 4x_2x_3^2, 5x_1x_3 - 10x_1x_4 - 5x_3^2, -8x_1x_2 + 3x_3^2 + 4x_2^2x_4\rangle$.

For all these examples, the term order in R and R^m $(3 \le m \le 4)$ is anti-graded revlex order and TOP order, respectively. We implement the two algorithms on the computer algebra system *Maple*, and the codes and examples are available on the web: http://www.mmrc.iss.ac.cn/~dwang/software.html.

Table 1: examples

ideal	signature-based method			classical method		
	J-pairs	discard	ratio	S-polys	discard	ratio
I_1	21	14	67%	28	6	21%
I_2	21	14	67%	21	9	43%
I_3	15	12	80%	15	8	53%
I_4	20	16	80%	21	10	48%
I_5	15	9	60%	15	4	27%
I_6	20	16	80%	21	6	29%
I_7	14	11	79%	15	4	27%
I_8	35	29	83%	28	9	32%
I_9	10	7	70%	15	6	40%
I_{10}	21	17	81%	66	28	43%

The second column and fifth column in Table 1 represents the total number of J-pairs and S-polynomials (abbreviated S-polys) generated during the calculation, respectively. The third column (sixth column) represents the useless J-pairs (useless S-polys) that are discarded. The fourth column (last column) shows the percentage of the number of discarded J-pairs (S-polys) to the number of the total J-pairs (S-polys). Experimental data in Table 1 suggests that the proposed algorithm is superior in practice in comparison with the classical Gröbner basis algorithm.

5 CONCLUDING REMARKS

The paper proposed an efficient algorithm to compute the standard bases in local ring. In the process of extending the GVW algorithm from polynomial ring to local ring, we solved two key problems. First, an infinite set has not a minimal element in local ring. Under the situation that $\prec_1$ is an anti-graded order in $k[X]$ and $\prec_2$ is a TOP order in $(k[X])^m$, we proved that the signature set $L(\text{lpp}(v_0))$ w.r.t. v_0 has a minimal element. Then we generalized the cover theorem to local ring to discard the useless J-pairs. Second, since the general division algorithm may not terminate in local ring, Mora normal form algorithm is used to do regular top-reduction, and the proposed algorithm terminates in finite steps.

Although we only consider the case that $\prec_2$ is a TOP order in $(k[X])^m$, if $\prec_2$ is an $\mathbf{f}$-weighted anti-degree followed by TOP or an $\mathbf{f}$-weighted $\prec_1$ followed by POT, Lemma 2.9 and Theorem 3.1 are also established. Moreover, an alternative method to compute the standard bases is using the Lazard's homogeneous idea. In the future work, we will consider the case of $\prec_1$ is not an anti-graded order in $k[X]$. We hope that the results of this paper will motivate new progress in this research topic.

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