5 CONCLUDING REMARKS

In this paper, we have studied a property of modules over a polynomial ring and applied it to multivariate polynomial matrix factorizations.

Modules over a polynomial ring $k[\mathbf{z}]$ are quite different from vector spaces over a field k. For instance, there exist many vectors in $\mathcal{M} \subset k[\mathbf{z}]^m$ that cannot be expressed as linear combinations of the elements in a maximum $k[\mathbf{z}]$-linearly independent vector set of $\mathcal{M}$. The essential reason is that the polynomial division algorithm cannot be done arbitrarily in $k[\mathbf{z}]$. Let $\vec{w} \in \mathcal{M}$ be a nonzero vector and $\mathcal{G}$ be a maximum $k[\mathbf{z}]$-linearly independent vector set of $\mathcal{M}$, then Theorem 3.1 tells us that h can be obtained from the greatest common divisors of maximal minors of the relevant matrices composed of $\vec{w}$ and $\mathcal{G}$. Furthermore, Corollary 3.3 shows that the polynomial constructed in Theorem 3.1 is minimal.

With the help of the above property, a connection (Theorem 3.5) between a rank-deficient matrix and any of its full row rank submatrices was established. Based on this relationship, the problem of general factorizations of rank-deficient matrices can be translated into that of full row rank matrices in regular case (Theorem 4.1). Then, a constructive algorithm (Algorithm 1) for computing general factorizations of rank-deficient matrices has been proposed. Moreover, the previous results, such as ZLP, MLP, FLP factorizations of full row rank matrices, can be extended to the rank-deficient case.

In order to enable everyone to understand Algorithm 1 more intuitively, we deliberately avoid tricks and optimizations, such as the check of regularity for f' w.r.t. some matrices in the algorithm. We have implemented the algorithm on *Maple* with k of characteristic 0. For interested readers, more examples can be generated by the codes at: http://www.mmrc.iss.ac.cn/~dwang/software.html.

Although we have proposed a new method for studying general factorizations of rank-deficient matrices, there is a requirement: f' is regular w.r.t. $\mathbf{F}_1$ in Theorem 4.1. This implies that Theorem 4.1 depends on Theorem 2.14. Therefore, when the condition of regularity in Theorem 2.14 fails, how can we construct a general factorization of a full row rank matrix? If we can solve this problem, then the general factorizations of rank-deficient matrices will be further developed.

ACKNOWLEDGMENTS

This research was supported by the National Natural Science Foundation of China under Grant No. 12001030 and No. 12171469, the National Key Research and Development Project 2020YFA0712300, and the Fundamental Research Funds for the Central Universities 2682022CX048.

REFERENCES

[1] N.K. Bose. 2003. *Multidimensional Systems Theory and Applications* (second ed.). Kluwer Academic Publishers, Dordrecht. With contributions by B. Buchberger and J.P. Guiver.

[2] N.K. Bose. 2017. *Applied Multidimensional Systems Theory*. Springer, Cham. Second edition, With a preface by W.K. Jenkins, C. Lagoa and U. Srinivas.

[3] C. Charoenlarpnopparut and N. Bose. 1999. Multidimensional FIR filter bank design using Gröbner bases. *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing* 46, 12 (1999), 1475–1486.

[4] D. Cox, J. Little, and D. O'shea. 2005. *Using Algebraic Geometry*. Springer, New York.

[5] J. Deng, F. Chen, and L. Shen. 2005. Computing μ-bases of rational curves and surfaces using polynomial matrix factorization. In *Proceedings of the 2005 ACM International Symposium on Symbolic and Algebraic Computation*. ACM, New York, 132–139.

[6] D. Eisenbud. 2013. *Commutative Algebra: with a view toward algebraic geometry*. New York: Springer.

[7] A. Fabiańska and A. Quadrat. 2007. Applications of the Quillen-Suslin theorem to multidimensional systems theory. https://wwwb.math.rwth-aachen.de/QuillenSuslin/. *In: H. Park and G. Regensburger, (Eds.), Gröbner Bases in Control Theory and Signal Processing, Radon Series on Computational and Applied Mathematics* 3 (2007), 23–106.

[8] E. Fornasini and M.E. Valcher. 1997. nD polynomial matrices with applications to multidimensional signal analysis. *Multidimensional Systems and Signal Processing* 8 (1997), 387–408.

[9] G. Greuel and G. Pfister. 2002. *A SINGULAR Introduction to Commutative Algebra*. Springer-Verlag.

[10] J. Guan, W. Li, and B. Ouyang. 2018. On rank factorizations and factor prime factorizations for multivariate polynomial matrices. *Journal of Systems Science and Complexity* 31, 6 (2018), 1647–1658.

[11] J. Guan, W. Li, and B. Ouyang. 2019. On minor prime factorizations for multivariate polynomial matrices. *Multidimensional Systems and Signal Processing* 30, 1 (2019), 493–502.

[12] J.P. Guiver and N.K. Bose. 1982. Polynomial matrix primitive factorization over arbitrary coefficient field and related results. *IEEE Transactions on Circuits and Systems* 29, 10 (1982), 649–657.

[13] Z. Lin. 1988. On matrix fraction descriptions of multivariable linear n-D systems. *IEEE Transactions on Circuits and Systems* 35, 10 (1988), 1317–1322.

[14] Z. Lin. 1999. Notes on n-D polynomial matrix factorizations. *Multidimensional Systems and Signal Processing* 10, 4 (1999), 379–393.

[15] Z. Lin and N.K. Bose. 2001. A generalization of Serre's conjecture and some related issues. *Linear Algebra Appl.* 338, 1-3 (2001), 125–138.

[16] Z. Lin, M.S. Boudellioua, and L. Xu. 2006. On the equivalence and factorization of multivariate polynomial matrices. In *Proceedings of the 2006 IEEE International Symposium on Circuits and Systems*. 4911–4914.

[17] Z. Lin, L. Xu, and H. Fan. 2005. On minor prime factorizations for n-D polynomial matrices. *IEEE Transactions on Circuits and Systems II: Express Briefs* 52, 9 (2005), 568–571.

[18] J. Liu, D. Li, and M. Wang. 2011. On general factorizations for n-D polynomial matrices. *Circuits, Systems, and Signal Processing* 30, 3 (2011), 553–566.

[19] J. Liu and M. Wang. 2010. Notes on factor prime factorization for n-D polynomial matrices. *Multidimensional Systems and Signal Processing* 21, 1 (2010), 87–97.

[20] J. Liu and M. Wang. 2013. New results on multivariate polynomial matrix factorizations. *Linear Algebra Appl.* 438, 1 (2013), 87–95.

[21] J. Liu and M. Wang. 2015. Further remarks on multivariate polynomial matrix factorizations. *Linear Algebra Appl.* 465 (2015), 204–213.

[22] D. Lu, X. Ma, and D. Wang. 2017. A new algorithm for general factorizations of multivariate polynomial matrices. In *Proceedings of the 2017 ACM International Symposium on Symbolic and Algebraic Computation*. ACM, New York, 277–284.

[23] D. Lu, D. Wang, and F. Xiao. 2020. Factorizations for a class of multivariate polynomial matrices. *Multidimensional Systems and Signal Processing* 31, 3 (2020), 989–1004.

[24] D. Lu, D. Wang, and F. Xiao. 2021. On Factor Left Prime Factorization Problems for Multivariate Polynomial Matrices. *Multidimensional Systems and Signal Processing* 32, 3 (2021), 975–992.

[25] M. Morf, B.C. Lévy, and S.Y. Kung. 1977. New results in 2-D systems theory, Part I: 2-D polynomial matrices, factorization, and coprimeness. *Proc. IEEE* 65, 6 (1977), 861–872.

[26] R. Piziak and P.L. Odell. 1999. Full rank factorization of matrices. *Mathematics Magazine* 72, 3 (1999), 193–201.

[27] J.F. Pommaret. 2001. Solving Bose conjecture on linear multidimensional systems. In *Proceedings of the 2001 European Control Conference*. 1653–1655.

[28] V. Srinivas. 2004. A generalized Serre problem. *Journal of Algebra* 278, 2 (2004), 621–627.

[29] V. Sule. 1994. Feedback stabilization over commutative rings: the matrix case. *SIAM Journal on Control and Optimization* 32, 6 (1994), 1675–1695.

[30] M. Wang. 2007. On factor prime factorizations for n-D polynomial matrices. *IEEE Transactions on Circuits and Systems I: Regular Papers* 54, 6 (2007), 1398–1405.

[31] M. Wang. 2008. Remarks on n-D polynomial matrix factprization problems. *IEEE Transactions on Circuits and Systems II: Express Briefs* 55, 1 (2008), 61–64.

[32] M. Wang and D. Feng. 2004. On Lin-Bose problem. *Linear Algebra Appl.* 390 (2004), 279–285.

[33] M. Wang and C.P. Kwong. 2005. On multivariate polynomial matrix factorization problems. *Mathematics of Control, Signals, and Systems* 17, 4 (2005), 297–311.

[34] J. Wood, E. Rogers, and D. Owens. 1998. A formal theory of matrix primeness. *Mathematics of Control, Signals, and Systems* 11 (1998), 40–78.

[35] F. Xiao, D. Lu, and D. Wang. 2022. Solving multivariate polynomial matrix Diophantine equations with Gröbner basis method. *Journal of Systems Science and Complexity* 35 (2022), 413–426.

[36] D.C. Youla and G. Gnavi. 1979. Notes on n-dimensional system theory. *IEEE Transactions on Circuits and Systems* 26, 2 (1979), 105–111.