Update $E_{12} = E_{12} \cup \{\mathrm{PSC}_2(\mathcal{X}, \mathcal{X}')\} = \{u_1 - 4u_2, -8u_1^3(u_1 - 2u_2)\}$. By check, $\mathbb{V}_{\mathbb{C}}(E_{12}) \backslash \mathbb{V}_{\mathbb{C}}(N_{12}) = \emptyset$. This branch has been done and

$$B_3 := B_3 \cup A_{12} \\ = \{(\{0\}, \{u_1^7 u_2^3 (u_1 - 4u_2)^3\}, x_1), (\{u_1 - 4u_2\}, \{-8u_1^4 u_2 (u_1 - 2u_2)\}, x_1)\}.$$

(4) For the branch $(\{0\}, \{u_1^7 u_2^3 (u_1 - 4u_2)^3\}, x_1)$, we compute the parametric GCDs of $\mathcal{X}$ and $\mathcal{X}'$ corresponding to $t = x_1$. That is $\{(\{0\}, \{u_2 u_1 (u_1 - 4u_2)\}, 1)\}$. Then $\overline{\mathcal{X}_t} = \mathcal{X}_t$. By Formula (3),(4), $g_1 = 4T^3 - \frac{2(u_1 - 2u_2)T}{u_1}$, $g_{11} = \frac{2(u_1 - 2u_2)}{u_1}T^2 - \frac{4u_2^2}{u_1^2}$, $g_{12} = -\frac{2u_1}{u_2}T^3 + 2T$. Let $\mathcal{X}_1 = \mathcal{X}_t = T^4 - \frac{u_1 - 2u_2}{u_1}T^2 + \frac{u_2^2}{u_1^2}$, we obtain the RUR with parameters of this branch is $(\{0\}, \{u_1 u_2 (u_1 - 4u_2)\}, \mathcal{X}_1, g_1, g_{11}, g_{12})$.

For the branch $(\{u_1 - 4u_2\}, \{-8u_1^4 u_2 (u_1 - 2u_2)\}, x_1)$, we compute the parametric GCDs of $\mathcal{X}, \mathcal{X}'$ corresponding to $t = x_1$. That is

$$(\{u_1 - 4u_2\}, \{(u_1 - 2u_2)u_1 u_2\}, 4T^2 u_2^2 - u_2^2)$$

By computation, $\overline{\mathcal{X}_t} = T^2 - \frac{3u_1 - 8u_2}{4u_1}$, $g_2 = 4T$, $g_{21} = \frac{2(u_1 - 2u_2)}{u_1}$, $g_{22} = -\frac{2u_1 T}{u_2}$. Let $\mathcal{X}_2 = \mathcal{X}_t$. In this case, we can reduce $\mathcal{X}_2, g_2, g_{21}, g_{22}$ by the relation $u_1 - 4u_2 = 0$. Then we have $\mathcal{X}_2 = T^4 - \frac{1}{2}T^2 + \frac{1}{16}$, $g_2 = 4T$, $g_{21} = 1$, $g_{22} = -8T$. Thus, the RUR with parameters of this branch is $(\{u_1 - 4u_2\}, \{(u_1 - 2u_2)u_1 u_2\}, \mathcal{X}_2, g_2, g_{21}, g_{22})$.

In conclusion, we obtain two RURs of I under the parameter branch $(\{0\}, \{u_1 u_2\})$. That is $(\{0\}, \{u_1 u_2 (u_1 - 4u_2)\}, \mathcal{X}_1, g_1, g_{11}, g_{12})$ and $(\{u_1 - 4u_2\}, \{(u_1 - 2u_2)u_1 u_2\}, \mathcal{X}_2, g_2, g_{21}, g_{22})$. Therefore,

(i) For all $\bar{u} = (\bar{u}_1, \bar{u}_2) \in \mathbb{C}^2 \backslash \mathbb{V}_{\mathbb{C}}(u_1 u_2 (u_1 - 4u_2))$,

$$\mathbb{V}_{\mathbb{C}}(I(\bar{u})) = \left\{ \left(\frac{g_{11}(\bar{u}, \beta)}{g_1(\bar{u}, \beta)}, \frac{g_{12}(\bar{u}, \beta)}{g_1(\bar{u}, \beta)} \right) \Big| \ \beta \in \mathbb{V}_{\mathbb{C}} \left(T^4 - \frac{\bar{u}_1 - 2\bar{u}_2}{\bar{u}_1} T^2 + \frac{\bar{u}_2^2}{\bar{u}_1^2} \right) \right\}$$
$$= \left\{ \left(\frac{(\bar{u}_1^2 - 2\bar{u}_1\bar{u}_2)\beta^2 - 2\bar{u}_2^2}{2\beta^3\bar{u}_1^2 - (\bar{u}_1 - 2\bar{u}_2)\bar{u}_1\beta}, \frac{-\bar{u}_1^2\beta^2 + \bar{u}_1\bar{u}_2}{2\beta^2\bar{u}_1\bar{u}_2 - \bar{u}_2(\bar{u}_1 - 2\bar{u}_2)} \right) \Big| \right.$$
$$\left. \beta \in \mathbb{V}_{\mathbb{C}} \left(T^4 - \frac{\bar{u}_1 - 2\bar{u}_2}{\bar{u}_1} T^2 + \frac{\bar{u}_2^2}{\bar{u}_1^2} \right) \right\}.$$

(ii) For all $\bar{u} = (\bar{u}_1, \bar{u}_2) \in \mathbb{V}_{\mathbb{C}}(u_1 - 4u_2) \backslash \mathbb{V}_{\mathbb{C}}(u_1 u_2)$,

$$\mathbb{V}_{\mathbb{C}}(I(\bar{u})) = \left\{ \left(\frac{1}{4\beta}, -2 \right) \Big| \ \beta \in \mathbb{V}_{\mathbb{C}} \left(T^4 - \frac{1}{2}T^2 + \frac{1}{16} \right) \right\},$$

5 CONCLUDING REMARKS

The rational univariate representation of zero-dimensional ideals with parameters has been considered in the paper. Because the number of zeros for zero-dimensional ideals with parameters under parametric specializations is different, the choosing of separating elements which is the premise and the key to computing the rational univariate representation is quite difficult. By means of comprehensive Gröbner systems to divide the parameter space, we make the ideal under each branch have the same number of zeros. Moreover, we extended the subresultant theorem to parametric cases, and based on it we choose the separating element corresponding to each branch, which further divides the parameter space. As a result, making use of rational univariate representation and parametric greatest common divisors we obtained a finite set of which each branch shares the same expression of rational univariate representation and proposed the algorithm for computing rational univariate representations of parametric zero-dimensional ideals.

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REFERENCES

[1] W. Auzinger and H.J. Stetter. 1988. An elimination algorithm for the computation of all zeros of a system of multivariate polynomial equations. *International Series of Numerical Mathematics* 86 (1988), 11–30.
[2] T. Becker and V. Weispfenning. 1993. *Gröbner Bases.* Springer-Verlag.
[3] D. Cox, J. Little, and D. O'shea. 2005. *Using Algebraic Geometry.* Springer.
[4] David A Cox, John Little, and Donal O'shea. 2006. *Using algebraic geometry.* Vol. 185. Springer Science & Business Media.
[5] M. Kalkbrener. 1997. On the Stability of Gröbner Bases Under Specializations. *Journal of Symbolic Computation* 24, 1 (1997), 51–58.
[6] D. Kapur, D. Lu, M. Monagan, Y. Sun, and D.K. Wang. 2018. An Efficient Algorithm for Computing Parametric Multivariate Polynomial GCD. In *Proceedings of the 2018 International Symposium on Symbolic and Algebraic Computation.* 239–246.
[7] D. Kapur, Y. Sun, and D.K. Wang. 2013. An efficient algorithm for computing a comprehensive Gröbner system of a parametric polynomial systems. *Journal of Symbolic Computation* 49 (2013), 27–44.
[8] X.D. Ma, Y. Sun, and D.K. Wang. 2012. Computing polynomial univariate representations of zero-dimensional ideals by Gröbner basis. *Science China Mathematics* 55, 6 (2012), 1293–1302.
[9] X.D. Ma, Y. Sun, D.K. Wang, and Y.S. Xue. 2017. On Checking Linear Dependence of Parametric Vectors. In *Intelligent Computing Theories and Application.* 188–196.
[10] Bhubaneswar Mishra. 1993. *Algorithmic algebra.* Springer-Verlag, New York.
[11] A. Montes. 2002. A new algorithm for discussing Gröbner basis with parameters. *Journal of Symbolic Computation* 33 (2002), 183–208.
[12] K. Nabeshima. 2007. A speed-up of the algorithm for computing comprehensive Gröbner systems. In *Proceedings of the 2007 International Symposium on Symbolic and Algebraic Computation.* 299–306.
[13] K. Nagasaka. 2017. Parametric Greatest Common Divisors using Comprehensive Gröbner Systems. In *Proceedings of the 2017 International Symposium on Symbolic and Algebraic Computation.* 341–348.
[14] M. Noro and K. Yokoyama. 1999. A modular method to compute the rational univariate representation of zero-dimensional ideals. *Journal of Symbolic Computation* 28, 1-2 (1999), 243–263.
[15] K. Ouchi and J. Keyser. 2008. Rational univariate reduction via toric resultants. *Journal of Symbolic Computation* 43, 11 (2008), 811–844.
[16] F. Rouillier. 1999. Solving zero-dimensional systems through the rational univariate representation. *Applicable Algebra in Engineering Communication and Computing* 9, 5 (1999), 433–461.
[17] M. Safey El Din, Z.H. Yang, and L.H. Zhi. 2018. On the complexity of computing real radicals of polynomial systems. In *Proceedings of the 2018 International Symposium on Symbolic and Algebraic Computation.* 351–358.
[18] É. Schost. 2003. Computing parametric geometric resolutions. *Applicable Algebra in Engineering Communication and Computing* 13, 5 (2003), 349–393.
[19] B.X. Shang, S.G. Zhang, C. Tan, and P. Xia. 2017. A simplified rational representation for positive-dimensional polynomial systems and SHEPWM equation solving. *Journal of Systems Science and Complexity* 30 (2017), 1470–1482.
[20] H.J. Stetter. 1996. Matrix eigenproblem are at the heart of polynomial system solving. *ACM SIGSAM Bull.* 30 (1996), 27–36.
[21] A. Suzuki and Y. Sato. 2006. A simple algorithm to compute comprehensive Gröbner bases using Gröbner bases. In *Proceedings of the 2006 International Symposium on Symbolic and Algebraic Computation.* 326–331.
[22] C. Tan. 2009. *The rational representation for solving polynomial systems (in Chinese).* Ph.D. Dissertation.
[23] C. Tan and S.C. Zhang. 2009. Separating element computation for the rational univariate representation with short coefficients in zero-dimensional algebraic varieties. *Journal of Jilin University (Science Edition)* 47 (2009), 174–178.
[24] C. Tan and S.G. Zhang. 2010. Computation of the rational representation for solutions of high-dimensional systems. *Communications in Mathematical Research* 26, 2 (2010), 119–130.
[25] V. Weispfenning. 1992. Comprehensive Gröbner bases. *Journal of Symbolic Computation* 14, 1 (1992), 1–29.
[26] W.T. Wu. 1984. *Basic Principles of Mechanical Theorem Proving in Geometries (in Chinese), Vol. I:Part of Elementary Geometries.* Science Press.
[27] F.H. Xiao, D. Lu, X.D. Ma, and D.K. Wang. 2021. An Improvement of the Rational Representation for High-Dimensional Systems. *Journal of Systems Science and Complexity* 34, 6 (2021), 2410–2427.
[28] G.X. Zeng and S.J. Xiao. 2010. Computing the rational univariate representations for zero-dimensional systems by Wu's method. *Science China Mathematics(Chinese)* 40, 10 (2010), 999–1016.