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New remarks on the factorization and equivalence problems for a class of multivariate polynomial matrices

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ABSTRACT

This paper is concerned with the factorization and equivalence problems of multivariate polynomial matrices. We present some new criteria for the existence of matrix factorizations for a class of multivariate polynomial matrices, and obtain a necessary and sufficient condition for the equivalence of a square polynomial matrix and a diagonal matrix. Based on the constructive proof of the new criteria, we give a factorization algorithm and prove the uniqueness of the factorization. We implement the algorithm on Maple, and two illustrative examples are given to show the effectiveness of the algorithm.

1. Introduction

Multidimensional systems have wide applications in image, signal processing, control of networked systems, and other areas (see, e.g., Bose, 1982; Bose et al., 2003). A multidimensional system may be represented by a multivariate polynomial matrix, and we can obtain some important properties of the system by studying the corresponding matrix. Symbolic computation provides many effective theories and algorithms, such as module theory and Gröbner basis algorithms (Cox et al., 2005;

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Lin et al., 2008), on multidimensional systems. Therefore, great progress has been made on the factorization and equivalence problems related to multivariate polynomial matrices over the past decades.

Up to now, the factorization problem for univariate and bivariate polynomial matrices has been completely solved by Morf et al. (1977); Guiver and Bose (1982); Liu and Wang (2013), but the case of more than two variables is still open. Youla and Gnavi (1979) first introduced three important concepts according to different properties of multivariate polynomial matrices, namely zero prime matrix factorization, minor prime matrix factorization and factor prime matrix factorization. When multivariate polynomial matrices satisfy several special properties, there are some results about the existence problem of zero prime matrix factorizations for the matrices (see, e.g., Charoenlarpnopparut and Bose, 1999; Lin, 1999a, 2001). After that, Lin and Bose (2001) proposed the famous Lin-Bose conjecture: a multivariate polynomial matrix admits a zero prime matrix factorization if all its maximal reduced minors generate a unit ideal. This conjecture was proved by Pommaret (2001); Srinivas (2004); Wang and Feng (2004); Liu et al. (2014), respectively. Wang and Kwong (2005) gave a necessary and sufficient condition for a multivariate polynomial matrix with full rank to have a minor prime matrix factorization. They extracted an algorithm from Pommaret's proof of the Lin-Bose conjecture, and examples showed the effectiveness of the algorithm. Guan et al. (2019) generalized the main results in Wang and Kwong (2005) to the case of multivariate polynomial matrices without full rank. For the existence problem of factor prime matrix factorizations for multivariate polynomial matrices with full rank, Wang (2007) and Liu and Wang (2010) introduced the concept of regularity and obtained a necessary and sufficient condition. Guan et al. (2018) gave an algorithm to judge whether a multivariate polynomial matrix with the greatest common divisor of all its maximal minors being square-free has a factor prime matrix factorization. However, the existence problem for factor prime matrix factorizations of multivariate polynomial matrices remains a challenging open problem so far.

Comparing to the factorization problem of multivariate polynomial matrices which has been widely investigated during the past years, less attention has been paid to the equivalence problem of multivariate polynomial matrices. For any given multidimensional system, our goal is to simplify it into a simpler equivalent form.

Since a univariate polynomial ring is a principal ideal domain, a univariate polynomial matrix is always equivalent to its Smith form. This implies that the equivalence problem of univariate polynomial matrices has been solved (see, e.g., Rosenbrock, 1970; Kailath, 1993). For any given bivariate polynomial matrix, conditions under which it is equivalent to its Smith form have been investigated by Frost and Storey (1978); Lee and Zak (1983); Frost and Boudellioua (1986). Note that the equivalence problem of two multivariate polynomial matrices is equivalent to the isomorphism problem of two finitely presented modules. Boudellioua and Quadrat (2010) and Cluzeau and Quadrat (2008, 2013, 2015) obtained some important results by using module theory and homological algebra. According to the works of Boudellioua and Quadrat (2010), Boudellioua (2012, 2014) designed some algorithms based on Maple to compute Smith forms for some classes of multivariate polynomial matrices. For the case of multivariate polynomial matrices with more than one variable, however, the equivalence problem is not yet fully solved due to the lack of a mature polynomial matrix theory (see, e.g., Kung et al., 1977; Morf et al., 1977; Pugh et al., 1998).

From our personal viewpoint, new ideas need to be injected into these areas to obtain new theoretical results and effective algorithms. Therefore, it would be significant to provide some new criteria to study the factorization and equivalence problems for some classes of multivariate polynomial matrices.

From the 1990s to the present, there is a class of multivariate polynomial matrices that has always attracted attention. That is,

$$\mathcal{M} = \{\mathbf{F} \in k[\mathbf{z}]^{l\times m} : d_l(\mathbf{F}) \text{ has a divisor } z_1 - f(\mathbf{z}_2) \text{ for some } f(\mathbf{z}_2) \in k[\mathbf{z}_2]\},$$

where $l \leq m$, $\mathbf{z} = \{z_1, \ldots, z_n\}$ with $n \geq 3$, $\mathbf{z}_2 = \{z_2, \ldots, z_n\}$ and $d_l(\mathbf{F})$ is the greatest common divisor of all the $l \times l$ minors of $\mathbf{F}$. People tried to solve the factorization and equivalence problems of multivariate polynomial matrices in $\mathcal{M}$. Let $\mathbf{F} \in \mathcal{M}$ and $h = z_1 - f(\mathbf{z}_2)$. Many factorization criteria on the existence of a matrix factorization for $\mathbf{F}$ with respect to h have been proposed (see, e.g., Lin, 1993; Lin et al., 2001, 2005; Wang, 2008; Liu et al., 2011; Lu et al., 2020a). When $l = m$ and $\det(\mathbf{F}) = h$, Lin et al. (2006) proved that $\mathbf{F}$ is equivalent to its Smith form. After that, Li et al. (2017) studied the equivalence problem of a square matrix $\mathbf{F}$ with $\det(\mathbf{F}) = h^r$ and a diagonal matrix, where $r \geq 1$.

Through research, there are still many multivariate polynomial matrices in $\mathcal{M}$ without satisfying previous factorization criteria or equivalence conditions, but they can be factorized with respect to h or equivalent to simpler forms. As a consequence, we continue to study the factorization and equivalence problems of multivariate polynomial matrices in $\mathcal{M}$.

This paper is an extension of Lu et al. (2020b), and the contributions listed following are new. 1) Under the assumption that h is not a divisor of the greatest common divisor of all the $(l-1) \times (l-1)$ minors of $\mathbf{F}$, we give a necessary and sufficient condition for the existence of a matrix factorization of $\mathbf{F}$ with respect to h. 2) We summarize all factorization criteria for the existence of a matrix factorization of $\mathbf{F}$ with respect to h, and study the relationships among them. 3) For the case that h is a divisor of the greatest common divisor of all the $(l-1) \times (l-1)$ minors of $\mathbf{F}$, we obtain a sufficient condition for the existence of a matrix factorization of $\mathbf{F}$ with respect to h^r, where $2 \leq r \leq l$. 4) Based on the new factorization criteria, we construct a new factorization algorithm and implement it on Maple.

The rest of the paper is organized as follows. After a brief introduction to matrix factorization and matrix equivalence in Section 2, we use two examples to propose two problems that we shall consider. We present in Section 3 two criteria for factorizing $\mathbf{F}$ with respect to h, and then study the relationships among all existing factorization criteria. A necessary and sufficient condition for the equivalence of a square polynomial matrix and a diagonal matrix is described in Section 4. In Section 5, we generalize the main result in Section 3 to a more general case. In Section 6, we construct a factorization algorithm and study the uniqueness of matrix factorizations by the algorithm, and use two examples to illustrate the effectiveness of the algorithm in Section 7. The paper contains a summary of contributions and some remarks in Section 8.

2. Preliminaries and problems

In this section we first recall some basic notions which will be used in the following sections, and then we use two examples to put forward two problems that we are considering.

2.1. Basic notions

We denote by k an algebraically closed field. Let $k[\mathbf{z}]$ and $k[\mathbf{z}_2]$ be the polynomial ring in variables $\mathbf{z}$ and $\mathbf{z}_2$ with coefficients in k, respectively. Let $k[\mathbf{z}]^{l \times m}$ be the set of $l \times m$ matrices with entries in $k[\mathbf{z}]$. Throughout the paper, we assume that $l \leq m$, and use uppercase bold letters to denote polynomial matrices. In addition, "w.r.t." stands for "with respect to".

Let $\mathbf{F} \in k[\mathbf{z}]^{l \times m}$, we use $d_i(\mathbf{F})$ to denote the greatest common divisor of all the $i \times i$ minors of $\mathbf{F}$ with the convention that $d_0(\mathbf{F}) = 1$, where $i = 1, \dots, l$. Let $f \in k[\mathbf{z}_2]$, then $\mathbf{F}(f, \mathbf{z}_2)$ denotes a polynomial matrix in $k[\mathbf{z}_2]^{l \times m}$ which is formed by transforming z_1 in $\mathbf{F}$ into f.

Definition 1 *(Lin, 1988; Sule, 1994).* Let $\mathbf{F} \in k[\mathbf{z}]^{l \times m}$ with rank r, where $1 \leq r \leq l$. For any given integer i with $1 \leq i \leq r$, let $a_1, \dots, a_\beta$ denote all the $i \times i$ minors of $\mathbf{F}$, where $\beta = \binom{l}{i} \cdot \binom{m}{i}$. Extracting $d_i(\mathbf{F})$ from $a_1, \dots, a_\beta$ yields $a_j = d_i(\mathbf{F}) \cdot b_j$ with $j = 1, \dots, \beta$, then $b_1, \dots, b_\beta$ are called the $i \times i$ reduced minors of $\mathbf{F}$.

Lin (1988) showed that reduced minors are important invariants for polynomial matrices.

Lemma 2. *Let $\mathbf{F}_1 \in k[\mathbf{z}]^{r \times t}$ be of full row rank, $b_1, \dots, b_\gamma$ be all the $r \times r$ reduced minors of $\mathbf{F}_1$, and $\mathbf{F}_2 \in k[\mathbf{z}]^{t \times (t-r)}$ be of full column rank, $\bar{b}_1, \dots, \bar{b}_\gamma$ be all the $(t-r) \times (t-r)$ reduced minors of $\mathbf{F}_2$, where $r < t$ and $\gamma = \binom{t}{r}$. If $\mathbf{F}_1\mathbf{F}_2 = \mathbf{0}_{r \times (t-r)}$, then $\bar{b}_i = \pm b_i$ for $i = 1, \dots, \gamma$, and signs depend on indices.*

Let $\mathbf{F} \in k[\mathbf{z}]^{l \times m}$ with rank r, where $1 \leq r \leq l$. Let $\bar{\mathbf{F}}_1, \dots, \bar{\mathbf{F}}_\eta \in k[\mathbf{z}]^{l \times r}$ be all the full column rank submatrices of $\mathbf{F}$, where $1 \leq \eta \leq \binom{m}{r}$. According to Lemma 2, it is easy to prove that $\bar{\mathbf{F}}_1, \dots, \bar{\mathbf{F}}_\eta$ have the same $r \times r$ reduced minors. Based on this phenomenon, we give the following concept.

Definition 3. Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ with rank r, and $\bar{\mathbf{F}} \in k[\mathbf{z}]^{l\times r}$ be any of full column rank submatrices of $\mathbf{F}$, where $1 \le r \le l$. Let $c_1, \ldots, c_\xi$ be all the $r \times r$ reduced minors of $\bar{\mathbf{F}}$, where $\xi = \binom{l}{r}$. Then $c_1, \ldots, c_\xi$ are called the $r \times r$ **column** reduced minors of $\mathbf{F}$.

We can define the $r \times r$ **row** reduced minors of $\mathbf{F}$ in the same way. To state conveniently problems and main conclusions of this paper, we introduce the following concepts and results.

Definition 4. Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank.

1. If all the $l \times l$ minors of $\mathbf{F}$ generate $k[\mathbf{z}]$, then $\mathbf{F}$ is said to be a zero left prime (ZLP) matrix.
2. If all the $l \times l$ minors of $\mathbf{F}$ are relatively prime, i.e., $d_l(\mathbf{F})$ is a nonzero constant in k, then $\mathbf{F}$ is said to be a minor left prime (MLP) matrix.
3. If for any polynomial matrix factorization $\mathbf{F} = \mathbf{F}_1\mathbf{F}_2$ with $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times l}$, $\mathbf{F}_1$ is necessarily a unimodular matrix, i.e., $\det(\mathbf{F}_1)$ is a nonzero constant in k, then $\mathbf{F}$ is said to be a factor left prime (FLP) matrix.

We refer to Youla and Gnavi (1979) for more details about the relationships among ZLP, MLP and FLP. Quillen (1976) and Suslin (1976) solved the Serre's conjecture raised by Serre (1955), respectively. This result is called Quillen-Suslin theorem, and it is as follows.

Lemma 5. *If $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ is a ZLP matrix, then a unimodular matrix $\mathbf{U} \in k[\mathbf{z}]^{m\times m}$ can be constructed such that $\mathbf{F}$ is its first l rows.*

There are many algorithms for the Quillen-Suslin theorem, we refer to Youla and Pickel (1984); Logar and Sturmfels (1992); Park (1995) for more details. Fabiańska and Quadrat (2007) first designed a Maple package, which is called QUILLENSUSLIN, to implement the Quillen-Suslin theorem.

Let W be a $k[\mathbf{z}]$-module generated by $\vec{u}_1, \ldots, \vec{u}_l \in k[\mathbf{z}]^{1\times m}$. The set of all $(b_1, \ldots, b_l) \in k[\mathbf{z}]^{1\times l}$ such that $b_1\vec{u}_1 + \cdots + b_l\vec{u}_l = \vec{0}$ is a $k[\mathbf{z}]$-module of $k[\mathbf{z}]^{1\times l}$, is called the (first) syzygy module of W, and denoted by $\mathrm{Syz}(W)$. Lin (1999b) proposed several interesting structural properties of syzygy modules. Let $\mathbf{F} = \left[\vec{u}_1^{\mathrm{T}}, \ldots, \vec{u}_l^{\mathrm{T}}\right]^{\mathrm{T}}$. The rank of W is defined as the rank of $\mathbf{F}$ that is denoted by $\mathrm{rank}(\mathbf{F})$. Guan et al. (2018) proved that the rank of W does not depend on the choice of generators of W.

Lemma 6. *With above notations. If $\mathrm{rank}(W) = r$ with $1 \le r \le l$, then the rank of $\mathrm{Syz}(W)$ is $l - r$.*

Proof. Let $k(\mathbf{z})$ be the fraction field of $k[\mathbf{z}]$, and $\mathrm{Syz}^*(W) = \{\vec{v} \in k(\mathbf{z})^{1\times l} : \vec{v} \cdot \mathbf{F} = \vec{0}\}$. Then, $\mathrm{Syz}^*(W)$ is a $k(\mathbf{z})$-vector space of dimension $l - r$. For any given $l - r + 1$ different vectors $\vec{v}_1, \ldots, \vec{v}_{l-r+1} \in k[\mathbf{z}]^{1\times l}$ in $\mathrm{Syz}(W)$, $\vec{v}_i \in \mathrm{Syz}^*(W)$ for each i, and they are $k(\mathbf{z})$-linearly dependent. This implies that $\vec{v}_1, \ldots, \vec{v}_{l-r+1}$ are $k[\mathbf{z}]$-linearly dependent. Thus $\mathrm{rank}(\mathrm{Syz}(W)) \le l - r$.

Assume that $\vec{p}_1, \ldots, \vec{p}_{l-r} \in k(\mathbf{z})^{1\times l}$ are $l - r$ vectors in $\mathrm{Syz}^*(W)$, and they are $k(\mathbf{z})$-linearly independent. For each j, we have $p_{j1}\vec{u}_1 + \cdots + p_{jl}\vec{u}_l = \vec{0}$, where $\vec{p}_j = (p_{j1}, \ldots, p_{jl})$. Multiplying both sides of the equation by the least common multiple of the denominators of $p_{j1}, \ldots, p_{jl}$, we obtain $\bar{p}_j = (\bar{p}_{j1}, \ldots, \bar{p}_{jl}) \in k[\mathbf{z}]$ such that $\bar{p}_{j1}\vec{u}_1 + \cdots + \bar{p}_{jl}\vec{u}_l = \vec{0}$. Then, $\bar{p}_j \in \mathrm{Syz}(W)$, where $j = 1, \ldots, l - r$. Moreover, $\bar{p}_1, \ldots, \bar{p}_{l-r}$ are $k[\mathbf{z}]$-linearly independent. Thus, $\mathrm{rank}(\mathrm{Syz}(W)) \ge l - r$.

As a consequence, the rank of $\mathrm{Syz}(W)$ is $l - r$ and the proof is completed. $\square$

Remark 7. The above lemma implies that the number of vectors in any given generators of $\mathrm{Syz}(W)$ is greater than or equal to $l - r$.

Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ with rank r, where $1 \le r \le l$. For each $1 \le i \le r$, we use $I_i(\mathbf{F})$ to denote the ideal generated by all the $i \times i$ minors of $\mathbf{F}$. For convenience, let $I_0(\mathbf{F}) = k[\mathbf{z}]$. Moreover, we denote the submodule of $k[\mathbf{z}]^{1\times m}$ generated by all the row vectors of $\mathbf{F}$ by $\mathrm{Im}(\mathbf{F})$.

Definition 8. Let W be a finitely generated $k[\mathbf{z}]$-module, and $k[\mathbf{z}]^{1\times l} \xrightarrow{\phi} k[\mathbf{z}]^{1\times m} \to W \to 0$ be a presentation of W, where ϕ acts on the right on row vectors, i.e., $\phi(\vec{u}) = \vec{u}\cdot\mathbf{F}$ for $\vec{u}\in k[\mathbf{z}]^{1\times l}$ with $\mathbf{F}$ being a presentation matrix corresponding to the linear mapping ϕ. Then the ideal $Fitt_j(W) = I_{m-j}(\mathbf{F})$ is called the j-th Fitting ideal of W. Here, we make the convention that $Fitt_j(W) = k[\mathbf{z}]$ for $j\geq m$, and that $Fitt_j(W) = 0$ for $j < \max\{m-l, 0\}$.

We remark that $Fitt_j(W)$ only depends on W (see, e.g., Greuel and Pfister, 2002; Eisenbud, 2013). In addition, the chain $0 = Fitt_{-1}(W) \subseteq Fitt_0(W) \subseteq \ldots \subseteq Fitt_m(W) = k[\mathbf{z}]$ of Fitting ideals is increasing. Cox et al. (2005) showed that one obtains the presentation matrix $\mathbf{F}$ for W by arranging the generators of $\mathrm{Syz}(W)$ as rows.

2.2. Matrix factorization problem

A matrix factorization of a multivariate polynomial matrix is formulated as follows.

Definition 9. Let $\mathbf{F}\in k[\mathbf{z}]^{l\times m}$ and $h_0 \mid d_l(\mathbf{F})$. $\mathbf{F}$ is said to admit a matrix factorization w.r.t. h_0 if $\mathbf{F}$ can be factorized as

$$\mathbf{F} = \mathbf{G}_1\mathbf{F}_1 \tag{1}$$

such that $\mathbf{G}_1\in k[\mathbf{z}]^{l\times l}$ with $\det(\mathbf{G}_1) = h_0$ and $\mathbf{F}_1\in k[\mathbf{z}]^{l\times m}$. In particular, Equation (1) is said to be a ZLP (MLP, FLP) matrix factorization if $\mathbf{F}_1$ is a ZLP (MLP, FLP) matrix.

Throughout the paper, let $h = z_1 - f(\mathbf{z}_2)$ with $f(\mathbf{z}_2)\in k[\mathbf{z}_2]$. This paper will address the following specific matrix factorization problem.

Problem 10. Let $\mathbf{F}\in\mathcal{M}$. Under what conditions does $\mathbf{F}$ have a matrix factorization w.r.t. h.

Several results related to Problem 10 have been given, and the latest progress on this problem was obtained by Lu et al. (2020a).

Lemma 11. *Let $\mathbf{F}\in\mathcal{M}$. If $h\nmid d_{l-1}(\mathbf{F})$ and the ideal generated by h and all the $(l-1)\times(l-1)$ reduced minors of $\mathbf{F}$ is $k[\mathbf{z}]$, then $\mathbf{F}$ admits a matrix factorization w.r.t. h.*

Although Lemma 11 gives a criterion to determine whether $\mathbf{F}$ has a matrix factorization w.r.t. h, we found that there exist some polynomial matrices in $\mathcal{M}$ which do not satisfy the conditions of Lemma 11, but still admit matrix factorizations w.r.t. h.

Example 12. Let

$$\mathbf{F} = \begin{bmatrix} -2z_1z_2^2 + z_1^2z_3 + z_2^2z_3 - z_1z_3^2 + z_2z_3^2 & z_1^3 - z_2^3 - z_1^2z_3 + z_2z_3^2 & z_1z_2 - z_2z_3 & z_2^2 \\ -z_1z_2 + z_3^2 & -z_2^2 + z_1z_3 & 0 & z_2 \end{bmatrix}$$

be a polynomial matrix in $\mathbb{C}[z_1, z_2, z_3]^{2\times 4}$, where $z_1 > z_2 > z_3$ and $\mathbb{C}$ is the complex field.

It is easy to compute that $d_2(\mathbf{F}) = z_2(z_1 - z_3)$ and $d_1(\mathbf{F}) = 1$. Let $h = z_1 - z_3$, then $h \mid d_2(\mathbf{F})$ implies that $\mathbf{F}\in\mathcal{M}$. Obviously, $h\nmid d_1(\mathbf{F})$. Since $d_1(\mathbf{F}) = 1$, the entries in $\mathbf{F}$ are all the 1×1 reduced minors of $\mathbf{F}$. Let $\prec_{\mathbf{z}}$ be the degree reverse lexicographic order, then the reduced Gröbner basis G of the ideal generated by h and all the 1×1 reduced minors of $\mathbf{F}$ w.r.t. $\prec_{\mathbf{z}}$ is $\{z_1 - z_3, z_2, z_3^2\}$. It follows from $G\neq\{1\}$ that Lemma 11 cannot be applied.

However, $\mathbf{F}$ admits a matrix factorization w.r.t. h, i.e., there exist $\mathbf{G}_1\in\mathbb{C}[z_1, z_2, z_3]^{2\times 2}$ and $\mathbf{F}_1\in\mathbb{C}[z_1, z_2, z_3]^{2\times 4}$ such that

$$\mathbf{F} = \mathbf{G}_1\mathbf{F}_1 = \begin{bmatrix} h & z_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} z_1z_3 - z_2^2 & z_1^2 - z_2z_3 & z_2 & 0 \\ -z_1z_2 + z_3^2 & -z_2^2 + z_1z_3 & 0 & z_2 \end{bmatrix},$$

where $\det(\mathbf{G}_1) = h$.

From the above example we see that Problem 10 is far from being resolved. So, in Section 3 we make a detailed analysis on this problem.

2.3. Matrix equivalence problem

Now we introduce the concept of the equivalence of two multivariate polynomial matrices.

Definition 13. Two polynomial matrices $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$ and $\mathbf{F}_2 \in k[\mathbf{z}]^{l\times m}$ are said to be equivalent if there exist two unimodular matrices $\mathbf{U} \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{V} \in k[\mathbf{z}]^{m\times m}$ such that

$$\mathbf{F}_1 = \mathbf{U}\mathbf{F}_2\mathbf{V}. \tag{2}$$

In fact, a univariate polynomial matrix is always equivalent to its Smith form. However, this result is not valid for the case of more than one variable, and there are many counter-examples (see, e.g., Lee and Zak, 1983; Boudellioua, 2013). Hence, people began to consider under what conditions multivariate polynomial matrices are equivalent to simpler forms. Li et al. (2017) investigated the equivalence problem for a class of multivariate polynomial matrices and obtained the following result.

Lemma 14. *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times l}$ with $\det(\mathbf{F}) = h^r$, where $h = z_1 - f(\mathbf{z}_2)$ and r is a positive integer. Then $\mathbf{F}$ is equivalent to $\mathrm{diag}(h^r, 1, \ldots, 1)$ if and only if h^r and all the $(l-1)\times(l-1)$ minors of $\mathbf{F}$ generate $k[\mathbf{z}]$.*

For a given square matrix that does not satisfy the condition of Lemma 14, we use the following example to illustrate that it can be equivalent to another diagonal matrix.

Example 15. Let

$$\mathbf{F} = \begin{bmatrix} z_1z_2 - z_2^2 + z_2z_3 + z_2 - z_3 - 1 & z_1z_2z_3 - z_2^2z_3 + z_1z_2 - z_2^2 + z_2z_3 - z_3 & z_1z_2z_3 - z_2^2z_3 \\ z_1z_2 - z_2^2 + z_1 - z_2 + z_3 + 1 & (z_1 - z_2)(z_2z_3 + 2z_2 + z_3 + 1) + z_3 & \mathbf{F}[2,3] \\ z_1 - z_2 & z_1z_3 - z_2z_3 + 2z_1 - 2z_2 & z_1z_3 - z_2z_3 + z_1 - z_2 \end{bmatrix}$$

be a polynomial matrix in $\mathbb{C}[z_1, z_2, z_3]^{3\times 3}$, where $\mathbf{F}[2,3] = z_1z_2z_3 - z_2^2z_3 + z_1z_2 - z_2^2 + z_1z_3 - z_2z_3$.

It is easy to compute that $\det(\mathbf{F}) = (z_1 - z_2)^2$. Let $h = z_1 - z_2$ and $\prec_{\mathbf{z}}$ be the degree reverse lexicographic order, then the reduced Gröbner basis G of the ideal generated by h^2 and all the 2×2 minors of $\mathbf{F}$ w.r.t. $\prec_{\mathbf{z}}$ is $\{z_1 - z_2\}$. It follows from $G \neq \{1\}$ that Lemma 14 cannot be applied.

However, $\mathbf{F}$ is equivalent to $\mathrm{diag}(h, h, 1)$, i.e., there exist two unimodular polynomial matrices $\mathbf{U} \in \mathbb{C}[z_1, z_2, z_3]^{3\times 3}$ and $\mathbf{V} \in \mathbb{C}[z_1, z_2, z_3]^{3\times 3}$ such that

$$\mathbf{F} = \mathbf{U}\cdot \mathrm{diag}(h, h, 1)\cdot \mathbf{V} = \begin{bmatrix} 0 & z_2 & z_2 - 1 \\ z_2 & z_2 + 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & z_3 + 1 & z_3 \\ z_3 + 1 & z_3 & 0 \end{bmatrix}.$$

Based on the phenomenon of Example 15, we consider the following matrix equivalence problem.

Problem 16. Let $\mathbf{F} \in k[\mathbf{z}]^{l\times l}$ with $\det(\mathbf{F}) = h^r$, where $h = z_1 - f(\mathbf{z}_2)$ and $1 \le r \le l$. What is the necessary and sufficient condition for the equivalence of $\mathbf{F}$ and $\mathrm{diag}(\underbrace{h, \ldots, h}_{r}, \underbrace{1, \ldots, 1}_{l-r})$?

3. Factorization for polynomial matrices

In this section, we first propose two criteria to judge whether $\mathbf{F} \in \mathcal{M}$ has a matrix factorization w.r.t. h, and then study the relationships among all existing factorization criteria.

3.1. A sufficient condition

We first recall two lemmas.

Lemma 17 *(Wang and Feng, 2004). Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ with rank r such that all the $r \times r$ reduced minors of $\mathbf{F}$ generate $k[\mathbf{z}]$. Then there exist $\mathbf{G}_1 \in k[\mathbf{z}]^{l\times r}$ and $\mathbf{F}_1 \in k[\mathbf{z}]^{r\times m}$ such that $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ with $\mathbf{F}_1$ being a ZLP matrix.*

Lemma 18 *(Lin et al., 2001). Let $p \in k[\mathbf{z}]$ and $f(\mathbf{z}_2) \in k[\mathbf{z}_2]$. Then $z_1 - f(\mathbf{z}_2)$ is a divisor of p if and only if $p(f, \mathbf{z}_2)$ is a zero polynomial in $k[\mathbf{z}_2]$.*

Now, we propose a sufficient condition to factorize $\mathbf{F}$ w.r.t. h.

Theorem 19. *Let $\mathbf{F} \in \mathcal{M}$ and $W = \mathrm{Im}(\mathbf{F}(f, \mathbf{z}_2))$. If $Fitt_{l-2}(W) = 0$ and $Fitt_{l-1}(W) = \langle d\rangle$ with $d \in k[\mathbf{z}_2] \setminus \{0\}$, then $\mathbf{F}$ admits a matrix factorization w.r.t. h.*

Proof. Let $k[\mathbf{z}_2]^{1\times s} \xrightarrow{\phi} k[\mathbf{z}_2]^{1\times l} \to W \to 0$ be a presentation of W, and $\mathbf{H} \in k[\mathbf{z}_2]^{s\times l}$ be a matrix corresponding to the linear mapping ϕ. Then $\mathrm{Syz}(W) = \mathrm{Im}(\mathbf{H})$.

It follows from $Fitt_{l-2}(W) = 0$ that all the 2×2 minors of $\mathbf{H}$ are zero polynomials. Then, $\mathrm{rank}(\mathbf{H}) \leq 1$. Moreover, $Fitt_{l-1}(W) = \langle d\rangle$ with $d \in k[\mathbf{z}_2]\setminus\{0\}$ implies that $\mathrm{rank}(\mathbf{H}) \geq 1$. As a consequence, we have $\mathrm{rank}(\mathbf{H}) = 1$.

Let $a_1, \dots, a_\beta \in k[\mathbf{z}_2]$ and $b_1, \dots, b_\beta \in k[\mathbf{z}_2]$ be all the 1×1 minors and 1×1 reduced minors of $\mathbf{H}$, respectively. Then, $a_i = d_1(\mathbf{H}) \cdot b_i$ for $i = 1, \dots, \beta$. Since $\langle a_1, \dots, a_\beta\rangle = \langle d\rangle$, it is obvious that $d \mid d_1(\mathbf{H})$. Moreover, we have $d = \sum_{i=1}^{\beta} c_i a_i$ for some $c_i \in k[\mathbf{z}_2]$. Thus $d = d_1(\mathbf{H}) \cdot (\sum_{i=1}^{\beta} c_i b_i)$. This implies that $d_1(\mathbf{H}) \mid d$. Hence $d = \delta \cdot d_1(\mathbf{H})$, where δ is a nonzero constant. Therefore, $\langle b_1, \dots, b_\beta\rangle = k[\mathbf{z}_2]$.

According to Lemma 17, there exist $\vec{u} \in k[\mathbf{z}_2]^{s\times 1}$ and $\vec{w} \in k[\mathbf{z}_2]^{1\times l}$ such that $\mathbf{H} = \vec{u}\vec{w}$ with $\vec{w}$ being a ZLP vector. It follows from $\mathrm{Syz}(W) = \mathrm{Im}(\mathbf{H})$ that $\vec{u}\vec{w}\mathbf{F}(f, \mathbf{z}_2) = \mathbf{0}_{s\times m}$. Since $\vec{u}$ is a column vector, we have $\vec{w}\mathbf{F}(f, \mathbf{z}_2) = \mathbf{0}_{1\times m}$.

Using the Quillen-Suslin theorem, we can construct a unimodular matrix $\mathbf{U} \in k[\mathbf{z}_2]^{l\times l}$ such that $\vec{w}$ is its first row. Let $\mathbf{F}_0 = \mathbf{U}\mathbf{F}$, then the first row of $\mathbf{F}_0(f, \mathbf{z}_2) = \mathbf{U}\mathbf{F}(f, \mathbf{z}_2)$ is zero vector. By Lemma 18, h is a common divisor of the polynomials in the first row of $\mathbf{F}_0$, thus

$$\mathbf{F}_0 = \mathbf{U}\mathbf{F} = \mathbf{D}\mathbf{F}_1 = \mathrm{diag}(h, \underbrace{1, \dots, 1}_{l-1}) \cdot \begin{bmatrix} \bar{f}_{11} & \bar{f}_{12} & \cdots & \bar{f}_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{f}_{l1} & \bar{f}_{l2} & \cdots & \bar{f}_{lm} \end{bmatrix}.$$

Consequently, we can now derive the matrix factorization of $\mathbf{F}$ w.r.t. h, i.e., $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$, where $\mathbf{G}_1 = \mathbf{U}^{-1}\mathbf{D} \in k[\mathbf{z}]^{l\times l}$, $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$ and $\det(\mathbf{G}_1) = h$. $\square$

Remark 20. In the above theorem, d is actually a nonzero constant. The reason is as follows. Since $\vec{w}\mathbf{F}(f, \mathbf{z}_2) = \mathbf{0}_{1\times m}$, we have $\vec{w} \in \mathrm{Syz}(W)$. As $\mathrm{Syz}(W) = \mathrm{Im}(\mathbf{H})$, this implies that there is a nonzero vector $\vec{v} \in k[\mathbf{z}_2]^{1\times s}$ such that $\vec{w} = \vec{v}\mathbf{H}$. It follows from $\mathbf{H} = \vec{u}\vec{w}$ that $\vec{v}\vec{u} = 1$. Therefore, $\vec{u}$ is a ZRP vector. Since $d_1(\mathbf{H}) = d_1(\vec{u}) \cdot d_1(\vec{w})$, $d_1(\mathbf{H})$ is a nonzero constant. Noting that $d = \delta \cdot d_1(\mathbf{H})$, where δ is a nonzero constant. Then, d is a nonzero constant.

According to Remark 20, Theorem 19 is equivalent to the following result.

Theorem 21. *Let $\mathbf{F} \in \mathcal{M}$ and $W = \mathrm{Im}(\mathbf{F}(f, \mathbf{z}_2))$. If $Fitt_{l-2}(W) = 0$ and $Fitt_{l-1}(W) = k[\mathbf{z}_2]$, then $\mathbf{F}$ admits a matrix factorization w.r.t. h.*

3.2. A necessary and sufficient condition for a special case

In Theorem 21, the conditions $Fitt_{l-2}(W) = 0$ and $Fitt_{l-1}(W) \neq 0$ imply that the rank of $\mathbf{F}(f, \mathbf{z}_2)$ is $l-1$. In the following, we first give a lemma about the necessary and sufficient condition for $\text{rank}(\mathbf{F}(f, \mathbf{z}_2)) = l-1$.

Lemma 22. *Let $\mathbf{F} \in \mathcal{M}$. Then $\text{rank}(\mathbf{F}(f, \mathbf{z}_2)) = l-1$ if and only if $h \nmid d_{l-1}(\mathbf{F})$.*

Proof. Since $h \mid d_l(\mathbf{F})$, we have $\text{rank}(\mathbf{F}(f, \mathbf{z}_2)) \leq l-1$. Let $a_1, \ldots, a_\gamma \in k[\mathbf{z}]$ be all the $(l-1) \times (l-1)$ minors of $\mathbf{F}$, then $a_1(f, \mathbf{z}_2), \ldots, a_\gamma(f, \mathbf{z}_2)$ are all the $(l-1) \times (l-1)$ minors of $\mathbf{F}(f, \mathbf{z}_2)$.

If $\text{rank}(\mathbf{F}(f, \mathbf{z}_2)) = l-1$, then there is at least one integer i such that $a_i(f, \mathbf{z}_2)$ is a nonzero polynomial. According to Lemma 18, h is not a divisor of a_i. Obviously, $h \nmid d_{l-1}(\mathbf{F})$.

Assume that $h \nmid d_{l-1}(\mathbf{F})$. If $\text{rank}(\mathbf{F}(f, \mathbf{z}_2)) < l-1$, then $a_j(f, \mathbf{z}_2) = 0$ for $j = 1, \ldots, \gamma$. This implies that h is a common divisor of $a_1, \ldots, a_\gamma$, which leads to a contradiction. $\square$

Lemma 23 *(Lin et al., 2005). Let $\mathbf{G} \in k[\mathbf{z}]^{l \times l}$ with $\det(\mathbf{G}) = h$, then there is a ZLP vector $\vec{w} \in k[\mathbf{z}_2]^{1 \times l}$ such that $\vec{w}\mathbf{G}(f, \mathbf{z}_2) = \mathbf{0}_{1 \times l}$.*

Now, we give a partial solution to Problem 10.

Theorem 24. *Let $\mathbf{F} \in \mathcal{M}$ with $h \nmid d_{l-1}(\mathbf{F})$. Then the following are equivalent:*

1. $\mathbf{F}$ *admits a matrix factorization w.r.t. h;*
2. *all the $(l-1) \times (l-1)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$.*

Proof. $1 \to 2$. Suppose $\mathbf{F}$ admits a matrix factorization w.r.t. h, then there exist $\mathbf{G}_1 \in k[\mathbf{z}]^{l \times l}$ and $\mathbf{F}_1 \in k[\mathbf{z}]^{l \times m}$ such that $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ with $\det(\mathbf{G}_1) = h$. Obviously, $\mathbf{F}(f, \mathbf{z}_2) = \mathbf{G}_1(f, \mathbf{z}_2)\mathbf{F}_1(f, \mathbf{z}_2)$. Since $\det(\mathbf{G}_1) = h$, by Lemma 23 there is a ZLP vector $\vec{w} \in k[\mathbf{z}_2]^{1 \times l}$ such that $\vec{w}\mathbf{G}_1(f, \mathbf{z}_2) = \mathbf{0}_{1 \times l}$. This implies that $\vec{w}\mathbf{F}(f, \mathbf{z}_2) = \mathbf{0}_{1 \times m}$. According to Lemma 22, we have $\text{rank}(\mathbf{F}(f, \mathbf{z}_2)) = l-1$. Using Lemma 2, all the $(l-1) \times (l-1)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ are equivalent to all the 1×1 reduced minors of $\vec{w}$. It follows that all the $(l-1) \times (l-1)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$.

$2 \to 1$. Since $\text{rank}(\mathbf{F}(f, \mathbf{z}_2)) = l-1$, there is a nonzero vector $\vec{w} = [w_1, \ldots, w_l] \in k[\mathbf{z}_2]^{1 \times l}$ such that $\vec{w}\mathbf{F}(f, \mathbf{z}_2) = \mathbf{0}_{1 \times m}$. As all the $(l-1) \times (l-1)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$, all the 1×1 reduced minors of $\vec{w}$ generate $k[\mathbf{z}_2]$ by Lemma 2. Assume that $w_0 \in k[\mathbf{z}_2]$ is the greatest common divisor of $w_1, \ldots, w_l$, then $\vec{w}/w_0$ is a ZLP vector. Using the Quillen-Suslin theorem, we can construct a unimodular matrix $\mathbf{U} \in k[\mathbf{z}_2]^{l \times l}$ such that $\vec{w}/w_0$ is its first row. This implies that there exist $\mathbf{D} \in k[\mathbf{z}]^{l \times l}$ and $\mathbf{F}_1 \in k[\mathbf{z}]^{l \times m}$ such that $\mathbf{U}\mathbf{F} = \mathbf{D}\mathbf{F}_1$, where $\mathbf{D} = \text{diag}(h, 1, \ldots, 1)$. Therefore, we obtain a matrix factorization of $\mathbf{F}$ w.r.t. h, i.e., $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$, where $\mathbf{G}_1 = \mathbf{U}^{-1}\mathbf{D}$ and $\det(\mathbf{G}_1) = h$. $\square$

3.3. Comparison among all existing factorization criteria

Let $\mathbf{F} \in \mathcal{M}$, and $a_1, \ldots, a_\beta \in k[\mathbf{z}]$ be all the $l \times l$ minors of $\mathbf{F}$. Since $h \mid d_l(\mathbf{F})$, there are $e_1, \ldots, e_\beta \in k[\mathbf{z}]$ such that $a_i = he_i$, $i = 1, \ldots, \beta$. Lin et al. (2001) proved that $\mathbf{F}$ has a matrix factorization w.r.t. h if $\langle h, e_1, \ldots, e_\beta \rangle = k[\mathbf{z}]$. The main idea is as follows. $\langle h, e_1, \ldots, e_\beta \rangle = k[\mathbf{z}]$ implies that $\text{rank}(\mathbf{F}(f, \mathbf{z}_2)) = l-1$ for every $\mathbf{z}_2 \in k^{n-1}$. It follows that $h \nmid d_{l-1}(\mathbf{F})$ and there is a ZLP vector $\vec{w} \in k[\mathbf{z}_2]^{1 \times l}$ such that $\vec{w}\mathbf{F}(f, \mathbf{z}_2) = \mathbf{0}_{1 \times m}$. Hence, the condition $\langle h, e_1, \ldots, e_\beta \rangle = k[\mathbf{z}]$ is a special case of Theorem 24.

When $d_l(\mathbf{F}) = h$, Lin et al. (2005) proved that $\mathbf{F}$ has an MLP matrix factorization w.r.t. h if and only if all the $(l-1) \times (l-1)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$. Moreover, Wang (2008) proved that $\mathbf{F}$ has an MLP matrix factorization w.r.t. h if and only if there is a ZLP vector $\vec{w} \in k[\mathbf{z}_2]^{1 \times l}$ such that $\vec{w}\mathbf{F}(f, \mathbf{z}_2) = \mathbf{0}_{1 \times m}$. It is easy to see that the above two results are equivalent. In fact, $d_l(\mathbf{F}) = h$ implies that $h \nmid d_{l-1}(\mathbf{F})$. Hence, these results are also a special case of Theorem 24.

Let $c_1, \ldots, c_\eta \in k[\mathbf{z}]$ be all the $(l-1) \times (l-1)$ minors of $\mathbf{F}$. Liu et al. (2011) proved that $\text{rank}(\mathbf{F}(f, \mathbf{z}_2)) = l-1$ for every $\mathbf{z}_2 \in k^{n-1}$ if and only if $\langle h, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$. Then, $\mathbf{F}$ has a matrix factorization w.r.t. h if $\langle h, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$. Although Liu et al. (2011) generalized the main result of Lin et al. (2001), $\langle h, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$ is still a special case of Theorem 24.

Let $b_1, \ldots, b_\eta \in k[\mathbf{z}]$ be all the $(l-1) \times (l-1)$ reduced minors of $\mathbf{F}$. Lu et al. (2020a) proved that $\mathbf{F}$ has a matrix factorization w.r.t. h if $h \nmid d_{l-1}(\mathbf{F})$ and $\langle h, b_1, \ldots, b_\eta \rangle = k[\mathbf{z}]$. We explain the difference between $\langle h, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$ and $\langle h, b_1, \ldots, b_\eta \rangle = k[\mathbf{z}]$. $\langle h, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$ implies that all the $(l-1) \times (l-1)$ minors of $\mathbf{F}(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$, and $\langle h, b_1, \ldots, b_\eta \rangle = k[\mathbf{z}]$ implies that all the $(l-1) \times (l-1)$ reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$. Therefore, the main result of Lu et al. (2020a) is a generalization of that of Liu et al. (2011). Under the assumption that $h \nmid d_{l-1}(\mathbf{F})$, there is no doubt that $\langle h, b_1, \ldots, b_\eta \rangle = k[\mathbf{z}]$ is a special case of Theorem 24.

Assume that $h \nmid d_{l-1}(\mathbf{F})$ and $\langle h, b_1, \ldots, b_\eta \rangle = k[\mathbf{z}]$. Suppose the rows of $\mathbf{H} \in k[\mathbf{z}_2]^{s \times l}$ represent generators of $\mathrm{Syz}(W)$, where $W = \mathrm{Im}(\mathbf{F}(f, \mathbf{z}_2))$. As $h \nmid d_{l-1}(\mathbf{F})$, we have $\mathrm{rank}(\mathbf{H}) = 1$. Suppose $\vec{w}' \in k[\mathbf{z}_2]^{1 \times l}$ is any row vector of $\mathbf{H}$. $\langle h, b_1, \ldots, b_\eta \rangle = k[\mathbf{z}]$ implies that the 1×1 reduced minors of $\vec{w}'$ generate $k[\mathbf{z}_2]$. It follows that the 1×1 row reduced minors of $\mathbf{H}$ generate $k[\mathbf{z}_2]$. According to Lemma 10 in Lu et al. (2021), there exist $\vec{u} \in k[\mathbf{z}_2]^{s \times 1}$ and $\vec{w} \in k[\mathbf{z}_2]^{1 \times l}$ such that $\mathbf{H} = \vec{u}\vec{w}$ with $\vec{w}$ being a ZLP vector. Based on Remark 20, we can prove that $I_1(\mathbf{H}) = k[\mathbf{z}_2]$. Therefore, we have $Fitt_{l-2}(W) = 0$ and $Fitt_{l-1}(W) = k[\mathbf{z}_2]$. In addition, Example 12 shows that Theorem 21 can solve some problems that the main result in Lu et al. (2020a) cannot solve. It follows that Theorem 21 is a generalization of the main result in Lu et al. (2020a).

On the one hand, $Fitt_{l-2}(W) = 0$ and $Fitt_{l-1}(W) \neq 0$ in Theorem 21 imply that $\mathrm{rank}(\mathbf{F}(f, \mathbf{z}_2)) = l - 1$. Hence, $h \nmid d_{l-1}(\mathbf{F})$. Moreover, $Fitt_{l-1}(W) = k[\mathbf{z}_2]$ implies that the 1×1 reduced minors of $\mathbf{H}$ generate $k[\mathbf{z}_2]$. According to Lemma 2, all the $(l-1) \times (l-1)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$.

On the other hand, If $h \nmid d_{l-1}(\mathbf{F})$ and all the $(l-1) \times (l-1)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$ in Theorem 24, we can use the method in the above paragraph to prove that $Fitt_{l-2}(W) = 0$ and $Fitt_{l-1}(W) = k[\mathbf{z}_2]$. As a consequence, $Fitt_{l-1}(W) = k[\mathbf{z}_2]$ in Theorem 21 is equivalent to the sufficient condition in Theorem 24 under the premise that $Fitt_{l-2}(W) = 0$. It is easy to see that the conditions of Theorem 24 are easier to verify in the actual calculation process, which can help us improve the computational efficiency.

Based on Lemma 17, Liu and Wang (2013) proposed a criterion for the existence of a matrix factorization of $\mathbf{F}$ w.r.t. h_0.

Lemma 25. *Let $\mathbf{F} \in k[\mathbf{z}]^{l \times m}$ be a full row rank matrix, $h_0 \in k[\mathbf{z}]$ be a divisor of $d_l(\mathbf{F})$, $a_1, \ldots, a_\beta \in k[\mathbf{z}]$ and $c_1, \ldots, c_\eta \in k[\mathbf{z}]$ be all the $l \times l$ minors and $(l-1) \times (l-1)$ minors of $\mathbf{F}$, respectively. There are $e_1, \ldots, e_\beta \in k[\mathbf{z}]$ such that $a_i = h_0 e_i$, where $i = 1, \ldots, \beta$. If $h_0, e_1, \ldots, e_\beta, c_1, \ldots, c_\eta$ generate $k[\mathbf{z}]$, then $\mathbf{F}$ has a matrix factorization w.r.t. h_0.*

In Lemma 25, $\mathbf{F}$ has no restriction and h_0 does not have to be of the form $z_1 - f(\mathbf{z}_2)$. Obviously, the main results of Lin et al. (2001) and Liu et al. (2011) are special cases of Lemma 25.

When $h_0 = z_1 - f(\mathbf{z}_2)$, however, we find that $\langle h_0, e_1, \ldots, e_\beta, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$ is equivalent to $\langle h_0, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$. That is, Lemma 25 is the same as the main result of Liu et al. (2011) for the case of $h_0 = z_1 - f(\mathbf{z}_2)$. Before proving this conclusion, we first introduce a lemma which was proposed by Lin et al. (2001).

Lemma 26. *Let $\mathbf{F} \in k[z_1]^{l \times m}$ be a univariate polynomial matrix with full row rank, and $d \in k[z_1]$ be the greatest common divisor of all the $l \times l$ minors of $\mathbf{F}$. If $z_{11} \in k$ is a simple zero of d, i.e., $z_1 - z_{11}$ is a divisor of d, but $(z_1 - z_{11})^2$ is not a divisor of d, then $\mathrm{rank}(\mathbf{F}(z_{11})) = l - 1$.*

Now, we can assert that the following conclusion is correct.

Proposition 27. *Let $\mathbf{F} \in \mathcal{M}$, $a_1, \ldots, a_\beta \in k[\mathbf{z}]$ and $c_1, \ldots, c_\eta \in k[\mathbf{z}]$ be all the $l \times l$ minors and $(l-1) \times (l-1)$ minors of $\mathbf{F}$, respectively. There are $e_1, \ldots, e_\beta \in k[\mathbf{z}]$ such that $a_i = h e_i$, $i = 1, \ldots, \beta$. Then, $\langle h, e_1, \ldots, e_\beta, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$ if and only if $\langle h, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$.*

Proof. On the one hand, it is easy to see that $\langle h, e_1, \ldots, e_\beta, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$ if $\langle h, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$.

On the other hand, assume that $\langle h, e_1, \ldots, e_\beta, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$. If $\langle h, c_1, \ldots, c_\eta \rangle \neq k[\mathbf{z}]$, then there exists a point $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n) \in k^n$ such that

$$\varepsilon_1 = f(\varepsilon_2, \ldots, \varepsilon_n) \text{ and } c_i(\vec{\varepsilon}) = 0, \ \ i = 1, \ldots, \eta.$$

It follows that $\mathrm{rank}(F(\vec{\varepsilon})) < l - 1$. Let $\tilde{\mathbf{F}} = \mathbf{F}(z_1, \varepsilon_2, \ldots, \varepsilon_n)$ be a univariate polynomial matrix with entries in $k[z_1]$, and $\tilde{a}_1, \ldots, \tilde{a}_\beta \in k[z_1]$ be all the $l \times l$ minors of $\tilde{\mathbf{F}}$. Obviously, we have

$$\tilde{a}_j = a_j(z_1, \varepsilon_2, \ldots, \varepsilon_n) = (z_1 - \varepsilon_1) \cdot e_j(z_1, \varepsilon_2, \ldots, \varepsilon_n), \ \ j = 1, \ldots, \beta.$$

Assume that $q \in k[z_1]$ is the greatest common divisor of $e_1(z_1, \varepsilon_2, \ldots, \varepsilon_n), \ldots, e_\beta(z_1, \varepsilon_2, \ldots, \varepsilon_n)$, then $d_l(\tilde{\mathbf{F}}) = (z_1 - \varepsilon_1) \cdot q$. It follows from $\langle h, e_1, \ldots, e_\beta, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$ that $\vec{\varepsilon}$ is not a common zero of the system $\{e_1 = 0, \ldots, e_\beta = 0\}$. Thus, ε_1 is not a zero of p. This implies that ε_1 is a simple zero of $d_l(\tilde{\mathbf{F}})$. According to Lemma 26, we have $\mathrm{rank}(\tilde{\mathbf{F}}(\varepsilon_1)) = l - 1$, which leads to a contradiction. Therefore, $\langle h, c_1, \ldots, c_\eta \rangle = k[\mathbf{z}]$. □

4. Equivalence for polynomial matrices

In this section, we first put forward a necessary and sufficient condition to solve Problem 16, and then use an example to illustrate the effectiveness of the matrix equivalence theorem.

We introduce a lemma, which is called the Binet-Cauchy formula (Strang, 1980).

Lemma 28. *Let $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$, where $\mathbf{G}_1 \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$. Then an $i \times i$ minor of $\mathbf{F}$ is*

$$\det\left(\mathbf{F}\begin{pmatrix} r_1 \cdots r_i \\ j_1 \cdots j_i \end{pmatrix}\right) = \sum_{1 \le s_1 < \cdots < s_i \le l} \det\left(\mathbf{G}_1\begin{pmatrix} r_1 \cdots r_i \\ s_1 \cdots s_i \end{pmatrix}\right) \cdot \det\left(\mathbf{F}_1\begin{pmatrix} s_1 \cdots s_i \\ j_1 \cdots j_i \end{pmatrix}\right).$$

In Lemma 28, $\mathbf{F}\begin{pmatrix} r_1 \cdots r_i \\ j_1 \cdots j_i \end{pmatrix}$ denotes an $i \times i$ submatrix consisting of the $r_1, \ldots, r_i$ rows and $j_1, \ldots, j_i$ columns of $\mathbf{F}$. Based on this lemma, we can obtain the following two results.

Lemma 29. *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ be of full row rank with $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$, where $\mathbf{G}_1 \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$. Then $d_i(\mathbf{F}_1) \mid d_i(\mathbf{F})$ and $d_i(\mathbf{G}_1) \mid d_i(\mathbf{F})$ for each $i \in \{1, \ldots, l\}$.*

Proof. We only prove $d_i(\mathbf{F}_1) \mid d_i(\mathbf{F})$, since the proof of $d_i(\mathbf{G}_1) \mid d_i(\mathbf{F})$ follows in a similar manner. For any given $i \in \{1, \ldots, l\}$, let $a_{i,1}, \ldots, a_{i,t_i}$ and $\bar{a}_{i,1}, \ldots, \bar{a}_{i,t_i}$ be all the $i \times i$ minors of $\mathbf{F}$ and $\mathbf{F}_1$ respectively, where $t_i = \binom{l}{i}\binom{m}{i}$. For each $a_{i,j}$, it is a $k[\mathbf{z}]$-linear combination of $\bar{a}_{i,1}, \ldots, \bar{a}_{i,t_i}$ by using Lemma 28, where $j = 1, \ldots, t_i$. Since $d_i(\mathbf{F}_1)$ is the greatest common divisor of $\bar{a}_{i,1}, \ldots, \bar{a}_{i,t_i}$, for each j we have $d_i(\mathbf{F}_1) \mid a_{i,j}$. Then, $d_i(\mathbf{F}_1) \mid d_i(\mathbf{F})$. □

Lemma 30. *Let $\mathbf{F}_1, \mathbf{F}_2 \in k[\mathbf{z}]^{l\times m}$ be of full row rank. If $\mathbf{F}_1$ and $\mathbf{F}_2$ are equivalent, then $d_i(\mathbf{F}_1) = d_i(\mathbf{F}_2)$ for each $i \in \{1, \ldots, l\}$.*

Proof. Since $\mathbf{F}_1$ and $\mathbf{F}_2$ are equivalent, there exist two unimodular matrices $\mathbf{U} \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{V} \in k[\mathbf{z}]^{m\times m}$ such that $\mathbf{F}_1 = \mathbf{U}\mathbf{F}_2\mathbf{V}$. For each $i \in \{1, \ldots, l\}$, it follows from Lemma 29 that $d_i(\mathbf{F}_2) \mid d_i(\mathbf{U}\mathbf{F}_2) \mid d_i(\mathbf{F}_1)$. Furthermore, we have $\mathbf{F}_2 = \mathbf{U}^{-1}\mathbf{F}_1\mathbf{V}^{-1}$ since $\mathbf{U}$ and $\mathbf{V}$ are two unimodular matrices. Similarly, we obtain $d_i(\mathbf{F}_1) \mid d_i(\mathbf{U}^{-1}\mathbf{F}_1) \mid d_i(\mathbf{F}_2)$. Therefore, $d_i(\mathbf{F}_1) = d_i(\mathbf{F}_2)$ up to multiplication by a nonzero constant. □

Lemma 31 *(Lu et al., 2017). Let $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$ with rank $l - r$. If all the $(l-r) \times (l-r)$ minors of $\mathbf{F}$ generate $k[\mathbf{z}]$, then there exists a ZLP matrix $\mathbf{H} \in k[\mathbf{z}]^{r\times l}$ such that $\mathbf{H}\mathbf{F} = \mathbf{0}_{r\times m}$.*

Combining Lemma 31 and the Quillen-Suslin theorem, we can now solve Problem 16.

Theorem 32. *Let $\mathbf{F} \in k[\mathbf{z}]^{l\times l}$ with $\det(\mathbf{F}) = h^r$, where $h = z_1 - f(\mathbf{z}_2)$ and $1 \le r \le l$. Then the following are equivalent:*

1. $\mathbf{F}$ *is equivalent to* $\mathrm{diag}(\underbrace{h,\dots,h}_{r},\underbrace{1,\dots,1}_{l-r})$*;*
2. $h \mid d_{l-r+1}(\mathbf{F})$ *and the ideal generated by h and all the* $(l-r)\times(l-r)$ *minors of* $\mathbf{F}$ *is* $k[\mathbf{z}]$.

Proof. For convenience, let $\mathbf{D} = \mathrm{diag}(h,\dots,h,1,\dots,1)$ and $\bar{\mathbf{F}} = \mathbf{F}(f,\mathbf{z}_2)$. Let $a_1,\dots,a_\beta$ be all the $(l-r)\times(l-r)$ minors of $\mathbf{F}$. It is obvious that $a_1(f,\mathbf{z}_2),\dots,a_\beta(f,\mathbf{z}_2)$ are all the $(l-r)\times(l-r)$ minors of $\bar{\mathbf{F}}$.

$2 \to 1$. It follows from $h \mid d_{l-r+1}(\mathbf{F})$ that $\mathrm{rank}(\bar{\mathbf{F}}) \le l-r$. Assume that there exists a point $(\varepsilon_2,\dots,\varepsilon_n) \in k^{1\times(n-1)}$ such that $a_i(f(\varepsilon_2,\dots,\varepsilon_n),\varepsilon_2,\dots,\varepsilon_n) = 0$, where $i = 1,\dots,\beta$. Let $\varepsilon_1 = f(\varepsilon_2,\dots,\varepsilon_n)$, then $(\varepsilon_1,\varepsilon_2,\dots,\ \varepsilon_n) \in k^{1\times n}$ is a common zero of the polynomial system $\{h = 0, a_1 = 0,\dots,a_\beta = 0\}$. This contradicts the fact that h and all the $(l-r)\times(l-r)$ minors of $\mathbf{F}$ generate $k[\mathbf{z}]$. Then, all the $(l-r)\times(l-r)$ minors of $\bar{\mathbf{F}}$ generate $k[\mathbf{z}_2]$. According to Lemma 31, there exists a ZLP matrix $\mathbf{H} \in k[\mathbf{z}_2]^{r\times l}$ such that $\mathbf{H}\bar{\mathbf{F}} = \mathbf{0}_{r\times l}$. Based on the Quillen-Suslin theorem, we can construct a unimodular matrix $\mathbf{U} \in k[\mathbf{z}_2]^{l\times l}$ such that $\mathbf{H}$ is its first r rows. Then, there is $\mathbf{V} \in k[\mathbf{z}]^{l\times l}$ such that $\mathbf{UF} = \mathbf{DV}$. Since $\det(\mathbf{F}) = h^r$ and $\mathbf{U}$ is a unimodular matrix, we have $\mathbf{F} = \mathbf{U}^{-1}\mathbf{DV}$ and $\mathbf{V}$ is a unimodular matrix. Therefore, $\mathbf{F}$ and $\mathbf{D}$ are equivalent.

$1 \to 2$. If $\mathbf{F}$ and $\mathbf{D}$ are equivalent, then there exist two unimodular matrices $\mathbf{U} \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{V} \in k[\mathbf{z}]^{l\times l}$ such that $\mathbf{F} = \mathbf{UDV}$. It follows from Lemma 30 that $d_{l-r+1}(\mathbf{F}) = d_{l-r+1}(\mathbf{D}) = h$. If $\langle h, a_1,\dots,a_\beta\rangle \ne k[\mathbf{z}]$, then there exists a point $\vec{\varepsilon} \in k^{1\times n}$ such that $h(\vec{\varepsilon}) = 0$ and $\mathrm{rank}(\mathbf{F}(\vec{\varepsilon})) < l-r$. Obviously, $\mathrm{rank}(\mathbf{D}(\vec{\varepsilon})) = l-r$ and $\mathrm{rank}(\mathbf{U}^{-1}(\vec{\varepsilon})) = \mathrm{rank}(\mathbf{V}^{-1}(\vec{\varepsilon})) = l$. Since $\mathbf{D} = \mathbf{U}^{-1}\mathbf{F}\mathbf{V}^{-1}$, we have $\mathrm{rank}(\mathbf{D}(\vec{\varepsilon})) \le \min\{\mathrm{rank}(\mathbf{U}^{-1}(\vec{\varepsilon})), \mathrm{rank}(\mathbf{F}(\vec{\varepsilon})), \mathrm{rank}(\mathbf{V}^{-1}(\vec{\varepsilon}))\}$, which leads to a contradiction. Therefore, $\langle h, a_1,\dots,a_\beta\rangle = k[\mathbf{z}]$ and the proof is completed. □

Remark 33. When $r = l$ in Theorem 32, we just need to check whether h is a divisor of $d_1(\mathbf{F})$.

Now, we use Example 15 to illustrate a constructive method which follows the proof process of $2 \to 1$ of Theorem 32 and we explain how to obtain the two unimodular matrices associated with equivalent matrices.

Example 34. Let $\mathbf{F}$ be the same polynomial matrix as in Example 15. It is easy to compute that $\det(\mathbf{F}) = (z_1 - z_2)^2$ and $d_2(\mathbf{F}) = z_1 - z_2$. Let $h = z_1 - z_2$, it is obvious that $h \mid d_2(\mathbf{F})$. The reduced Gröbner basis of the ideal generated by h and all the 1×1 minors of $\mathbf{F}$ w.r.t. $\prec_{\mathbf{z}}$ is $\{1\}$. Then, $\mathbf{F}$ is equivalent to $\mathrm{diag}(h,h,1)$. Note that

$$\mathbf{F}(z_2,z_2,z_3) = \begin{bmatrix} (z_3+1)(z_2-1) & z_3(z_2-1) & 0 \\ z_3+1 & z_3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$\mathrm{rank}(\mathbf{F}(z_2,z_2,z_3)) = 1$. Let $W = \mathrm{Im}(\mathbf{F}(z_2,z_2,z_3))$. We compute a system of generators of the syzygy module of W, and obtain

$$\mathbf{H} = \begin{bmatrix} 1 & -z_2+1 & z_2^2 - z_2 \\ -1 & z_2 - 1 & -z_2^2 + z_2 + 1 \end{bmatrix}$$

such that $\mathbf{H}\cdot\mathbf{F}(z_2,z_2,z_3) = \mathbf{0}_{2\times 3}$. It is easy to check that $\mathbf{H}$ is a ZLP matrix. Then, a unimodular matrix $\mathbf{U} \in k[\mathbf{z}_2]^{3\times 3}$ can be constructed such that $\mathbf{H}$ is its first 2 rows by using the Maple package QUILLENSUSLIN, where

$$\mathbf{U} = \begin{bmatrix} 1 & -z_2+1 & z_2^2 - z_2 \\ -1 & z_2 - 1 & -z_2^2 + z_2 + 1 \\ -1 & z_2 & -z_2^2 \end{bmatrix}.$$

Now we can extract h from the first 2 rows of $\mathbf{UF}$, and get

$$\mathbf{F} = \mathbf{U}^{-1} \cdot \mathrm{diag}(h, h, 1) \cdot \mathbf{V} = \begin{bmatrix} 0 & z_2 & z_2 - 1 \\ z_2 & z_2 + 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & z_3 + 1 & z_3 \\ z_3 + 1 & z_3 & 0 \end{bmatrix}.$$

5. Generalizations

We construct the following two sets of polynomial matrices:

$$\mathcal{M}_1 = \{\mathbf{F} \in \mathcal{M} : h \nmid d_{l-1}(\mathbf{F})\} \text{ and } \mathcal{M}_2 = \{\mathbf{F} \in \mathcal{M} : h \mid d_{l-1}(\mathbf{F})\}.$$

Let $\mathbf{F} \in \mathcal{M}$ and $h = z_1 - f(\mathbf{z}_2)$ be given, then $\mathbf{F} \in \mathcal{M}_1$ or $\mathbf{F} \in \mathcal{M}_2$. If $\mathbf{F} \in \mathcal{M}_1$, we can use Theorem 24 to judge whether $\mathbf{F}$ has a matrix factorization w.r.t. h. If $\mathbf{F} \in \mathcal{M}_2$, we need to propose some criteria to factorize $\mathbf{F}$.

Since $d_0(\mathbf{F}) \mid d_1(\mathbf{F}) \mid \cdots \mid d_{l-1}(\mathbf{F}) \mid d_l(\mathbf{F})$, there exists a unique integer r with $1 \le r \le l$ such that $h \mid d_{l-r+1}(\mathbf{F})$ but $h \nmid d_{l-r}(\mathbf{F})$. Based on this fact, we subdivide $\mathcal{M}_2$ into the following sets:

$$\mathcal{M}_{2,r} = \{\mathbf{F} \in \mathcal{M}_2 : h \mid d_{l-r+1}(\mathbf{F}) \text{ but } h \nmid d_{l-r}(\mathbf{F})\},\ r = 2, \ldots, l.$$

Lemma 35. *Let $\mathbf{F} \in \mathcal{M}_2$. Then $\mathrm{rank}(\mathbf{F}(f, \mathbf{z}_2)) = l - r$ with $2 \le r \le l$ if and only if $\mathbf{F} \in \mathcal{M}_{2,r}$.*

The proof of Lemma 35 is basically the same as that of Lemma 22, so it is omitted here. Inspired by Theorem 24 and Theorem 32, we propose the following result for the existence of a matrix factorization of $\mathbf{F} \in \mathcal{M}_{2,r}$ w.r.t. h^r, where $2 \le r < l$.

Theorem 36. *Let $\mathbf{F} \in \mathcal{M}_{2,r}$ with $2 \le r < l$, then the following are equivalent:*

1. *there exist $\mathbf{G}_1 \in k[\mathbf{z}]^{l\times l}$ and $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$ such that $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ with $\mathbf{G}_1$ being equivalent to $\mathrm{diag}(\underbrace{h, \ldots, h}_{r}, \underbrace{1, \ldots, 1}_{l-r})$;*
2. *all the $(l-r) \times (l-r)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$.*

Proof. $1 \to 2$. Since $\mathbf{G}_1$ and $\mathrm{diag}(h, \ldots, h, 1, \ldots, 1)$ are equivalent, we have $h \mid d_{l-r+1}(\mathbf{G}_1)$ and $\langle h, g_1, \ldots, g_\eta\rangle = k[\mathbf{z}]$ by Theorem 32, where $g_1, \ldots, g_\eta$ are all the $(l-r) \times (l-r)$ minors of $\mathbf{G}_1$. This implies that all the $(l-r) \times (l-r)$ minors of $\mathbf{G}_1(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$. According to Lemma 31, we can construct a ZLP matrix $\mathbf{W} \in k[\mathbf{z}_2]^{r\times l}$ such that $\mathbf{W}\mathbf{G}_1(f, \mathbf{z}_2) = \mathbf{0}_{r\times l}$. It follows from $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ that $\mathbf{W}\mathbf{F}(f, \mathbf{z}_2) = \mathbf{0}_{r\times m}$. Since $\mathbf{W}$ is a ZLP matrix, all the $(l-r) \times (l-r)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$.

$2 \to 1$. From Lemma 35, there exists a full row rank matrix $\mathbf{H} \in k[\mathbf{z}_2]^{r\times l}$ such that $\mathbf{HF}(f, \mathbf{z}_2) = \mathbf{0}_{r\times m}$. Since all the $(l-r) \times (l-r)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$, all the $r \times r$ reduced minors of $\mathbf{H}$ generate $k[\mathbf{z}_2]$ by Lemma 2. Using Lemma 17, $\mathbf{H}$ has a ZLP matrix factorization $\mathbf{H} = \mathbf{H}_1\mathbf{H}_2$, where $\mathbf{H}_1 \in k[\mathbf{z}_2]^{r\times r}$, and $\mathbf{H}_2 \in k[\mathbf{z}_2]^{r\times l}$ is a ZLP matrix. As $\mathbf{H}_1$ is a full column rank matrix, it follows from $\mathbf{HF}(f, \mathbf{z}_2) = \mathbf{0}_{r\times m}$ that $\mathbf{H}_2\mathbf{F}(f, \mathbf{z}_2) = \mathbf{0}_{r\times m}$. Using the Quillen-Suslin theorem, we can construct a unimodular matrix $\mathbf{U} \in k[\mathbf{z}_2]^{l\times l}$ such that $\mathbf{H}_2$ is its first r rows. This implies that there is $\mathbf{F}_1 \in k[\mathbf{z}]^{l\times m}$ such that $\mathbf{UF} = \mathbf{DF}_1$, where $\mathbf{D} = \mathrm{diag}(h, \ldots, h, 1, \ldots, 1)$ with $\det(\mathbf{D}) = h^r$. Therefore, we obtain a matrix factorization of $\mathbf{F}$ w.r.t. h^r, i.e., $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ with $\mathbf{G}_1 = \mathbf{U}^{-1}\mathbf{D}$. Obviously, $\mathbf{G}_1$ is equivalent to $\mathbf{D}$. □

Remark 37. In Theorem 36, the matrix factorization $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ must satisfy that $\mathbf{G}_1$ is equivalent to $\mathrm{diag}(h, \ldots, h, 1, \ldots, 1)$. Since there exist many polynomial matrices such that their matrix factorizations do not satisfy this requirement, the condition "all the $(l-r) \times (l-r)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$" is only a sufficient condition for the existence of a matrix factorization of $\mathbf{F} \in \mathcal{M}_{2,r}$ w.r.t. h^r, where $2 \le r < l$.

Algorithm 1: Factorization algorithm.

Input : $\mathbf{F} \in \mathcal{M}$, $h = z_1 - f(\mathbf{z}_2)$ and a monomial order $\prec_{\mathbf{z}_2}$ in $k[\mathbf{z}_2]$.
Output: a matrix factorization of $\mathbf{F}$ w.r.t. h^r, where $1 \leq r \leq l$.

1 **begin**
2 compute the rank $l - r$ of $\mathbf{F}(f, \mathbf{z}_2)$;
3 **if** $r = l$ **then**
4 extract h from each row of $\mathbf{F}$ and obtain $\mathbf{F}_1$, i.e., $\mathbf{F} = \mathrm{diag}(h, \dots, h) \cdot \mathbf{F}_1$;
5 **return** $\mathrm{diag}(h, \dots, h)$ and $\mathbf{F}_1$.
6 compute a reduced Gröbner basis $\mathcal{G}$ of all the $(l - r) \times (l - r)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ w.r.t. $\prec_{\mathbf{z}_2}$;
7 **if** $\mathcal{G} \neq \{1\}$ **then**
8 **if** $r = 1$ **then**
9 **return** $\mathbf{F}$ has no matrix factorization w.r.t. h.
10 **else**
11 **return** unable to judge.
12 compute a ZLP matrix $\mathbf{H} \in k[\mathbf{z}_2]^{r \times l}$ such that $\mathbf{H}\mathbf{F}(f, \mathbf{z}_2) = \mathbf{0}_{r \times m}$;
13 construct a unimodular matrix $\mathbf{U} \in k[\mathbf{z}_2]^{l \times l}$ such that $\mathbf{H}$ is its first r rows;
14 compute $\mathbf{F}_1 \in k[\mathbf{z}]^{l \times m}$ such that $\mathbf{U}\mathbf{F} = \mathrm{diag}(h, \dots, h, 1, \dots, 1) \cdot \mathbf{F}_1$;
15 **return** $\mathbf{U}^{-1} \cdot \mathrm{diag}(h, \dots, h, 1, \dots, 1)$ and $\mathbf{F}_1$.

Theorem 38. *Let $\mathbf{F} \in \mathcal{M}_{2,l}$, then h is a common divisor of all entries in $\mathbf{F}$. We can extract h from each row of $\mathbf{F}$ and obtain a matrix factorization of $\mathbf{F}$ w.r.t. h^l.*

Let $k[\bar{\mathbf{z}}_i] = k[z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n]$ and $h_i = z_i - f(\bar{\mathbf{z}}_i)$, where $f(\bar{\mathbf{z}}_i) \in k[\bar{\mathbf{z}}_i]$ and $1 \leq i \leq n$. We construct the following sets of polynomial matrices:

$$\mathcal{M}^{(i,r)} = \{\mathbf{F} \in k[\mathbf{z}]^{l \times m} : h_i \mid d_{l-r+1}(\mathbf{F}) \ \text{but}\ h_i \nmid d_{l-r}(\mathbf{F})\},\ r = 1, \dots, l.$$

Then, we can get the following corollary.

Corollary 39. *Let $\mathbf{F} \in \mathcal{M}^{(i,r)}$, where $1 \leq i \leq n$ and $1 \leq r \leq l$. If all the $(l - r) \times (l - r)$ column reduced minors of $\mathbf{F}(z_1, \dots, z_{i-1}, f, z_{i+1}, \dots, z_n)$ generate $k[\bar{\mathbf{z}}_i]$, then $\mathbf{F}$ admits a matrix factorization w.r.t. h_i^r.*

6. Factorization algorithm and uniqueness of factorizations

In this section, we first propose an algorithm to factorize $\mathbf{F} \in \mathcal{M}$ w.r.t. h^r, where $1 \leq r \leq l$. And then, we study the uniqueness of matrix factorizations by the algorithm.

6.1. Factorization algorithm

According to Theorem 24, Theorem 36 and Theorem 38, we construct an algorithm to factorize polynomial matrices in $\mathcal{M}$.

Theorem 40. *Algorithm 1 is correct.*

Proof. The proof follows directly from Theorem 24, Theorem 36, Remark 37 and Theorem 38. □

Before proceeding further, let us remark on Algorithm 1.

- It follows from $\mathcal{G} \neq \{1\}$ in Step 7 that all the $(l - r) \times (l - r)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ do not generate $k[\mathbf{z}_2]$.
- Under the assumption that $\mathcal{G} \neq \{1\}$ and $r = 1$, the algorithm in Step 9 returns that "$\mathbf{F}$ has no matrix factorization w.r.t. h" by Theorem 24. When $\mathcal{G} \neq \{1\}$ and $1 < r < l$, the algorithm in Step 11 returns that "unable to judge" by Remark 37.

- We explain how to calculate a ZLP matrix $\mathbf{H}$ in Step 12. We first compute a Gröbner basis $\mathcal{G}^*$ of the syzygy module of $\mathbf{F}(f, \mathbf{z}_2)$. As $\text{rank}(\mathbf{F}(f, \mathbf{z}_2)) = l - r$, we can select r $k[\mathbf{z}_2]$-linearly independent vectors from $\mathcal{G}^*$ and form $\mathbf{H}_0 \in k[\mathbf{z}_2]^{r\times l}$ with full row rank. According to Lemma 2, all the $r \times r$ reduced minors of $\mathbf{H}_0$ generate $k[\mathbf{z}_2]$. Then, $\mathbf{H}_0$ has a ZLP matrix factorization by Lemma 17. Hence, we second use the Maple package QUILLENSUSLIN to compute a ZLP matrix factorization of $\mathbf{H}_0$ and obtain a ZLP matrix $\mathbf{H}$.
- In Step 13 we use QUILLENSUSLIN again to construct a unimodular matrix. Since QUILLENSUSLIN is a Maple package, we implement the factorization algorithm on Maple. Codes and examples are available on the website: http://www.mmrc.iss.ac.cn/~dwang/software.html.

6.2. Uniqueness of matrix factorizations

Liu and Wang (2015) studied the uniqueness problem of polynomial matrix factorizations. They pointed out that for a non-regular divisor h_0 of $\mathbf{F} \in k[\mathbf{z}]^{l\times m}$, under the condition that there exists a matrix factorization $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1$ with $\det(\mathbf{G}_1) = h_0$, $\text{Im}(\mathbf{F}_1)$ is not uniquely determined. In other words, when $\mathbf{F} = \mathbf{G}_1\mathbf{F}_1 = \mathbf{G}_2\mathbf{F}_2$ with $\det(\mathbf{G}_1) = \det(\mathbf{G}_2) = h_0$, $\text{Im}(\mathbf{F}_1)$ and $\text{Im}(\mathbf{F}_2)$ might not be the same.

Let $\mathbf{F} \in \mathcal{M}$. Suppose $h = z_1 - f(\mathbf{z}_2)$ and $\prec_{\mathbf{z}_2}$ are given. We use Algorithm 1 to factorize $\mathbf{F}$ w.r.t. h^r, where $1 \le r \le l$. Assume that all the $(l-r) \times (l-r)$ column reduced minors of $\mathbf{F}(f, \mathbf{z}_2)$ generate $k[\mathbf{z}_2]$, then we need to compute a ZLP matrix and construct a unimodular matrix. Due to the different choices of a ZLP matrix and a unimodular matrix, we will get different matrix factorizations of $\mathbf{F}$ w.r.t. h^r. Hence, in the following we study the uniqueness of matrix factorizations by Algorithm 1.

Theorem 41. *Let $\mathbf{F} \in \mathcal{M}$ satisfy $\mathbf{F} = \mathbf{U}_1^{-1}\mathbf{D}\mathbf{F}_1 = \mathbf{U}_2^{-1}\mathbf{D}\mathbf{F}_2$, where $\mathbf{U}_1, \mathbf{U}_2 \in k[\mathbf{z}_2]^{l\times l}$ are two unimodular matrices, and $\mathbf{D} = \text{diag}(\underbrace{h, \ldots, h}_{r}, \underbrace{1, \ldots, 1}_{l-r})$. Then, $\text{Im}(\mathbf{F}_1) = \text{Im}(\mathbf{F}_2)$.*

Proof. Let $\mathbf{F}_1 = \left[\vec{u}_1^{\text{T}}, \ldots, \vec{u}_l^{\text{T}}\right]^{\text{T}}$ and $\mathbf{F}_2 = \left[\vec{v}_1^{\text{T}}, \ldots, \vec{v}_l^{\text{T}}\right]^{\text{T}}$, where $\vec{u}_1, \ldots, \vec{u}_l, \vec{v}_1, \ldots, \vec{v}_l \in k[\mathbf{z}]^{1\times m}$. So, $\text{Im}(\mathbf{F}_1) = \langle \vec{u}_1, \ldots, \vec{u}_l\rangle$ and $\text{Im}(\mathbf{F}_2) = \langle \vec{v}_1, \ldots, \vec{v}_l\rangle$.

Let $\mathbf{F}_{01} = \mathbf{U}_1\mathbf{F}$ and $\mathbf{F}_{02} = \mathbf{U}_2\mathbf{F}$. Then $\mathbf{F}_{01} = \mathbf{D}\mathbf{F}_1$ and $\mathbf{F}_{02} = \mathbf{D}\mathbf{F}_2$. It follows that $\mathbf{F}_{01} = \left[h\vec{u}_1^{\text{T}}, \ldots, h\vec{u}_r^{\text{T}}, \vec{u}_{r+1}^{\text{T}}, \ldots, \vec{u}_l^{\text{T}}\right]^{\text{T}}$ and $\mathbf{F}_{02} = \left[h\vec{v}_1^{\text{T}}, \ldots, h\vec{v}_r^{\text{T}}, \vec{v}_{r+1}^{\text{T}}, \ldots, \vec{v}_l^{\text{T}}\right]^{\text{T}}$. Since $\mathbf{U}_1$ and $\mathbf{U}_2$ are two unimodular matrices in $k[\mathbf{z}_2]^{l\times l}$, we have $\mathbf{F}_{01} = \mathbf{U}_1\mathbf{U}_2^{-1}\mathbf{F}_{02}$. This implies that there exist polynomials $a_{i1}, \ldots, a_{il} \in k[\mathbf{z}_2]$ such that

$$h\vec{u}_i = h \cdot (\sum_{j=1}^{r} a_{ij}\vec{v}_j) + \sum_{j=r+1}^{l} a_{ij}\vec{v}_j,$$

where $i = 1, \ldots, r$. Then, for each i setting z_1 of the above equation to $f(\mathbf{z}_2)$, we have

$$a_{i(r+1)}\vec{v}_{r+1}(f, \mathbf{z}_2) + \cdots + a_{il}\vec{v}_l(f, \mathbf{z}_2) = \vec{0}.$$

As $\text{rank}(\mathbf{F}(f, \mathbf{z}_2)) = l - r$ and $\text{rank}(\mathbf{F}_{02}(f, \mathbf{z}_2)) = \text{rank}(\mathbf{F}(f, \mathbf{z}_2))$, we have that $\vec{v}_{r+1}(f, \mathbf{z}_2), \ldots, \vec{v}_l(f, \mathbf{z}_2)$ are $k[\mathbf{z}_2]$-linearly independent. This implies that $a_{i(r+1)} = \cdots = a_{il} = 0$. Hence,

$$\vec{u}_i = a_{i1}\vec{v}_1 + \cdots + a_{ir}\vec{v}_r,$$

where $i = 1, \ldots, r$. Obviously, $\vec{u}_j$ is a $k[\mathbf{z}]$-linear combination of $\vec{v}_1, \ldots, \vec{v}_l$, where $j = r+1, \ldots, l$. As a consequence, $\langle \vec{u}_1, \ldots, \vec{u}_l\rangle \subset \langle \vec{v}_1, \ldots, \vec{v}_l\rangle$. We can use the same method to prove that $\langle \vec{v}_1, \ldots, \vec{v}_l\rangle \subset \langle \vec{u}_1, \ldots, \vec{u}_l\rangle$.

Therefore, we have $\text{Im}(\mathbf{F}_1) = \text{Im}(\mathbf{F}_2)$. □

Based on Theorem 41, we can now derive the conclusion: the output $\mathbf{F}_1$ of Algorithm 1 is unique, i.e., $\text{Im}(\mathbf{F}_1)$ is uniquely determined.

7. Examples

We use two examples to illustrate the calculation process of Algorithm 1. We first return to Example 12.

Example 42. Let

$$\mathbf{F} = \begin{bmatrix} -2z_1z_2^2 + z_1^2z_3 + z_2^2z_3 - z_1z_3^2 + z_2z_3^2 & z_1^3 - z_2^3 - z_1^2z_3 + z_2z_3^2 & z_1z_2 - z_2z_3 & z_2^2 \\ -z_1z_2 + z_3^2 & -z_2^2 + z_1z_3 & 0 & z_2 \end{bmatrix}$$

be a polynomial matrix in $\mathbb{C}[z_1, z_2, z_3]^{2\times 4}$.

It is easy to compute that $d_2(\mathbf{F}) = z_2(z_1 - z_3)$ and $d_1(\mathbf{F}) = 1$. Let $\mathbf{F}$, $h = z_1 - z_3$ and $\prec_{z_2,z_3}$ be the input of Algorithm 1, where $\prec_{z_2,z_3}$ is the degree reverse lexicographic order.

Note that

$$\mathbf{F}(z_3, z_2, z_3) = \begin{bmatrix} -z_2^2z_3 + z_2z_3^2 & -z_2^3 + z_2z_3^2 & 0 & z_2^2 \\ -z_2z_3 + z_3^2 & -z_2^2 + z_3^2 & 0 & z_2 \end{bmatrix},$$

$\mathrm{rank}(\mathbf{F}(z_3, z_2, z_3)) = 1$ and $r = 1$. All the 1×1 column reduced minors of $\mathbf{F}(z_3, z_2, z_3)$ are $z_2, 1$. Since the reduced Gröbner basis of $\langle z_2, 1\rangle$ w.r.t. $\prec_{z_2,z_3}$ is $\{1\}$, $\mathbf{F}$ has a matrix factorization w.r.t. h.

Let $W = \mathrm{Im}(\mathbf{F}(z_3, z_2, z_3))$. Then we compute a reduced Gröbner basis of the syzygy module of W, and obtain

$$\mathbf{H} = \begin{bmatrix} 1 & -z_2 \end{bmatrix}.$$

It is easy to check that $\mathbf{H}$ is a ZLP matrix. $\mathbf{H}$ can be extended as the first row of a unimodular matrix

$$\mathbf{U} = \begin{bmatrix} 1 & -z_2 \\ 0 & 1 \end{bmatrix}$$

by using the package QUILLENSUSLIN. We extract h from the first row of $\mathbf{UF}$, and get

$$\mathbf{UF} = \mathbf{DF}_1 = \begin{bmatrix} z_1 - z_3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1z_3 - z_2^2 & z_1^2 - z_2z_3 & z_2 & 0 \\ -z_1z_2 + z_3^2 & -z_2^2 + z_1z_3 & 0 & z_2 \end{bmatrix}.$$

Then, $\mathbf{F}$ has a matrix factorization w.r.t. h:

$$\mathbf{F} = \mathbf{G}_1\mathbf{F}_1 = (\mathbf{U}^{-1}\mathbf{D})\mathbf{F}_1 = \begin{bmatrix} z_1 - z_3 & z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1z_3 - z_2^2 & z_1^2 - z_2z_3 & z_2 & 0 \\ -z_1z_2 + z_3^2 & -z_2^2 + z_1z_3 & 0 & z_2 \end{bmatrix},$$

where $\det(\mathbf{G}_1) = \det(\mathbf{U}^{-1}\mathbf{D}) = h$.

At this moment, $d_2(\mathbf{F}_1) = z_2$. We reuse Algorithm 1 to judge whether $\mathbf{F}_1$ has a matrix factorization w.r.t. z_2. Note that

$$\mathbf{F}_1(z_1, 0, z_3) = \begin{bmatrix} z_1z_3 & z_1^2 & 0 & 0 \\ z_3^2 & z_1z_3 & 0 & 0 \end{bmatrix},$$

$\mathrm{rank}(\mathbf{F}_1(z_1, 0, z_3)) = 1$ and $r = 1$. All the 1×1 column reduced minors of $\mathbf{F}_1(z_1, 0, z_3)$ are z_1, z_3, and the reduced Gröbner basis $\mathcal{G}$ of $\langle z_1, z_3\rangle$ is $\{z_1, z_3\}$. Since $\mathcal{G} \neq \{1\}$ and $r = 1$, $\mathbf{F}_1$ has no matrix factorization w.r.t. z_2.

Remark 43. In Example 42, we can first judge whether $\mathbf{F}$ has a matrix factorization w.r.t. z_2. Note that

$$\mathbf{F}(z_1, 0, z_3) = \begin{bmatrix} z_1z_3(z_1 - z_3) & z_1^2(z_1 - z_3) & 0 & 0 \\ z_3^2 & z_1z_3 & 0 & 0 \end{bmatrix},$$

$\mathrm{rank}(\mathbf{F}(z_1, 0, z_3)) = 1$ and $r = 1$. All the 1×1 column reduced minors of $\mathbf{F}(z_1, 0, z_3)$ are $z_1(z_1 - z_3), z_3$, and do not generate $k[z_1, z_3]$. This implies that $\mathbf{F}$ has no matrix factorization w.r.t. z_2.

According to the above calculations, we have the following conclusion: $\mathbf{F}$ has a matrix factorization w.r.t. $z_1 - z_3$, but does not have a matrix factorization w.r.t. z_2.

Example 44. Let

$$\mathbf{F}=\begin{bmatrix} z_1^2-z_1z_2 & z_2z_3+z_3^2+z_2+z_3 & -z_2z_3-z_2 \\ z_1z_2-z_2^2 & -z_1z_3+z_2z_3 & z_1^3-z_1^2z_2+z_1z_2-z_2^2 \\ 0 & z_2+z_3 & -z_2 \end{bmatrix}$$

be a polynomial matrix in $\mathbb{C}[z_1,z_2,z_3]^{3\times 3}$.

It is easy to compute that $d_3(\mathbf{F})=-z_1(z_1-z_2)^2(z_1^2z_2+z_1^2z_3+z_2^2)$, $d_2(\mathbf{F})=z_1-z_2$ and $d_1(\mathbf{F})=1$. Let $\mathbf{F}$, $h=z_1-z_2$ and $\prec_{z_2,z_3}$ be the input of Algorithm 1, where $\prec_{z_2,z_3}$ is the degree reverse lexicographic order.

Note that

$$\mathbf{F}(z_2,z_2,z_3)=\begin{bmatrix} 0 & (z_2+z_3)(z_3+1) & -z_2(z_3+1) \\ 0 & 0 & 0 \\ 0 & z_2+z_3 & -z_2 \end{bmatrix},$$

$\mathrm{rank}(\mathbf{F}(z_2,z_2,z_3))=1$ and $r=2$. Obviously, all the 1×1 column reduced minors of $\mathbf{F}(z_2,z_2,z_3)$ are $z_3+1, 1$. Since the reduced Gröbner basis of $\langle z_3+1,1\rangle$ w.r.t. $\prec_{z_2,z_3}$ is $\{1\}$, $\mathbf{F}$ has a matrix factorization w.r.t. h^2.

Let $W=\mathrm{Im}(\mathbf{F}(z_2,z_2,z_3))$. Then we compute a reduced Gröbner basis of the syzygy module of W, and obtain

$$\mathbf{H}=\begin{bmatrix} 1 & 0 & -z_3-1 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is easy to check that the reduced Gröbner basis of all the 2×2 minors of $\mathbf{H}$ w.r.t. $\prec_{z_2,z_3}$ is $\mathcal{G}=\{1\}$. Then, $\mathbf{H}$ is a ZLP matrix. We use the package QUILLENSUSLIN to construct a unimodular matrix

$$\mathbf{U}=\begin{bmatrix} 1 & 0 & -z_3-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that $\mathbf{H}$ is the first 2 rows of $\mathbf{U}$. We extract h from the first 2 rows of $\mathbf{UF}$, and get

$$\mathbf{UF}=\mathbf{DF}_1=\begin{bmatrix} z_1-z_2 & 0 & 0 \\ 0 & z_1-z_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} z_1 & 0 & 0 \\ z_2 & -z_3 & z_1^2+z_2 \\ 0 & z_2+z_3 & -z_2 \end{bmatrix}.$$

Then, we obtain a matrix factorization of $\mathbf{F}$ w.r.t. h^2:

$$\mathbf{F}=\mathbf{G}_1\mathbf{F}_1=(\mathbf{U}^{-1}\mathbf{D})\mathbf{F}_1=\begin{bmatrix} z_1-z_2 & 0 & z_3+1 \\ 0 & z_1-z_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} z_1 & 0 & 0 \\ z_2 & -z_3 & z_1^2+z_2 \\ 0 & z_2+z_3 & -z_2 \end{bmatrix},$$

where $\det(\mathbf{G}_1)=\det(\mathbf{U}^{-1}\mathbf{D})=h^2$.

At this moment, $d_3(\mathbf{F}_1)=-z_1(z_1^2z_2+z_1^2z_3+z_2^2)$. We reuse Algorithm 1 to judge whether $\mathbf{F}_1$ has a matrix factorization w.r.t. z_1. Similarly, we obtain

$$\mathbf{F}_1=\mathbf{G}_2\mathbf{F}_2=\begin{bmatrix} z_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ z_2 & -z_3 & z_1^2+z_2 \\ 0 & z_2+z_3 & -z_2 \end{bmatrix},$$

where $\det(\mathbf{G}_2)=z_1$.

Therefore, we obtain a matrix factorization of $\mathbf{F}$ w.r.t. $z_1(z_1-z_2)^2$, i.e.,

$$\mathbf{F}=\mathbf{GF}_2=(\mathbf{G}_1\mathbf{G}_2)\mathbf{F}_2=\begin{bmatrix} z_1(z_1-z_2) & 0 & z_3+1 \\ 0 & z_1-z_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ z_2 & -z_3 & z_1^2+z_2 \\ 0 & z_2+z_3 & -z_2 \end{bmatrix},$$

where $\det(\mathbf{G})=z_1(z_1-z_2)^2$.

Remark 45. In Example 44, we can first judge whether $\mathbf{F}$ has a matrix factorization w.r.t. z_1. Note that

$$\mathbf{F}(0, z_2, z_3) = \begin{bmatrix} 0 & (z_2+z_3)(z_3+1) & -z_2(z_3+1) \\ -z_2^2 & z_2z_3 & -z_2^2 \\ 0 & z_2+z_3 & -z_2 \end{bmatrix},$$

$\operatorname{rank}(\mathbf{F}(0, z_2, z_3)) = 2$ and $r = 1$. All the 2×2 column reduced minors of $\mathbf{F}(0, z_2, z_3)$ are $z_3 + 1, 1$, and generate $k[z_2, z_3]$. This implies that $\mathbf{F}$ has a matrix factorization w.r.t. z_1.

According to the above calculations, we have the following conclusion: $\mathbf{F}$ has a matrix factorization w.r.t. z_1, $z_1 - z_2$, $z_1(z_1 - z_2)$, $(z_1 - z_2)^2$ and $z_1(z_1 - z_2)^2$, respectively.

8. Concluding remarks

In this paper, we point out two directions of research in which multivariate polynomial matrices have been explored. The first is concerned with the factorization problem for a class of multivariate polynomial matrices, and the second direction is devoted to the investigation of the equivalence problem of a square polynomial matrix and a diagonal matrix.

The main contributions of this paper include: 1) some new factorization criteria are given to factorize $\mathbf{F} \in \mathcal{M}$ w.r.t. h^r, and the relationships among all existing factorization criteria have been studied; 2) a necessary and sufficient condition is proposed to judge whether a square polynomial matrix with the determinant being h^r is equivalent to the diagonal matrix $\operatorname{diag}(h, \ldots, h, 1, \ldots, 1)$; 3) based on new criteria, a factorization algorithm is given and the output of the algorithm is proved to be unique; 4) the algorithm is implemented on Maple, and two examples are given to illustrate the effectiveness of the algorithm.

A sufficient condition is obtained for the existence of a matrix factorization of $\mathbf{F}$ w.r.t. h^r $(1 < r < l)$. At this moment, how to establish a necessary and sufficient condition for $\mathbf{F}$ admitting a matrix factorization w.r.t. h^r is the question that remains for further investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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