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On Minor Left Prime Factorization Problem for Multivariate Polynomial Matrices*

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Abstract A new necessary and sufficient condition for the existence of minor left prime factorizations of multivariate polynomial matrices without full row rank is presented. The key idea is to establish a relationship between a matrix and any of its full row rank submatrices. Based on the new result, the authors propose an algorithm for factorizing matrices and have implemented it on the computer algebra system Maple. Two examples are given to illustrate the effectiveness of the algorithm, and experimental data shows that the algorithm is efficient.

Keywords Free modules, Gröbner bases, minor left prime (MLP), multivariate polynomial matrices, polynomial matrix factorizations.

1 Introduction

Multivariate polynomial matrix factorization is one of the most important operations in multidimensional systems, signal processing, and other related areas[1, 2]. The factorization problems of multivariate polynomial matrices have been extensively investigated and numerous algorithms have been developed to compute factorizations of multivariate polynomial matrices. Since the factorization problems have been solved for univariate and bivariate polynomial

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matrices[3–5], we only consider the case where the number of variables is greater than or equal to three.

Using three important concepts proposed by Youla and Gnavi[6], there have been many publications studying matrix factorizations. Lin[7] first proposed the existence problem for zero prime factorizations of multivariate polynomial matrices. Charoenlarpnopparut and Bose[8] first used Gröbner bases of modules to compute zero prime matrix factorizations of multivariate polynomial matrices. After that, Lin, et al.[9] introduced some applications of Gröbner bases in the broad field of signals and systems. Lin and Bose[10] put forward the famous Lin-Bose conjecture which was solved in [11, 12]. Wang and Kwong[13] focused on the existence problem for minor prime factorizations of multivariate polynomial matrices, and gave a necessary and sufficient condition. Wang[14] designed an algorithm to compute factor prime factorizations of a class of multivariate polynomial matrices.

In linear algebra as well as multidimensional systems, the factorization problems of multivariate polynomial matrices without full row rank are important and deserve some attention[6, 7]. Up to now, few results have been achieved on factorizations of multivariate polynomial matrices without full row rank[10, 15, 16]. Therefore, this paper focuses on factorization problems of multivariate polynomial matrices without full row rank. We try to use local properties (i.e., the full rank submatrix of a matrix) to study the existence for minor prime factorizations of multivariate polynomial matrices without full row rank.

The rest of the paper is organized as follows. In Section 2, we introduce some basic concepts and present the problem that we are considering. In Section 3, we present a new necessary and sufficient condition for the existence of minor left prime factorizations of multivariate polynomial matrices without full row rank. In Section 4, we construct an algorithm based on the new result, and use two examples to illustrate the effectiveness of the algorithm. The comparative performance of our algorithm and Guan's algorithm are provided by experimental data in Section 5. We end with some concluding remarks in Section 6.

2 Preliminaries and Problem

Let n be the number of variables, and $\boldsymbol{z}$ be the n variables $z_1, \cdots, z_n$. Let $k[\boldsymbol{z}]$ be the polynomial ring in $\boldsymbol{z}$ over k, where k is a field. Let $k[\boldsymbol{z}]^{l\times m}$ denote the set of $l \times m$ matrices with entries in $k[\boldsymbol{z}]$. Throughout this paper, we assume that $l \le m$. Let $\boldsymbol{F} \in k[\boldsymbol{z}]^{l\times m}$ with rank r, we use $d_r(\boldsymbol{F})$ to denote the greatest common divisor of all the $r \times r$ minors of $\boldsymbol{F}$, and $I_r(\boldsymbol{F})$ to represent the ideal generated by all the $r \times r$ minors of $\boldsymbol{F}$, where $1 \le r \le l$. In addition, we use $\rho(\boldsymbol{F})$ and $\mathrm{Syz}(\boldsymbol{F})$ to denote the submodule (of $k[\boldsymbol{z}]^{1\times m}$) generated by the rows and the syzygy module of $\boldsymbol{F}$, respectively. Superscript $^{\mathrm{T}}$ denotes transposition.

We first recall the most important concept in the paper.

Definition 2.1 ([6]) Let $\boldsymbol{F} \in k[\boldsymbol{z}]^{l\times m}$ be of full row rank. Then $\boldsymbol{F}$ is said to be a minor left prime (MLP) matrix if all the $l \times l$ minors of $\boldsymbol{F}$ are relatively prime, that is, $d_l(\boldsymbol{F})$ is a nonzero constant.

Let $\boldsymbol{F} \in k[\boldsymbol{z}]^{m\times l}$ be of full column rank with $m \ge l$, an MRP matrix can be similarly defined. We refer to [6] for more details about the concepts of zero left prime (ZLP) matrices and factor left prime (FLP) matrices.

An MLP factorization of a multivariate polynomial matrix is formulated as follows.

Definition 2.2 Let $\boldsymbol{F} \in k[\boldsymbol{z}]^{l\times m}$ with rank r, where $1 \leq r \leq l$. $\boldsymbol{F}$ is said to admit an MLP factorization if $\boldsymbol{F}$ can be factorized as

$$\boldsymbol{F} = \boldsymbol{G}_0\boldsymbol{F}_0 \tag{1}$$

such that $\boldsymbol{G}_0 \in k[\boldsymbol{z}]^{l\times r}$, and $\boldsymbol{F}_0 \in k[\boldsymbol{z}]^{r\times m}$ is an MLP matrix.

When Youla and Gnavi[6] studied the structure of n-dimensional linear systems, they obtained the following MLP factorization lemma by using matrix theory.

Lemma 2.3 ([6]) *Let*

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \in k[\boldsymbol{z}]^{l\times m}$$

with rank r, where $\boldsymbol{A}_{11} \in k[\boldsymbol{z}]^{r\times r}$ with $\det(\boldsymbol{A}_{11}) \neq 0$, *$\boldsymbol{A}_{12} \in k[\boldsymbol{z}]^{r\times(m-r)}$, $\boldsymbol{A}_{21} \in k[\boldsymbol{z}]^{(l-r)\times r}$, $\boldsymbol{A}_{22} \in k[\boldsymbol{z}]^{(l-r)\times(m-r)}$, and $1 \leq r \leq l$. If $[\boldsymbol{A}_{11}\ \ \boldsymbol{A}_{12}]$ is an MLP matrix, then $\boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}$ is a multivariate polynomial matrix and $\boldsymbol{A}$ has an MLP factorization*

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I}_{r\times r} \\ \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \end{bmatrix}. \tag{2}$$

In order to state conveniently the problem and the main result of this paper, we introduce the following concepts and conclusions.

Definition 2.4 ([7]) Let $\boldsymbol{F} \in k[\boldsymbol{z}]^{l\times m}$ with rank r, where $1 \leq r \leq l$. Let $a_1, \cdots, a_\beta$ denote all the $r \times r$ minors of $\boldsymbol{F}$, where $\beta = \binom{l}{r} \cdot \binom{m}{r}$. Extracting $d_r(\boldsymbol{F})$ from $a_1, \cdots, a_\beta$ yields

$$a_j = d_r(\boldsymbol{F}) \cdot b_j, \quad j = 1, \cdots, \beta. \tag{3}$$

Then, $b_1, \cdots, b_\beta$ are called all the reduced minors of $\boldsymbol{F}$.

Definition 2.5 ([17]) Let $\boldsymbol{F} \in k[\boldsymbol{z}]^{l\times m}$ with rank r, and $\overline{\boldsymbol{F}} \in k[\boldsymbol{z}]^{l\times r}$ be an arbitrary full column rank submatrix of $\boldsymbol{F}$, where $1 \leq r \leq l$. Let $c_1, \cdots, c_\xi$ be all the reduced minors of $\overline{\boldsymbol{F}}$, where $\xi = \binom{l}{r}$. Then $c_1, \cdots, c_\xi$ are called all the **column** reduced minors of $\boldsymbol{F}$.

Definition 2.6 ([18]) Let $\mathcal{K}$ be a submodule of $k[\boldsymbol{z}]^{1\times m}$, and J be a nonzero ideal of $k[\boldsymbol{z}]$. We define

$$\mathcal{K} : J = \{\vec{u} \in k[\boldsymbol{z}]^{1\times m} \mid J\vec{u} \subseteq \mathcal{K}\}, \tag{4}$$

where $J\vec{u}$ is the set $\{f\vec{u} \mid f \in J\}$.

Obviously, $\mathcal{K} \subseteq \mathcal{K} : J$. Let $\{f_1, \cdots, f_s\} \subset k[\boldsymbol{z}]$ be a Gröbner basis of J, then

$$\mathcal{K} : J = \mathcal{K} : \langle f_1, \cdots, f_s\rangle = (\mathcal{K} : f_1) \cap \cdots \cap (\mathcal{K} : f_s). \tag{5}$$

Here, we write $\mathcal{K} : \langle f\rangle$ as $\mathcal{K} : f$ for any $f \in k[\boldsymbol{z}]$.

Let $\boldsymbol{F} \in k[\boldsymbol{z}]^{l\times m}$ with rank r, and $f \in k[\boldsymbol{z}]$ be a nonzero polynomial, where $1 \le r \le l$. Wang and Kwong[13] proved that there is a one-to-one correspondence between the two modules: $\rho(\boldsymbol{F}) : f$ and $\mathrm{Syz}([\boldsymbol{F}^{\mathrm{T}}, -f\cdot \boldsymbol{I}_{m\times m}]^{\mathrm{T}})$. That is, we compute a Gröbner basis $\{[\vec{g}_1, \vec{f}_1], \cdots, [\vec{g}_s, \vec{f}_s]\}$ of $\mathrm{Syz}([\boldsymbol{F}^{\mathrm{T}}, -f\cdot \boldsymbol{I}_{m\times m}]^{\mathrm{T}})$, then $\{\vec{f}_1, \cdots, \vec{f}_s\}$ is a system of generators of $\rho(\boldsymbol{F}) : f$, where $[\vec{g}_i, \vec{f}_i] \in k[\boldsymbol{z}]^{1\times(l+m)}$ and $i = 1, \cdots, s$. Suppose $\boldsymbol{F}' \in k[\boldsymbol{z}]^{s\times m}$ is composed of $\{\vec{f}_1, \cdots, \vec{f}_s\}$. Lu et al.[17] proved that $\rho(\boldsymbol{F}) : f$ is a free module of rank r if and only if all the column reduced minors of $\boldsymbol{F}'$ generate the unit ideal $k[\boldsymbol{z}]$.

Wang and Kwong[13] proposed a necessary and sufficient condition for MLP factorizations of multivariate polynomial matrices with full row rank.

Lemma 2.7 ([13]) *Let $\boldsymbol{F} \in k[\boldsymbol{z}]^{l\times m}$ be of full row rank. Then the following are equivalent:*

1) *$\boldsymbol{F}$ has an MLP factorization;*

2) *$\rho(\boldsymbol{F}) : d_l(\boldsymbol{F})$ is a free module of rank l.*

Guan, et al.[16] generalized Lemma 2.7 to the case of multivariate polynomial matrices without full row rank.

Lemma 2.8 ([16]) *Let $\boldsymbol{F} \in k[\boldsymbol{z}]^{l\times m}$ with rank r, where $1 \le r \le l$. Then the following are equivalent:*

1) *$\boldsymbol{F}$ has an MLP factorization;*

2) *$\rho(\boldsymbol{F}) : I_r(\boldsymbol{F})$ is a free module of rank r.*

Remark 2.9 Although Lemma 2.8 is different from Lemma 2.7 for the case of $r = l$, Guan, et al.[16] have proven that $\rho(\boldsymbol{F}) : I_l(\boldsymbol{F}) = \rho(\boldsymbol{F}) : d_l(\boldsymbol{F})$.

Let $a_1, \cdots, a_\beta \in k[\boldsymbol{z}]$ be all the $r\times r$ minors of $\boldsymbol{F}$, then $I_r(\boldsymbol{F}) = \langle a_1, \cdots, a_\beta\rangle$, where $\beta = \binom{l}{r}\cdot\binom{m}{r}$. From Equation (5) we have

$$\rho(\boldsymbol{F}) : I_r(\boldsymbol{F}) = (\rho(\boldsymbol{F}) : a_1) \cap \cdots \cap (\rho(\boldsymbol{F}) : a_\beta). \tag{6}$$

When we verify whether $\rho(\boldsymbol{F}) : I_r(\boldsymbol{F})$ is a free module of rank r, we need to do the following calculation. First, we compute a Gröbner basis $\{\overline{a}_1, \cdots, \overline{a}_\gamma\}$ of $I_r(\boldsymbol{F})$, where $\gamma \le \beta$. Then,

$$\rho(\boldsymbol{F}) : I_r(\boldsymbol{F}) = (\rho(\boldsymbol{F}) : \overline{a}_1) \cap \cdots \cap (\rho(\boldsymbol{F}) : \overline{a}_\gamma). \tag{7}$$

Second, we obtain a system $\mathcal{G}_i$ of generators of $\rho(\boldsymbol{F}) : \overline{a}_i$ by computing a Gröbner basis of $\mathrm{Syz}([\boldsymbol{F}^{\mathrm{T}}, -\overline{a}_i\cdot \boldsymbol{I}_{m\times m}]^{\mathrm{T}})$, where $i = 1, \cdots, \gamma$. Third, we compute a Gröbner basis $\mathcal{G}$ of $\mathcal{G}_1 \cap \cdots \cap \mathcal{G}_\gamma$. Finally, we compute a Gröbner basis G of the ideal generated by all the column reduced minors of the matrix that is composed of the elements in $\mathcal{G}$. If $G = \{1\}$, then $\rho(\boldsymbol{F}) : I_r(\boldsymbol{F})$ is a free module of rank r; otherwise not.

The classical method to compute a Gröbner basis of the intersection of modules is to introduce new variables. Given that the complexity of Gröbner basis computations is heavily influenced by the number of variables and the total degrees of polynomials[19, 20], it can be seen that the calculation amount of $\rho(\boldsymbol{F}) : I_r(\boldsymbol{F})$ is very large. Therefore, we consider the following problem.

Problem 2.10 Is there a simpler condition that can replace $\rho(\boldsymbol{F}) : I_r(\boldsymbol{F})$ in Lemma 2.8?

3 Necessary and Sufficient Condition

Let $\boldsymbol{F} \in k[\boldsymbol{z}]^{l\times m}$ with rank r, where $1 \le r \le l$. We use Lemma 2.3 to establish a relationship between $\boldsymbol{F}$ and an arbitrary full row rank submatrix of $\boldsymbol{F}$, and then solve Problem 2.10.

Theorem 3.1 *Let $\boldsymbol{F} \in k[\boldsymbol{z}]^{l\times m}$ with rank r, and $\boldsymbol{F}_1 \in k[\boldsymbol{z}]^{r\times m}$ be an arbitrary full row rank submatrix of $\boldsymbol{F}$, where $1 \le r \le l$. Then the following are equivalent:*

1) $\boldsymbol{F}$ has an MLP factorization;

2) $\rho(\boldsymbol{F}_1) : d_r(\boldsymbol{F}_1)$ is a free module of rank r.

Proof 1) $\to$ 2). Suppose $\boldsymbol{F}$ has an MLP factorization. Then there exist $\boldsymbol{G}_0 \in k[\boldsymbol{z}]^{l\times r}$ and $\boldsymbol{F}_0 \in k[\boldsymbol{z}]^{r\times m}$ such that $\boldsymbol{F} = \boldsymbol{G}_0\boldsymbol{F}_0$ with $\boldsymbol{F}_0$ being an MLP matrix. Without loss of generality, we assume that the first r rows of $\boldsymbol{F}$ are $k[\boldsymbol{z}]$-linearly independent. Let $\boldsymbol{F}_1 \in k[\boldsymbol{z}]^{r\times m}$ be composed of the first r rows of $\boldsymbol{F}$, then

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{01} \\ \boldsymbol{G}_{02} \end{bmatrix} \boldsymbol{F}_0, \tag{8}$$

where $\boldsymbol{G}_{01} \in k[\boldsymbol{z}]^{r\times r}$ is the first r rows of $\boldsymbol{G}_0$. From Equation (8) we have

$$\boldsymbol{F}_1 = \boldsymbol{G}_{01}\boldsymbol{F}_0. \tag{9}$$

According to Lemma 2.7, $\rho(\boldsymbol{F}_1) : d_r(\boldsymbol{F}_1)$ is a free module of rank r.

2) $\to$ 1). Assume that $\rho(\boldsymbol{F}_1) : d_r(\boldsymbol{F}_1)$ is a free module of rank r. Using Lemma 2.7, there exist $\boldsymbol{G}_{11} \in k[\boldsymbol{z}]^{r\times r}$ and $\boldsymbol{F}_{11} \in k[\boldsymbol{z}]^{r\times m}$ such that $\boldsymbol{F}_1 = \boldsymbol{G}_{11}\boldsymbol{F}_{11}$ with $\boldsymbol{F}_{11}$ being an MLP matrix. Since $\boldsymbol{F}_1$ is an arbitrary $r\times m$ submatrix of $\boldsymbol{F}$, there exists an elementary transformation matrix $\boldsymbol{U} \in k^{l\times l}$ such that $\boldsymbol{F}_1$ is the first r rows of $\overline{\boldsymbol{F}}$, where $\overline{\boldsymbol{F}} = \boldsymbol{U}\boldsymbol{F}$. Let $\overline{\boldsymbol{F}} = [\boldsymbol{F}_1^{\mathrm{T}}\ \boldsymbol{C}^{\mathrm{T}}]^{\mathrm{T}}$, where $\boldsymbol{C} \in k[\boldsymbol{z}]^{(l-r)\times m}$ is the last $(l-r)$ rows of $\overline{\boldsymbol{F}}$. Then,

$$\overline{\boldsymbol{F}} = \boldsymbol{U}\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{11}\boldsymbol{F}_{11} \\ \boldsymbol{C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{0}_{r\times(l-r)} \\ \boldsymbol{0}_{(l-r)\times r} & \boldsymbol{I}_{(l-r)\times(l-r)} \end{bmatrix} \begin{bmatrix} \boldsymbol{F}_{11} \\ \boldsymbol{C} \end{bmatrix}. \tag{10}$$

Because $\boldsymbol{F}_{11} \in k[\boldsymbol{z}]^{r\times m}$ is a full row rank matrix, there exists another elementary transformation matrix $\boldsymbol{V} \in k^{m\times m}$ such that the first r columns of $\overline{\boldsymbol{F}}_{11}$ are $k[\boldsymbol{z}]$-linearly independent, where $\overline{\boldsymbol{F}}_{11} = \boldsymbol{F}_{11}\boldsymbol{V}$. It follows from $\det(\boldsymbol{V}) = \pm 1$ that $\overline{\boldsymbol{F}}_{11}\boldsymbol{V}^{-1} = \boldsymbol{F}_{11}$. According to the Binet-Cauchy formula, we obtain $d_r(\overline{\boldsymbol{F}}_{11}) \mid d_r(\boldsymbol{F}_{11})$. This implies that $d_r(\overline{\boldsymbol{F}}_{11})$ is a nonzero constant. Therefore, $\overline{\boldsymbol{F}}_{11}$ is an MLP matrix. Suppose that

$$\begin{bmatrix} \boldsymbol{F}_{11} \\ \boldsymbol{C} \end{bmatrix} \boldsymbol{V} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}, \tag{11}$$

where $\boldsymbol{A}_{11} \in k[\boldsymbol{z}]^{r\times r}$, $\boldsymbol{A}_{12} \in k[\boldsymbol{z}]^{r\times(m-r)}$, $\boldsymbol{A}_{21} \in k[\boldsymbol{z}]^{(l-r)\times r}$, and $\boldsymbol{A}_{22} \in k[\boldsymbol{z}]^{(l-r)\times(m-r)}$. Then, $\det(\boldsymbol{A}_{11}) \neq 0$ and $[\boldsymbol{A}_{11}\ \boldsymbol{A}_{12}]$ is an MLP matrix. By Lemma 2.3, we get

$$\begin{bmatrix} \boldsymbol{F}_{11} \\ \boldsymbol{C} \end{bmatrix} \boldsymbol{V} = \begin{bmatrix} \boldsymbol{I}_{r\times r} \\ \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{r\times r} \\ \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1} \end{bmatrix} \overline{\boldsymbol{F}}_{11}. \tag{12}$$

Combining Equation (10) and Equation (12), we have

$$\boldsymbol{UFV} = \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{0}_{r\times(l-r)} \\ \boldsymbol{0}_{(l-r)\times r} & \boldsymbol{I}_{(l-r)\times(l-r)} \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_{r\times r} \\ \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1} \end{bmatrix} \overline{\boldsymbol{F}}_{11} = \begin{bmatrix} \boldsymbol{G}_{11} \\ \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1} \end{bmatrix} \overline{\boldsymbol{F}}_{11}. \tag{13}$$

As $\boldsymbol{U}$ and $\boldsymbol{V}$ are two elementary transformation matrices, from Equation (13) we can derive

$$\boldsymbol{F} = \boldsymbol{U}^{-1} \begin{bmatrix} \boldsymbol{G}_{11} \\ \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1} \end{bmatrix} \overline{\boldsymbol{F}}_{11}\boldsymbol{V}^{-1} = \boldsymbol{U}^{-1} \begin{bmatrix} \boldsymbol{G}_{11} \\ \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1} \end{bmatrix} \boldsymbol{F}_{11}. \tag{14}$$

Let

$$\boldsymbol{G}_0 = \boldsymbol{U}^{-1} \begin{bmatrix} \boldsymbol{G}_{11} \\ \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1} \end{bmatrix}$$

and $\boldsymbol{F}_0 = \boldsymbol{F}_{11}$, then $\boldsymbol{F} = \boldsymbol{G}_0\boldsymbol{F}_0$. Thus, $\boldsymbol{F}$ has an MLP factorization, and the proof is completed. ▌

Remark 3.2 Theorem 3.1 is the same as Lemma 2.7 for the case of $r = l$.

4 Algorithm and Examples

Let $\boldsymbol{F} \in k[\boldsymbol{z}]^{l\times m}$ with rank r, and $\boldsymbol{F}_1 \in k[\boldsymbol{z}]^{r\times m}$ be an arbitrary full row rank submatrix of $\boldsymbol{F}$, where $1 \le r \le l$. Suppose $\rho(\boldsymbol{F}_1) : d_r(\boldsymbol{F}_1)$ is a free module of rank r, then $\boldsymbol{F}$ has an MLP factorization. Now, we need to design an algorithm to compute $\boldsymbol{F}_0 \in k[\boldsymbol{z}]^{r\times m}$ and $\boldsymbol{G}_0 \in k[\boldsymbol{z}]^{l\times r}$ such that $\boldsymbol{F} = \boldsymbol{G}_0\boldsymbol{F}_0$ with $\boldsymbol{F}_0$ being an MLP matrix.

Computing free bases of free modules is a crucial step in the process of matrix factorizations. Fabiańska and Quadrat[21] first designed a Maple package, which is called QUILLENSUSLIN, to compute free bases of free modules. Thus, we can use the QUILLENSUSLIN to compute a free basis of $\rho(\boldsymbol{F}_1) : d_r(\boldsymbol{F}_1)$ and then form $\boldsymbol{F}_0$ by this free basis. As $\boldsymbol{F}_0$ is a full row rank matrix, we compute the generalized right inverse $\boldsymbol{F}_0^{-1} \in k(\boldsymbol{z})^{m\times r}$ of $\boldsymbol{F}_0$ over $k(\boldsymbol{z})$ such that $\boldsymbol{F}_0\boldsymbol{F}_0^{-1} = \boldsymbol{I}_{r\times r}$. Since we ensure that there is a unique solution to the equation $\boldsymbol{F} = \boldsymbol{G}_0\boldsymbol{F}_0$, we get $\boldsymbol{G}_0 = \boldsymbol{F}\boldsymbol{F}_0^{-1}$.

Now, we can propose the following constructive algorithm to compute MLP factorizations of polynomial matrices without full row rank.

From Algorithm 1 we have $\rho(\boldsymbol{F}_1') = \rho(\boldsymbol{F}_1) : d_r(\boldsymbol{F}_1)$ in Step 5. Moreover, $G \neq \{1\}$ in Step 6 implies that $\rho(\boldsymbol{F}_1) : d_r(\boldsymbol{F}_1)$ is not a free module of rank r. If $s = r$ in Step 5, then $\boldsymbol{F}_1'$ is a full row rank matrix. It follows that $\rho(\boldsymbol{F}_1) : d_r(\boldsymbol{F}_1)$ is a free module of rank r and the rows of $\boldsymbol{F}_1'$ constitute a free basis of $\rho(\boldsymbol{F}_1) : d_r(\boldsymbol{F}_1)$. In this case, we do not need to compute the Gröbner basis G and perform the calculation from Step 10.

We use the two examples in [16] to illustrate the calculation process of Algorithm 1.

Algorithm 1: MLP factorization

Input: $\boldsymbol{F} \in k[\boldsymbol{z}]^{l\times m}$.

Output: An MLP factorization of $\boldsymbol{F}$.

compute the rank r of $\boldsymbol{F}$;

perform elementary row transformations on $\boldsymbol{F}$, such that the first r rows of $\overline{\boldsymbol{F}}$ are $k[\boldsymbol{z}]$-linearly independent, where $\overline{\boldsymbol{F}} = \boldsymbol{U}\boldsymbol{F}$ and $\boldsymbol{U} \in k^{l\times l}$ is an elementary transformation matrix;

compute $d_r(\boldsymbol{F}_1)$, where $\boldsymbol{F}_1$ is composed of the first r rows of $\overline{\boldsymbol{F}}$;

compute a Gröbner basis $\{[\vec{g}_1, \vec{f}_1], \cdots, [\vec{g}_s, \vec{f}_s]\}$ of $\mathrm{Syz}([\boldsymbol{F}_1^{\mathrm{T}}, -d_r(\boldsymbol{F}_1)\cdot \boldsymbol{I}_{m\times m}]^{\mathrm{T}})$;

compute a Gröbner basis G of the ideal generated by all the column reduced minors of $\boldsymbol{F}_1' \in k[\boldsymbol{z}]^{s\times m}$, where $\boldsymbol{F}_1'$ is composed of $\{\vec{f}_1, \cdots, \vec{f}_s\}$;

if $G \neq \{1\}$ **then**

  return $\boldsymbol{F}$ has no MLP factorizations.

end

use the QUILLENSUSLIN to compute a free basis of $\rho(\boldsymbol{F}_1')$ and use it to make up $\boldsymbol{F}_0 \in k[\boldsymbol{z}]^{r\times m}$;

compute the right inverse $\boldsymbol{F}_0^{-1}$ of $\boldsymbol{F}_0$, and let $\boldsymbol{G}_0 := \boldsymbol{U}^{-1}\overline{\boldsymbol{F}}\boldsymbol{F}_0^{-1}$;

return $\boldsymbol{F}_0$ and $\boldsymbol{G}_0$.

Example 4.1 Let

$$\boldsymbol{F} = \begin{bmatrix} z_1^2z_2 + z_1^2 & z_1 & 0 \\ z_1z_3^2 - z_1z_3 & 0 & z_2z_3 - z_2 + z_3 - 1 \\ 2z_1^2z_2z_3 - z_1^2z_2 + z_1^2z_3 - z_1^2 & z_1z_3 - z_1 & z_1z_2^2 + z_1z_2 \end{bmatrix}$$

be a multivariate polynomial matrix in $\mathbb{C}[z_1, z_2, z_3]^{3\times 3}$, where $z_1 > z_2 > z_3$ and $\mathbb{C}$ is the complex field.

It is easy to see that the rank of $\boldsymbol{F}$ is 2, and the first 2 rows of $\boldsymbol{F}$ are $\mathbb{C}[z_1, z_2, z_3]$-linearly independent. Let $\boldsymbol{F}_1 \in \mathbb{C}[z_1, z_2, z_3]^{2\times 3}$ be composed of the first 2 rows of $\boldsymbol{F}$, then $d_2(\boldsymbol{F}_1) = z_1z_3 - z_1$. We compute a Gröbner basis of $\mathrm{Syz}([\boldsymbol{F}_1^{\mathrm{T}}, -d_2(\boldsymbol{F}_1)\cdot \boldsymbol{I}_{3\times 3}]^{\mathrm{T}})$ and obtain

$$\{[0,\ z_1,\ z_1z_3,\ 0,\ z_2 + 1],\ [z_3 - 1,\ 0,\ z_1z_2 + z_1,\ 1,\ 0]\}.$$

Now, we get a system of generators of $\rho(\boldsymbol{F}_1) : d_2(\boldsymbol{F}_1)$ as follows

$$\{[z_1z_3,\ 0,\ z_2 + 1],\ [z_1z_2 + z_1,\ 1,\ 0]\}.$$

Let

$$\boldsymbol{F}_1' = \begin{bmatrix} z_1z_3 & 0 & z_2 + 1 \\ z_1z_2 + z_1 & 1 & 0 \end{bmatrix}.$$

Since $\mathrm{rank}(\boldsymbol{F}_1') = 2$, $\boldsymbol{F}_1'$ is a full row rank matrix. Then, $\rho(\boldsymbol{F}_1) : d_2(\boldsymbol{F}_1)$ is a free module of rank 2, and the rows of $\boldsymbol{F}_1'$ constitute a free basis of $\rho(\boldsymbol{F}_1) : d_2(\boldsymbol{F}_1)$. Let $\boldsymbol{F}_0 = \boldsymbol{F}_1'$, we compute the generalized right inverse $\boldsymbol{F}_0^{-1}$ of $\boldsymbol{F}_0$ and get

$$\boldsymbol{G}_0 = \boldsymbol{F}\boldsymbol{F}_0^{-1} = \begin{bmatrix} 0 & z_1 \\ z_3 - 1 & 0 \\ z_1 z_2 & z_1 z_3 - z_1 \end{bmatrix}.$$

Therefore, $\boldsymbol{F}$ has an MLP factorization:

$$\boldsymbol{F} = \boldsymbol{G}_0 \boldsymbol{F}_0 = \begin{bmatrix} 0 & z_1 \\ z_3 - 1 & 0 \\ z_1 z_2 & z_1 z_3 - z_1 \end{bmatrix} \begin{bmatrix} z_1 z_3 & 0 & z_2 + 1 \\ z_1 z_2 + z_1 & 1 & 0 \end{bmatrix}.$$

Example 4.2 Let

$$\boldsymbol{F} = \begin{bmatrix} z_1 z_2 + z_1 - z_2 - 1 & 0 & z_3 \\ z_2 + 1 & z_2 + 1 & z_1 - 1 \\ z_1 z_2 + z_1 & z_2 + 1 & z_1 + z_3 - 1 \end{bmatrix}$$

be a multivariate polynomial matrix in $\mathbb{C}[z_1, z_2, z_3]^{3\times 3}$, where $z_1 > z_2 > z_3$ and $\mathbb{C}$ is the complex field.

It is easy to see that the rank of $\boldsymbol{F}$ is 2, and the first 2 rows of $\boldsymbol{F}$ are $\mathbb{C}[z_1, z_2, z_3]$-linearly independent. Let $\boldsymbol{F}_1 \in \mathbb{C}[z_1, z_2, z_3]^{2\times 3}$ be composed of the first 2 rows of $\boldsymbol{F}$, then $d_2(\boldsymbol{F}_1) = z_2 + 1$. We compute a Gröbner basis of $\mathrm{Syz}([\boldsymbol{F}_1^{\mathrm{T}}, -d_2(\boldsymbol{F}_1) \cdot \boldsymbol{I}_{3\times 3}]^{\mathrm{T}})$ and obtain a system of generators of $\rho(\boldsymbol{F}_1) : d_2(\boldsymbol{F}_1)$ as follows

$$\{[z_2 + 1,\ z_2 + 1,\ z_1 - 1],\ [z_1 z_2 + z_1 - z_2 - 1,\ 0,\ z_3],\ [z_1^2 - 2z_1 - z_3 + 1,\ -z_3,\ 0]\}.$$

Let

$$\boldsymbol{F}_1' = \begin{bmatrix} z_2 + 1 & z_2 + 1 & z_1 - 1 \\ z_1 z_2 + z_1 - z_2 - 1 & 0 & z_3 \\ z_1^2 - 2z_1 - z_3 + 1 & -z_3 & 0 \end{bmatrix},$$

then all the column reduced minors of $\boldsymbol{F}_1'$ are $z_2 + 1, z_1 - 1, z_3$. It is easy to compute that the Gröbner basis of $\langle z_1 - 1, z_2 + 1, z_3 \rangle$ is $\{z_1 - 1, z_2 + 1, z_3\}$. This implies that $\rho(\boldsymbol{F}_1) : d_2(\boldsymbol{F}_1)$ is not a free module of rank 2. Then, $\boldsymbol{F}$ has no MLP factorizations.

5 Comparative Performance

The above two examples show that the calculation by Algorithm 1 is simpler than that by the algorithm, which is called Guan's algorithm, proposed in [16]. To illustrate the advantages of our algorithm, we first compare the main differences between the two algorithms.

The symbols in Table 1 are the same as those in Lemma 2.8 and Theorem 3.1. From Table 1, we can get the following preliminary conclusions: First, the calculation of the quotient module of Algorithm 1 is faster than that of Guan's algorithm in almost all cases (Please see Section 2 for specific reasons); second, Guan, et al.[16] used the traditional method to verify whether a module is a free module, that is, they need to compute the r-th Fitting ideal (Please refer to [22, 23] for the specific calculation method) of a system of generators of $\rho(\boldsymbol{F}) : I_r(\boldsymbol{F})$ which is more time-consuming than the method by calculating the column reduced minors of a system of generators of $\rho(\boldsymbol{F}_1) : d_r(\boldsymbol{F}_1)$ to judge whether a module is a free module.

Table 1 The comparison of two MLP factorization algorithms

Main step	Guan's algorithm	Algorithm 1
Quotient module	$\rho(\boldsymbol{F}) : I_r(\boldsymbol{F})$	$\rho(\boldsymbol{F}_1) : d_r(\boldsymbol{F}_1)$
Verification method of free modules	Fitting ideals	Column reduced minors

Next, we will show from some specific experimental data that Algorithm 1 is more efficient than Guan's algorithm. The two algorithms have been implemented by us on the computer algebra system Maple. The implementations of the two algorithms have been tried on a number of examples including the two examples in Section 4. Please see the Appendix for all examples. For interested readers, more comparative examples can be generated by the codes at: `http://www.mmrc.iss.ac.cn/~dwang/software.html`.

In Table 2, timings were obtained on an Intel(R) Xeon(R) CPU E7-4809 v2 @ 1.90GHz and 756GB of RAM, and each time is an average of 100 repetitions of the corresponding algorithm. As is evident from Table 2, our algorithm performs better than Guan's algorithm, especially when the size of entries in matrices becomes larger and larger.

Table 2 Comparative performance of MLP factorization algorithms

Example	Guan's algorithm t_1 (sec)	Algorithm 1 t_2 (sec)	Time comparison t_1/t_2
$\boldsymbol{F}_1$	0.257	0.037	6.95
$\boldsymbol{F}_2$	0.263	0.044	5.98
$\boldsymbol{F}_3$	0.132	0.058	2.28
$\boldsymbol{F}_4$	0.407	0.063	6.46
$\boldsymbol{F}_5$	3.060	0.151	20.26
$\boldsymbol{F}_6$	4.275	0.283	15.11
$\boldsymbol{F}_7$	9.037	0.330	27.38
$\boldsymbol{F}_8$	17.306	0.549	31.52

6 Concluding Remarks

In this paper, we have given a new necessary and sufficient condition for the existence of MLP factorizations of multivariate polynomial matrices. All cases with matrices being full row rank and non-full row rank are considered. Based on the new result, a constructive algorithm for computing MLP factorizations has been proposed. We have implemented Algorithm 1 and Guan's algorithm on Maple, and the experimental data in Table 2 suggests that Algorithm 1 is superior in practice in comparison with Guan's algorithm. This is due to the fact that we can determine whether a multivariate polynomial matrix has an MLP factorization through less calculations.

Conflict of Interest

The authors declare no conflict of interest.

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Appendix

For all examples in Table 2, the monomial orders used on $k[\boldsymbol{z}]$ and $k[\boldsymbol{z}]^{1\times m}$ are degree reverse lexicographic order and position over term, respectively. k is the complex field $\mathbb{C}$, and $z_1 > z_2 > z_3$.

1) $\boldsymbol{F}_1 \in \mathbb{C}[z_1, z_2, z_3]^{3\times 3}$ is as follows, and it has no MLP factorizations.

$$\boldsymbol{F}_1 = \begin{bmatrix} z_1z_2 + z_1 - z_2 - 1 & 0 & z_3 \\ z_2 + 1 & z_2 + 1 & z_1 - 1 \\ z_1z_2 + z_1 & z_2 + 1 & z_1 + z_3 - 1 \end{bmatrix}.$$

2) $\boldsymbol{F}_2 \in \mathbb{C}[z_1, z_2, z_3]^{3\times 3}$ is as follows, and it has no MLP factorizations.

$$\boldsymbol{F}_2 = \begin{bmatrix} z_1z_2 - z_2 & 0 & z_3 + 1 \\ 0 & z_1z_2 - z_2 & z_1^2 - 2z_1 + 1 \\ z_1^2z_2 - z_1z_2 & z_1z_2^2 - z_2^2 & z_1^2z_2 - 2z_1z_2 + z_1z_3 + z_1 + z_2 \end{bmatrix}.$$

3) $\boldsymbol{F}_3 \in \mathbb{C}[z_1, z_2, z_3]^{3\times 3}$ is as follows, and it has an MLP factorization.

$$\boldsymbol{F}_3 = \begin{bmatrix} z_1 z_2^2 & z_1 z_3^2 & z_2^2 z_3 + z_3^3 \\ z_1 z_2 & 0 & z_2 z_3 \\ 0 & z_1^2 z_3 & z_1 z_3^2 \end{bmatrix}.$$

4) $\boldsymbol{F}_4 \in \mathbb{C}[z_1, z_2, z_3]^{3\times 3}$ is as follows, and it has an MLP factorization.

$$\boldsymbol{F}_4 = \begin{bmatrix} z_1^2 z_2 + z_1^2 & z_1 & 0 \\ z_1 z_3^2 - z_1 z_3 & 0 & z_2 z_3 - z_2 + z_3 - 1 \\ 2z_1^2 z_2 z_3 - z_1^2 z_2 + z_1^2 z_3 - z_1^2 & z_1 z_3 - z_1 & z_1 z_2^2 + z_1 z_2 \end{bmatrix}.$$

5) $\boldsymbol{F}_5 \in \mathbb{C}[z_1, z_2, z_3]^{3\times 3}$ is as follows, and it has an MLP factorization.

$$\boldsymbol{F}_5 \begin{cases} \boldsymbol{F}_5[1,1] = z_1^2 - z_1, \quad \boldsymbol{F}_5[1,2] = -z_2 z_3 + z_1 - z_3, \quad \boldsymbol{F}_5[1,3] = z_1 z_3 - 2z_1 - z_3, \\ \boldsymbol{F}_5[2,1] = z_1^3 z_2 z_3 - z_1^3 z_3 - z_1^2 z_2 z_3 - z_1^2 z_2 + 2z_1^2 z_3 + z_1^2 + z_1 z_2 - z_1 z_3 - z_1, \\ \boldsymbol{F}_5[2,2] = -z_1^2 z_2^2 z_3 - z_1 z_2^2 z_3^2 - z_1^2 z_2 z_3 - z_1 z_2 z_3^2 + z_1 z_2^2 - 2z_1^2 z_3 \\ \qquad\qquad + z_2^2 z_3 - z_2 z_3^2 + z_1 z_2 + z_1 z_3 + z_2 z_3 - z_3^2 + 2z_1, \\ \boldsymbol{F}_5[2,3] = z_1^2 z_2 z_3^2 - 3z_1^2 z_2 z_3 - z_1^2 z_3^2 - z_1 z_2 z_3^2 + z_1^2 z_3 - z_1 z_2 z_3 \\ \qquad\qquad + z_1 z_3^2 + 3z_1 z_2 - z_1 z_3 + z_2 z_3 - z_3^2 - z_1, \\ \boldsymbol{F}_5[3,1] = z_1^2 z_2^3 - z_1^2 z_2^2 - z_1 z_2^3 + z_1 z_2^2 + z_1 z_2 - z_2, \\ \boldsymbol{F}_5[3,2] = -z_1 z_2^4 - z_2^4 z_3 - z_1 z_2^3 - z_2^3 z_3 - 2z_1 z_2^2 + z_2^3 + 2z_2^2 + 2z_2, \\ \boldsymbol{F}_5[3,3] = z_1 z_2^3 z_3 - 3z_1 z_2^3 - z_1 z_2^2 z_3 - z_2^3 z_3 + z_1 z_2^2 + z_2^2 + z_2 z_3 - z_2. \end{cases}$$

6) $\boldsymbol{F}_6 \in \mathbb{C}[z_1, z_2, z_3]^{3\times 3}$ is as follows, and it has an MLP factorization.

$$\boldsymbol{F}_6 \begin{cases} \boldsymbol{F}_6[1,1] = z_1^3 z_2^2 + 2z_1^3 z_2 + z_1^3 - z_1 z_2^2 - z_1 z_2 z_3 - z_1 z_2 - z_1 z_3 \\ \qquad\qquad + z_2 z_3 - z_2 + z_3 - 1, \\ \boldsymbol{F}_6[1,2] = -z_1 z_2^3 z_3 + z_1^2 z_2^2 - 3z_1 z_2^2 z_3 - 2z_2^3 z_3 + 2z_1^2 z_2 + z_1 z_2^2 \\ \qquad\qquad + z_2^3 - 3z_1 z_2 z_3 - 6z_2^2 z_3 + z_1^2 + 2z_1 z_2 + 3z_2^2 - z_1 z_3 \\ \qquad\qquad - 7z_2 z_3 + z_1 + 4z_2 - 3z_3 + 2, \\ \boldsymbol{F}_6[1,3] = z_1^2 z_2^2 z_3 - 2z_1^2 z_2^2 + 2z_1^2 z_2 z_3 - 4z_1^2 z_2 - 2z_1 z_2^2 + z_1^2 z_3 - 2z_2^2 z_3 \\ \qquad\qquad - z_2 z_3^2 - 2z_1^2 - 4z_1 z_2 + z_2^2 - z_2 z_3 - z_3^2 - 2z_1 + z_3 - 1, \\ \boldsymbol{F}_6[2,1] = z_1^2 z_3 - z_1^2 + 2z_1 z_2 - z_1 z_3 - 2z_2 + 1, \\ \boldsymbol{F}_6[2,2] = z_1 z_2^2 + 2z_2^3 - z_2 z_3^2 + 2z_1 z_2 + 3z_2^2 + z_1 z_3 + 2z_2 z_3 \\ \qquad\qquad - z_3^2 + 2z_2 + 2z_3 - 2, \\ \boldsymbol{F}_6[2,3] = z_1 z_3^2 + z_1 z_2 + 2z_2^2 - 3z_1 z_3 + 2z_2 z_3 - z_3^2 + 3z_1 - 3z_2 + z_3 + 1, \\ \boldsymbol{F}_6[3,1] = z_1^2 z_2 z_3 + z_1^2 z_2 + 2z_1^2 z_3 - z_1 z_2 z_3 + 2z_1^2 - z_1 z_2 - 2z_1 z_3 - 2z_1, \\ \boldsymbol{F}_6[3,2] = -z_2^2 z_3^2 + z_1 z_2 z_3 - z_2^2 z_3 - 3z_2 z_3^2 + z_1 z_2 + 2z_1 z_3 \\ \qquad\qquad - 3z_2 z_3 - 2z_3^2 + 2z_1 - 2z_3, \\ \boldsymbol{F}_6[3,3] = z_1 z_2 z_3^2 - z_1 z_2 z_3 + 2z_1 z_3^2 - z_2 z_3^2 - 2z_1 z_2 - 2z_1 z_3 \\ \qquad\qquad - z_2 z_3 - 2z_3^2 - 4z_1 - 2z_3. \end{cases}$$

7) $\boldsymbol{F}_7 \in \mathbb{C}[z_1, z_2, z_3]^{3\times 4}$ is as follows, and it has an MLP factorization.

$$\boldsymbol{F}_7 \begin{cases} \boldsymbol{F}_7[1,1] = 2z_1^3z_2^3z_3^2 - z_1^3z_2^3z_3 + 2z_1^4z_3^3 - z_1^4z_3^2 + z_1^3z_2z_3^2 + z_1^3z_3^3 \\ \qquad + 3z_1z_2^3z_3 + 2z_1^3z_3^2 + z_1z_2^2z_3^2 + 2z_1z_2^2z_3 + 4z_1^2z_3^2 \\ \qquad - z_1^2z_3 + z_1z_2z_3 + z_1z_3^2 + 2z_1z_3 + 2z_3, \\ \boldsymbol{F}_7[1,2] = 2z_1z_2^3z_3^2 - z_1z_2^3z_3 + 2z_1^2z_3^3 - z_1^2z_3^2 + 2z_3^2 - z_3, \\ \boldsymbol{F}_7[1,3] = z_1^3z_3^2 + z_1z_2^2z_3 + z_1z_3, \\ \boldsymbol{F}_7[1,4] = 2z_1z_2^3z_3^2 + z_1^3z_3^3 - z_1z_2^3z_3 + z_1z_2^2z_3^2 + 2z_1^2z_3^3 \\ \qquad - z_1^2z_3^2 + z_1z_3^2 + 2z_3^2 - z_3, \\ \boldsymbol{F}_7[2,1] = -2z_1^2z_3^2 + z_1^2z_3 - z_1z_2z_3 - z_1z_3^2 - 2z_1z_3 - 2z_3, \\ \boldsymbol{F}_7[2,2] = -2z_3^2 + z_3, \quad \boldsymbol{F}_7[2,3] = -z_1z_3, \\ \boldsymbol{F}_7[2,4] = -z_1z_3^2 - 2z_3^2 + z_3, \\ \boldsymbol{F}_7[3,1] = -2z_1^4z_2^3z_3 + z_1^4z_2^3 - 2z_1^5z_3^2 + z_1^5z_3 - z_1^4z_2z_3 - z_1^4z_3^2 \\ \qquad - 3z_1^2z_2^3 - 2z_1^4z_3 - z_1^2z_2^2z_3 - 2z_1^2z_2^2 - 4z_1^3z_3 - 2z_1^2z_2z_3 \\ \qquad + z_1^3 - z_1^2z_3 - 2z_1^2 - 2z_1 - 3z_2 - z_3 - 2, \\ \boldsymbol{F}_7[3,2] = -2z_1^2z_2^3z_3 + z_1^2z_2^3 - 2z_1^3z_3^2 + z_1^3z_3 \\ \qquad - 2z_1z_3 - 2z_2z_3 + z_1 + z_2, \\ \boldsymbol{F}_7[3,3] = -z_1^4z_3 - z_1^2z_2^2 - z_1^2 - 1, \\ \boldsymbol{F}_7[3,4] = -2z_1^2z_2^3z_3 - z_1^4z_3^2 + z_1^2z_2^3 - z_1^2z_2^2z_3 - 2z_1^3z_3^2 + z_1^3z_3 \\ \qquad - z_1^2z_3 - 2z_1z_3 - 2z_2z_3 + z_1 + z_2 - z_3. \end{cases}$$

8) $\boldsymbol{F}_8 \in \mathbb{C}[z_1, z_2, z_3]^{3\times 3}$ is as follows, and it has an MLP factorization.

$$\boldsymbol{F}_8 \begin{cases} \boldsymbol{F}_8[1,1] = z_1^3z_3^2 - z_1^3z_3 + z_1^2z_2z_3 - z_1^2z_3^2 + z_2z_3^3 + z_3^4 - z_1^2z_2 \\ \qquad - z_1z_2z_3 - z_2z_3^2 - 2z_3^3 + z_1^2 + z_1z_2 + z_1z_3 + z_2z_3 \\ \qquad + 2z_3^2 - z_1 - z_2 - 2z_3 + 1, \\ \boldsymbol{F}_8[1,2] = z_1^4z_2z_3 + z_1^3z_2^2 - z_1^3z_2z_3 + z_1z_2^2z_3^2 + z_1z_2z_3^3 - z_1^3z_2 - z_1^2z_2^2 \\ \qquad + z_1^3z_3 - z_1z_2z_3^2 + 2z_1^2z_2 + z_1z_2^2 - z_1^2z_3 + z_1z_2z_3 + z_2z_3^2 \\ \qquad + z_3^3 - z_1^2 - 2z_1z_2 - z_3^2 + z_1 + z_2 + z_3 - 1, \\ \boldsymbol{F}_8[1,3] = 2z_1^3z_2 - z_1^3 - 2z_1^2z_2 + z_2^2z_3 + z_2z_3^2 + z_1^2 \\ \qquad + z_1z_2 + z_1z_3 - z_2z_3 - z_1, \\ \boldsymbol{F}_8[2,1] = -z_1z_2z_3^3 + z_1^2z_3^2 + z_1z_2z_3^2 - z_2^2z_3^2 - z_1^2z_3 + z_1z_2z_3 + z_2^2z_3 \\ \qquad - z_3^3 - z_1z_2 + 2z_3^2 - 2z_3 + 1, \\ \boldsymbol{F}_8[2,2] = -z_1^2z_2^2z_3^2 + z_1^3z_2z_3 - z_1z_2^3z_3 + z_1^2z_2^2 - 2z_1z_2z_3^2 + z_1^2z_3 \\ \qquad + z_1z_2z_3 - z_2^2z_3 - z_3^2 + z_3 - 1, \\ \boldsymbol{F}_8[2,3] = -2z_1z_2^2z_3 + 2z_1^2z_2 + z_1z_3 - z_2z_3 - z_1, \\ \boldsymbol{F}_8[3,1] = z_1z_2^2z_3^2 + z_1z_3^4 - z_1z_2^2z_3 + z_2^3z_3 + 2z_1z_2z_3^2 - 3z_1z_3^3 \\ \qquad - z_3^4 - z_2^3 - 2z_1z_2z_3 + z_2^2z_3 + 5z_1z_3^2 + z_3^3 - z_2^2 \\ \qquad - 5z_1z_3 - z_3^2 + 2z_1 - z_3 + 2, \\ \boldsymbol{F}_8[3,2] = z_1^2z_2^3z_3 + z_1^2z_2z_3^3 + z_1z_2^4 + 2z_1^2z_2^2z_3 - 2z_1^2z_2z_3^2 - z_1z_2z_3^3 \\ \qquad + z_1z_2^3 + 3z_1^2z_2z_3 + z_1z_2^2z_3 + z_1z_3^3 - z_2^3 + z_1z_2z_3 - 2z_1z_3^2 \\ \qquad - 2z_1^2z_2 + z_3^3 - 2z_1z_2 + z_2^2 + 3z_1z_3 - 2z_1 - z_3 - 2, \\ \boldsymbol{F}_8[3,3] = 2z_1z_2^3 + z_1z_2z_3^2 + 3z_1z_2^2 + z_1^2z_3 - 2z_1z_2z_3 \\ \qquad - z_2z_3^2 - 2z_1^2 + 2z_1z_2 - z_1z_3 - 2z_1. \end{cases}$$