Completing Parametric Unimodular Rows to Unimodular Matrices

Ligeng Fan
State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences
Beijing, China
School of Mathematical Sciences, University of Chinese Academy of Sciences
Beijing, China
fanligeng@amss.ac.cn

Dingkang Wang
State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences
Beijing, China
School of Mathematical Sciences, University of Chinese Academy of Sciences
Beijing, China
dwang@mmrc.iss.ac.cn

Fanghui Xiao
MOE-LCSM, School of Mathematics and Statistics, Hunan Normal University
Changsha, China
xiaofanghui@hunnu.edu.cn

Xiaopeng Zheng
College of Mathematics and Computer Science, Shantou University
Shantou, China
xiaopengzheng@stu.edu.cn

Abstract

Serre's conjecture, stating that every finitely generated projective module over a polynomial ring is free, was proven by Quillen and Suslin independently in 1976. An equivalent form of the Quillen-Suslin theorem says, "Every unimodular row over a polynomial ring can be completed to a unimodular matrix." In this paper, we generalize the Quillen-Suslin theorem to the parametric case and present an algorithm to construct the unimodular completion matrix system for any polynomial vector with parameters. Specifically, we first determine the conditions on the parameters under which the vector is unimodular using the comprehensive Gröbner system. Furthermore, we use a constructive method to find a finite partition of the parameter space such that, for each branch, the vector under specializations can be completed into a unimodular matrix in the same form. Since the method is constructive, we present an explicit algorithm to construct the unimodular completion matrix system for any polynomial vector with parameters. The correctness and termination of the algorithm have been proven, and an example is provided to demonstrate how the algorithm works.

CCS Concepts

• **Computing methodologies** → **Symbolic and algebraic algorithms**; **Algebraic algorithms**.

Keywords

Quillen-Suslin theorem, parametric polynomial vector, unimodular row, unimodular completion matrix system, comprehensive Gröbner system.

ISSAC '25, Guanajuato, Mexico

ACM ISBN 979-8-4007-2075-8/25/07
https://doi.org/10.1145/3747199.3747554

ACM Reference Format:
Ligeng Fan, Dingkang Wang, Fanghui Xiao, and Xiaopeng Zheng. 2025. Completing Parametric Unimodular Rows to Unimodular Matrices. In *International Symposium on Symbolic and Algebraic Computation (ISSAC '25), July 28–August 01, 2025, Guanajuato, Mexico.* ACM, New York, NY, USA, 9 pages. https://doi.org/10.1145/3747199.3747554

1 Introduction

Serre, in his influential paper [21] "Faisceaux algébriques cohérents", remarked that the corresponding question was not known for algebraic vector bundles: "It is not known whether there exist projective A-modules of finite type which are not free." Here A is a polynomial ring over a field. Serre made some progress towards a solution in 1957 when he proved that every finitely generated projective module P over a polynomial ring over a field was stably free, i.e. $P \oplus A^t \simeq A^s$. In view of this, Serre's conjecture becomes the following: does "stably free" imply "free" over $k[x_1, \ldots, x_n]$? In fact, in the aspect of polynomial matrix, the question is then equivalent to whether any unimodular row over a polynomial ring can be completed to a unimodular matrix (refer to [27, Proposition 20]).

In 1976, Quillen [19] and Suslin [23] proved independently that Serre's conjecture is true and gave a positive answer to the above problem, which means that a polynomial vector $(f_1, \ldots, f_m)$ can be completed to a unimodular matrix if $f_1, \ldots, f_m$ generate the unit ideal. This theorem is commonly referred as Quillen-Suslin theorem.

Another common equivalent form of Quillen-Suslin theorem says that if $f_1, \ldots, f_m$ generate the unit ideal, then there exists a $m \times m$ unimodular polynomial matrix U such that

$$(f_1, \ldots, f_m) \cdot U = (1, 0, \ldots, 0).$$

The equivalence between these two statements comes from an easy observation that U^{-1} is unimodular with $(f_1, \ldots, f_m)$ as its first row.

Historically, the original proof on Serre's conjecture is much sophisticated. Thanks to Vaserstein and Suslin's subsequent work using the technique of the completion of unimodular rows, we can now have this much elementary and concise expression. Their work also established the study of unimodular rows as a new interesting theme for research in commutative algebra, which have many applications, such as control theory [28]. More references related to Serre's conjecture can be found in Lam's book [13].

The constructive version of the Quillen-Suslin theorem is presented by [20] for $A = k[x_1, x_2]$, where k is a field, and by [15] for $A = \mathbb{C}[x_1, \ldots, x_n]$. Inspired by the methods proposed by Logar and Sturmfels [15], the algorithm of the constructive versions of Quillen-Suslin theorem was presented in [5] and has been implemented in computer algebra system *Maple* for $A = \mathbb{Q}[x_1, \ldots, x_n]$.

Many engineering problems are parameterized and have to be repeatedly solved for different values of parameters [4]. A parametric system in mathematics typically refers to a set of parameters that can describe a family of objects or solutions to a given problem. By adjusting parameters, we can generate and study different instances of a system. This flexibility is crucial in fields like control theory, coding theory and machine leaning.

Comprehensive Gröbner system (CGS) is one of the most efficient methods for solving parametric polynomial systems. The definition of comprehensive Gröbner systems was introduced by Weispfenning [26]. Since then, the comprehensive Gröbner system plays an important role in various parametric computer algebra problems [8, 9, 25] and has applications including automated geometry theorem proving [3] and discovery [17]. Many algorithms for computing the comprehensive Gröbner system have been proposed [7, 16, 18, 24].

In this paper, we will extend the Quillen-Suslin theorem to the parametric case. Given $\mathbf{f}(U,X) = (f_1(U,X), \ldots, f_m(U,X)) \in k[U,X]^m$ with variables X and parameters U, the considered problem is to find finite sets $E_i, N_i \subset k[U]$, $i \in \{1, \ldots, l\}$ such that for all $\mathbf{a} \in \mathbb{V}(E_i)\backslash\mathbb{V}(N_i)$, $\mathbf{f}(\mathbf{a}, X) = (f_1(\mathbf{a}, X), \ldots, f_m(\mathbf{a}, X)) \in L[X]^m$ is either all unimodular or all non-unimodular, where L denotes an algebraically closed field containing k and $\mathbb{V}(E_i)$ is the variety of E_i in L^s. Furthermore, for the subsets $\mathbb{V}(E_i)\backslash\mathbb{V}(N_i)$ within that the vector is unimodular for all parameter values, complete this row vector into a polynomial matrix with parameters, such that for all parameter values within those subsets, the matrix is a unimodular matrix. That is, for all $\mathbf{a} \in \mathbb{V}(E_i)\backslash\mathbb{V}(N_i)$, if $(f_1(\mathbf{a}, X), \ldots, f_m(\mathbf{a}, X)) \in L[X]^m$ is unimodular, complete it into a unimodular matrix $\mathbf{M}_i(\mathbf{a}, X)$.

The main idea is as follows. By means of comprehensive Gröbner systems, we first partition the entire parameter space into a finite number of constructible subsets and select the unimodular branches where the given vector is unimodular under parametric specializations. In order to complete the unimodular rows into unimodular matrices under parametric specializations, subsequently, we need to decompose these constructible subsets into a finite disjoint union of irreducible ones (i.e., for each of them, the equation-constrained part generates a prime ideal) so as to construct the fraction fields induced by the prime ideals. More importantly, at this point, we can apply the Quillen-Suslin theorem to each constructed field to compute the completion matrix. Finally, for each branch, by pulling the matrix back to the original polynomial ring with a further partition of the parameter space, we iteratively obtain the unimodular completion matrix system. In other words, we find a finite partition of the parameter space such that, for each branch, the parametric vector under specialization can be completed into a unimodular matrix in the same form.

The structure of the paper is as follows. In Section 2 we give some preliminaries on the constructible set, unimodular property and parameter system. In Section 3, we state each step of our method in details and prove some conclusions used in the analysis of our constructions. We present the algorithm in Section 4 and list an example in Section 5. A conclusion is given in the final section.

2 Preliminaries

In this section, we will introduce some basic notations and definitions to prepare for the following discussion.

Let k be an arbitrary field, and let L denote an algebraically closed field containing k. Define $k[U][X]$ as the polynomial ring in the parametric variables $U = \{u_1, u_2, \ldots, u_s\}$ and the variables $X = \{x_1, x_2, \ldots, x_n\}$, where U and X are disjoint sets. We generally use the letters d, f, g, h to denote polynomials, boldface letters such as $\mathbf{a}, \mathbf{f}, \mathbf{x}$ for vectors, and boldface capital letters such as $\mathbf{P}, \mathbf{H}, \mathbf{M}$ for matrices.

2.1 Constructible set and function field

For a polynomial set E , the ideal generated by the elements of E is denoted by $\langle E\rangle$. We will add a corner mark if it is necessary to emphasize the concrete polynomial ring which the ideal belongs to. i.e., $\langle E\rangle_{k[U][X]}$ stands for an ideal in $k[U][X]$ generated by the entries of E.

For a set $F \subseteq k[U]$, the affine (algebraic) variety defined by F in L^s is denoted by $\mathbb{V}(F)$. i.e., $\mathbb{V}(F) = \{\mathbf{a} \in L^s \mid f(\mathbf{a}) = 0, \text{ for all } f \in F\}$. Conversely, for any variety V, we use $\mathbb{I}(V)$ to denote the ideal of all polynomials that are zero on V, i.e., $\mathbb{I}(V) = \{f \in k[U] \mid f(\mathbf{a}) = 0, \text{ for all } \mathbf{a} \in V\}$.

For an ideal I in $k[U]$, we denote its radical by $\sqrt{I}$. By the Hilbert Nullstellensatz, there is a one-to-one correspondence between varieties in L^s and radical ideals in $k[U]$. A variety is **irreducible** if it cannot be written as the union of two proper varieties. It is well-known that a variety is irreducible if and only if the corresponding radical ideal is prime.

Next, we review the definition of (principle) constructible sets, which plays an important part in describing the different branches in a parameter system.

Definition 2.1. A set $C \subseteq L^s$ is called a **principal constructible set** if $C = \mathbb{V}(E) \setminus \mathbb{V}(N)$ for subsets E, N in $k[U]$. $C \subseteq L^s$ is called a **constructible set** if it can be written as finite union of principal constructible sets.

For simplicity in the paper, we call a principle constructible set $\mathbb{V}(E) \setminus \mathbb{V}(N)$ irreducible if $\mathbb{V}(E)$ is irreducible. By the above definition, any constructible set can be written as a finite union of principal constructible sets. Furthermore, according to the following lemma [6, Proposition 7.22], with some modifications, we can ensure that the principal constructible sets in the decomposition are disjoint.

Lemma 2.2. *Every constructible set can be decomposed as a finite disjoint union of principal constructible sets.*

Remark 2.3. *Above Lemma 2.2 's proof is constructive, involving only basic operations on varieties. Thus we can extract an algorithmic process for splitting a constructible set into disjoint unions.*

If V is a variety, then the quotient ring $k[U]/\mathbb{I}(V)$ is called the **coordinate ring** of V, denoted by $k[V]$. The elements of the coordinate ring $k[V]$ are also called the **regular functions** or the **polynomial functions on the variety**. If V is irreducible, then $\mathbb{I}(V)$ is prime, so the coordinate ring is an integral domain. By the coordinate ring, we can construct a field corresponding to the irreducible variety, which is called the function field of V.

Definition 2.4. Let V be an irreducible variety and $k[V]$ be the coordinate ring of V. The field of fractions of $k[V]$ is called the **function field** of V, denoted by $k(V)$.

Remark 2.5. *Let V be an irreducible variety. Then by the definition of function field, we have*

$$k(V) = \{[f]/[g] : [f], [g] \in k[V],\ g \notin \mathbb{I}(V)\}.$$

Let V be an irreducible variety. Let $\mathcal{R} = k[V]$ and $\mathcal{K} = k(V)$. The canonical map π is defined as

$$\pi : k[U] \to \mathcal{R}, \quad a \mapsto [a].$$

For a polynomial $f \in k[U][X]$, the canonical map can be extended as

$$\pi : k[U][X] \to \mathcal{R}[X], \quad \sum_{\alpha} c_\alpha X^\alpha \mapsto \sum_{\alpha} [c_\alpha] X^\alpha.$$

For a polynomial vector $\mathbf{f} \in k[U][X]^m$, the canonical map can be further extended as

$$\pi : k[U][X]^m \to \mathcal{R}[X]^m \quad (f_1, \ldots, f_m) \mapsto (\pi(f_1), \ldots, \pi(f_m)).$$

2.2 Comprehensive Gröbner system

In this section, we introduce the comprehensive Gröbner system, which generalizes Gröbner bases for an ideal to parametric cases.

In a parametric polynomial system with parameters $U = \{u_1, \ldots, u_s\}$, the full parametric space is L^s. To describe the different properties of the system under different parameter values, we have to partition the L^s into disjoint branches. In this paper we will always use constructible sets to partition L^s. That is, by saying a (parametric) branch, we mean a constructible set in L^s.

A **specialization** induced by $\mathbf{a} = (a_1, \ldots, a_s) \in L^s$ is a homomorphism $\sigma_{\mathbf{a}} : k[U] \to L$ which sends $h(U)$ to $h(\mathbf{a})$. We can naturally extend the definition of $\sigma_{\mathbf{a}}$ to $k[U][X] \to L[X]$ by applying $\sigma_{\mathbf{a}}$ coefficient-wise. For $f(U, X) \in k[U][X]$, we may also write specialization as $f(\mathbf{a}, X)$. We can also define specialization for a set (resp. matrix) in $k[U][X]$ by taking specialization at every entries and return a set (resp. matrix) in $L[X]$.

As stated in Introduction, the main purpose of this paper is to generalize the Quillen-Suslin theorem to parametric case. For the parametric polynomial system, a very useful tool is the comprehensive Gröbner system. In fact, the first step in our algorithm is to compute the comprehensive Gröbner system.

We first briefly introduce Gröbner bases. Let $>$ be a monomial order in $k[X]$, the leading term, leading coefficient and leading monomial of a polynomial $f \in k[X]$ with respect to $>$ are denoted by $\mathrm{LT}(f)$, $\mathrm{LC}(f)$ and $\mathrm{LM}(f)$ respectively.

Definition 2.6. Let $>$ be a monomial order in $k[X]$ and I be an ideal in $k[X]$.

(1) We denote by $\mathrm{LM}(I)$ the monomial ideal generated by the leading terms of $f \in I$ with respect to $>$.
(2) A finite collection $G = \{g_1, \ldots, g_t\} \subset I$ is called a **Gröbner basis** of I if

$$\langle \mathrm{LM}(I) \rangle = \langle \mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_t) \rangle.$$

We now introduce the definition of the comprehensive Gröbner system for a parametric polynomial system.

Definition 2.7. Let F be a subset of $k[U][X]$, S be a subset of L^s, $G_1, \ldots, G_l$ be subsets of $k[U][X]$, and $A_1, \ldots, A_l$ be algebraically constructible subsets of L^s such that $S = \bigcup_{i=1}^{l} A_i$. A finite set $\mathcal{G} = \{(A_1, G_1), ..., (A_l, G_l)\}$ is called a **comprehensive Gröbner system (CGS)** on S for F if $\sigma_{\mathbf{a}}(G_i)$ is a Gröbner basis of the ideal $\langle \sigma_{\mathbf{a}}(F) \rangle \subseteq L[X]$ for $\mathbf{a} \in A_i$ and $i = 1, \ldots, l$. Each (A_i, G_i) is called a branch of $\mathcal{G}$. In particular, if $S = L^s$, then $\mathcal{G}$ is called a comprehensive Gröbner system for F.

Definition 2.8. A comprehensive Gröbner system $\mathcal{G} = \{(A_1, G_1), \ldots, (A_l, G_l)\}$ on S for F is said to be **minimal**, if for every $i = 1, \ldots, l$,

(1) $A_i \neq \emptyset$, and furthermore, for each $j = 1, \ldots, l, A_i \cap A_j = \emptyset$ whenever $i \neq j$;
(2) $\sigma_{\mathbf{a}}(G_i)$ is a minimal Gröbner basis of $\langle \sigma_{\mathbf{a}}(F) \rangle \subset L[X]$ for $\mathbf{a} \in A_i$;
(3) for each $g \in G_i, \sigma_{\mathbf{a}}(\mathrm{LC}_X(g)) \neq 0$ for any $\mathbf{a} \in A_i$, where $\mathrm{LC}_X(g)$ is the leading term of g viewed as a polynomial with variables X.

More details about the comprehensive Gröbner system can be found in [10, 11, 26]. In [10, 11], an efficient algorithm for computing the comprehensive Gröbner System is also presented.

2.3 Unimodular row completing and Quillen-Suslin theorem

In this subsection, we begin with the definition of some special modules.

Definition 2.9. Let A be a commutative ring and let M be a module over A.

(1) M is **free** if $M \simeq A^r$, for some $r \in \mathbb{N}$;
(2) M is **stable free** if $M \oplus A^s \simeq A^r$, for some $r, s \in \mathbb{N}$;
(3) M is **projective** if $M \oplus N \simeq A^r$, for some $r \in \mathbb{N}$ and A-module N.

In [21], Serre conjectured that every finitely generated projective module over a polynomial ring is free. He made progress toward a solution in 1957 when he proved that every finitely generated projective module over a polynomial ring over a field is stably free. However, the problem of whether a stably free module is free remained open until 1976. And this problem is closely related to the unimodular extension property of a unimodular row.

Definition 2.10. Let A be a commutative ring.

(1) A vector $\mathbf{f} = (f_1, f_2, \ldots, f_m) \in A^m$ is said to be a **unimodular row** over A if $f_1, f_2, \ldots, f_m$ generate the unit ideal.

(2) A matrix $\mathbf{M} \in A^{m\times m}$ is said to be a **unimodular matrix** over A if its determinant is a unit in A. The set of all $m \times m$ unimodular matrices over A is denoted by $\mathrm{GL}_m(A)$.
(3) A vector $\mathbf{f} = (f_1, f_2, \ldots, f_m) \in A^m$ is said to be **unimodular completable** if there exists a matrix in $\mathrm{GL}_m(A)$ with first row $\mathbf{f}$.
(4) A matrix $\mathbf{M}$ is said to be a **unimodular completion matrix** of $\mathbf{f}$ if $\mathbf{M} \in \mathrm{GL}_m(A)$ with first row $\mathbf{f}$.

In fact, we have the following theorem, which implies that if any unimodular row is completable over A, then every stably free module over A is free. This theorem is well-known, and its proof can be found in standard textbooks such as [14].

THEOREM 2.11. *Let A be a commutative ring. If any unimodular row in A is completable, every stably free module over A is free.*

In 1976, Quillen and Suslin prove that any unimodular row in the polynomial ring is completable and solve the Serre's Conjecture.

THEOREM 2.12 (QUILLEN-SUSLIN). *Let k be a field and let $\mathbf{f}$ be a unimodular vector in $k[X]^m$. Then $\mathbf{f}$ is unimodular completable.*

REMARK 2.13. *Actually, Quillen-Suslin theorem still holds when the entries of a unimodular row are polynomials over a principal ideal domain, but we will only use the case when k is a field.*

Example 2.14. Let

$$\begin{aligned}\mathbf{f}(x,y) &= \left(0, x^2+y^2+xy, x^2+x-y, y^2-y-2\right)\\ &= (f_1, f_2, f_3, f_4) \in \mathbb{Q}[x,y]^{1\times 4}.\end{aligned}$$

By computing the Gröbner basis of $\langle f_1, f_2, f_3, f_4\rangle$ under any monomial order, we have

$$\langle f_1, f_2, f_3, f_4\rangle = \mathbb{Q}[x,y],$$

that is, $\mathbf{f}$ is unimodular over $\mathbb{Q}[x,y]$.

According to Quillen-Suslin theorem (i.e., Theorem 2.12), we can complete $\mathbf{f}$ to a unimodular matrix over $\mathbb{Q}[x,y]$:

$$\begin{pmatrix} 0 & x^2+y^2+xy & x^2+x-y & y^2-y-2 \\ 0 & x^2+2x+y & -1 & x^2+x+y-1 \\ 1 & 0 & 0 & 0 \\ 0 & x^2-4xy+y^2+x-3y-14 & x^2+6x-y+3 & -5xy+y^2-4x-4y-10 \end{pmatrix}$$

whose determinant is $28 \in \mathbb{Q} \setminus \{0\}$.

In the following content of the paper, we will apply the Quillen-Suslin theorem over function fields of irreducible varieties.

The main idea for the algorithmic version of the Quillen-Suslin theorem [15] is by induction on n, i.e. the number of the variables. Each round we eliminate a variable and maintain the unimodular relations. Specifically, we compute the unimodular matrices at finitely many suitable local rings (this process is also known as Horrocks theorem, see [13, Chapter 4]) and patching together these local solutions to reach the target. The whole process can be done using the Gröbner basis and resultant method.

2.4 Unimodular completion matrix system

As mentioned in Section 2.3, the problem of completing unimodular rows is crucial for Serre's conjecture, and it has been a central problem in classical K-theory. In this paper, we will consider the completion of parametric vectors. More precisely, we will define the unimodular completion matrix system of a parametric vector as follows.

Definition 2.15. Let $\mathbf{f}(U,X) \in k[U][X]^m$ be a vector with parameters U. A finite collection $\mathcal{M} = \{(A_0, \emptyset), (A_1, \mathbf{M}_1), \ldots, (A_l, \mathbf{M}_l)\}$ is called a **unimodular completion matrix system** of $\mathbf{f}$ if it satisfies

(1) $A_0, A_1, \ldots, A_l$ are disjoint constructible subsets of L^s such that $\bigcup_{i=0}^{l} A_i = L^s$.
(2) for any $\mathbf{a} \in A_0$, $\mathbf{f}(\mathbf{a}, X)$ is not unimodular and cannot be completed to a unimodular matrix.
(3) for any $\mathbf{a} \in A_i$, the matrix $\mathbf{M}_i(\mathbf{a}, X)$ is a unimodular completion matrix of $\mathbf{f}(\mathbf{a}, X)$ for $i = 1, \ldots, l$.

The following example is used to explain Definition 2.15.

Example 2.16. Let $\mathbf{f}(u,x,y) \in \mathbb{Q}[u][x,y]^{1\times 4}$ be a parametric vector defined as

$$\left(-uy-y, x^2+ux+y^2+xy+x, x^2-ux-y, y^2+uy-u^2+u\right).$$

By the computation, the unimodular completion matrix system of $\mathbf{f}$ is

$$\begin{aligned}\{(\mathbb{V}(u(u-1)), \emptyset), (\mathbb{C} \setminus \mathbb{V}(u(u-1)(u+1)(2u+1), \mathbf{M}_1),\\ (\mathbb{V}(u+1), \mathbf{M}_2), (\mathbb{V}(2u+1), \mathbf{M}_3)\},\end{aligned}$$

which implies that for any $a \in \mathbb{C}$, we have

(1) if $(a-1)\,a = 0$, then $\mathbf{f}(a,x,y)$ can not be completed to a unimodular matrix.
(2) if $(a-1)a \neq 0$ and $(a+1)(2a+1) \neq 0$, then $\mathbf{f}(a,x,y)$ can be completed to a unimodular matrix $\mathbf{M}_1$ over $\mathbb{C}[x,y]$, where

$$\mathbf{M}_1 = \begin{pmatrix} -ay-y & x^2+ax+y^2+xy+x & x^2-ax-y & y^2+ay-a^2+a \\ a+1 & -y-x & 1 & -y-a \\ 0 & -x & 2a-x+1 & 0 \\ 0 & -a-x-1 & a-x & 0 \end{pmatrix}$$

and $\det(\mathbf{M}_1) = a(2a+1)(a-1)(a+1)^2$.
(3) if $a+1 = 0$, then $\mathbf{f}(a,x,y)$ can be completed to a unimodular matrix $\mathbf{M}_2$ over $\mathbb{C}[x,y]$, where

$$\mathbf{M}_2 = \begin{pmatrix} 0 & x^2+y^2+xy & x^2+x-y & y^2-y-2 \\ 0 & x^2+2x+y & -1 & x^2+x+y-1 \\ 1 & 0 & 0 & 0 \\ 0 & x^2-4xy+y^2+x-3y-14 & x^2+6x-y+3 & -5xy+y^2-4x-4y-10 \end{pmatrix}$$

and $\det(\mathbf{M}_2) = 28$.
(4) if $2a+1 = 0$, then then $\mathbf{f}(a,x,y)$ can be completed to a unimodular matrix $\mathbf{M}_3$ over $\mathbb{C}[x,y]$, where

$$\mathbf{M}_3 = \begin{pmatrix} -\frac{y}{2} & x^2+xy+y^2+\frac{1}{2}x & x^2+\frac{1}{2}x-y & y^2-\frac{1}{2}y-\frac{3}{4} \\ -1 & 2x+2y & -2 & -1+2y \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and $\det(\mathbf{M}_3) = \frac{3}{4}$.

3 Unimodular completion of parametric vectors

In this section, we will state our main theorem formally, and sketch the solving steps. The main result is as follows.

THEOREM 3.1. *For any polynomial vector $\mathbf{f}(U,X) = (f_1, \ldots, f_m) \in k[U][X]^m$, the unimodular completion matrix system of $\mathbf{f}$ is constructible.*

In Theorem 3.1, the constructibility of the unimodular completion matrix system of $\mathbf{f}$ means that it can be explicitly computed using a constructive algorithm. In the following, we sketch the processes to construct the unimodular completion matrix system for any polynomial vector with parameters:

Step 1 Find a constructible set $C \subset L^s$ such that for any $\mathbf{a} = (a_1, \ldots, a_s) \in L^s$, $\mathbf{f}(\mathbf{a}, X)$ is unimodular if and only if $\mathbf{a} \in C$. If $C = \emptyset$, return $\{(L^s, \emptyset)\}$ and we are done.

Step 2 Split C into finite disjoint union of non-empty principal constructible sets and each of these subset has the form $\mathbb{V}(E) \setminus \mathbb{V}(N)$ where $\langle E \rangle$ is a prime ideal.

Step 3 For each separated subset $C_i = \mathbb{V}(E_i) \setminus \mathbb{V}(N_i) \subseteq C$ computed in Step 2, construct the function field $\mathcal{K} = k(\mathbb{V}(E_i))$ of the irreducible variety $\mathbb{V}(E_i)$. Let $\pi(\mathbf{f}) = (\pi(f_1), \ldots, \pi(f_m)) \in \mathcal{K}[X]^m$ and $\pi(\mathbf{f})$ is unimodular over $\mathcal{K}[X]$. Complete $\mathbf{f}_{\mathcal{K}}$ to a unimodular matrix $\mathbf{M}_i^{(0)}$ over $\mathcal{K}[X]$ using Quillen-Suslin algorithm.

Step 4 Multiply the denominators and pull back the entries of $\mathbf{M}_i^{(0)}$ to the original ring $k[U][X]$. We get a new matrix $\mathbf{M}_i \in k[U][X]^{m \times m}$ and a polynomial $d_i \in k[U]$, such that for any $\mathbf{a} \in \mathbb{V}(E_i) \setminus \mathbb{V}(N \times \{d_i\})$, $\mathbf{M}_i(\mathbf{a}, X)$ is a unimodular completion matrix of $\mathbf{f}(\mathbf{a}, X)$ (Theorem 3.5).

Step 5 Consider $C_i' = \mathbb{V}(E_i \cup \{d_i\}) \setminus \mathbb{V}(N)$ and repeat Steps 2-4 for C_i' if C_i' is not empty. This process will terminate within a finite number of steps thanks to the Noetherian property.

Through iterations, we can get the desired unimodular completion matrix system of $\mathbf{f}$. In the following subsections, we will explain each step in details. We state the algorithm for computing the unimodular completion matrix system for a parametric vector and prove its termination in the next section.

3.1 Compute unimodular branches

For m-tuples $\mathbf{f}(U, X) = (f_1, \ldots, f_m) \in k[U][X]^m$, it may happen that for different choices $\mathbf{a}, \mathbf{b} \in L^s$, $\mathbf{f}(\mathbf{a}, X)$ is unimodular over $L[X]$ and $\mathbf{f}(\mathbf{b}, X)$ is not. Therefore, determining the unimodular parametric branches is prerequisite and necessary, which translates into determining when $f_1(U, X), \ldots, f_m(U, X)$ can generate the unit ideal in $L[X]$ after specialization. This can be solved easily by computing the comprehensive Gröbner system. Specifically, we have the following:

Lemma 3.2. *Let $\mathbf{f}(U, X) = (f_1, \ldots, f_m) \in k[U][X]^m$ be a polynomial vector. There exists a constructible set C such that for any $\mathbf{a} \in L^s$, $\mathbf{f}(\mathbf{a}, X)$ is unimodular if and only if $\mathbf{a} \in C$.*

Proof. We only need to compute the minimal comprehensive Gröbner system $\mathcal{G}$ for $\{f_1, \ldots, f_m\}$. Choose the branch (A_i, G_i) of $\mathcal{G}$ for which $G_i = \{1\}$ and label them as $(A_1, G_1), \ldots, (A_l, G_l)$. If no such branch exists, let $C = \emptyset$. Otherwise, let

$$C = \bigcup_{i=1}^{l} \mathbb{V}(E_i) \setminus \mathbb{V}(N_i),$$

where $A_i = \mathbb{V}(E_i) \setminus \mathbb{V}(N_i)$ for $i = 1, \ldots l$. By definition of minimal comprehensive Gröbner systems, for any $\mathbf{a} \in L^s$, the minimal Gröbner basis of $f_1(\mathbf{a}, X), \ldots, f_m(\mathbf{a}, X)$ is $\{1\}$ if and only if $\mathbf{a} \in C$. Since $f_1(\mathbf{a}, X), \ldots, f_m(\mathbf{a}, X)$ generate the unit ideal if and only if the minimal Gröbner basis of them is $\{1\}$, we conclude that $\mathbf{f}(\mathbf{a}, X)$ is unimodular if and only if $\mathbf{a} \in C$. ◻

3.2 Decompose into irreducible branches

By Lemma 3.2, we can construct a constructible set C which is a finite disjoint union of principal constructible sets. For each principal constructible set $\mathbb{V}(E) \setminus \mathbb{V}(N)$, we need to construct the function field of $\mathbb{V}(E)$ in Step 3. However, in order to construct the function field, the variety $\mathbb{V}(E)$ must be irreducible. Therefore, we need to further decompose every separated principal constructible sets appearing in C into a finite union of disjoint irreducible parts, ensuring that the equation - constrained part of each part define an irreducible variety.

Lemma 3.3. *Let C be a constructible set. Then C can be decomposed into a finite union of disjoint principal constructible sets (i.e. $C = \bigcup_{i=1}^{l} \mathbb{V}(E_i) \setminus \mathbb{V}(N_i)$), where each $\mathbb{V}(E_i)$ is irreducible.*

Proof. According to Lemma 2.2, we can decompose C into a finite union of disjoint principal constructible sets. If we can prove that any principal constructible set can be decomposed into the irreducible forms as stated, then we can finish the proof.

We may assume that $C = \mathbb{V}(E) \setminus \mathbb{V}(N)$ is a principal constructible set and $\mathbb{V}(E) = \bigcup_{i=1}^{l} \mathbb{V}(E_i)$ is the irreducible decomposition with each $\mathbb{V}(E_i)$ irreducible.

Therefore, we have the disjoint decomposition of $\mathbb{V}(E)$:

$$\begin{aligned} \mathbb{V}(E) =& \mathbb{V}(E_1) \cup (\mathbb{V}(E_2) \setminus \mathbb{V}(E_1)) \cup (\mathbb{V}(E_3) \setminus \mathbb{V}(E_1 \times E_2)) \\ & \cup \cdots \cup (\mathbb{V}(E_l) \setminus \mathbb{V}(\prod_{i=1}^{l-1} E_i)), \end{aligned}$$

where $A \times B = \{fg \mid f \in A, g \in B\}$. Removing $\mathbb{V}(N)$ from both sides of the equation , we have the disjoint decomposition of C:

$$\begin{aligned} C =& (\mathbb{V}(E_1) \setminus \mathbb{V}(N)) \cup (\mathbb{V}(E_2) \setminus \mathbb{V}(N \times E_1)) \\ & \cup (\mathbb{V}(E_3) \setminus \mathbb{V}(N \times E_1 \times E_2)) \cup \cdots \cup (\mathbb{V}(E_l) \setminus \mathbb{V}(N \times \prod_{i=1}^{l-1} E_i)) \end{aligned}$$

which is exactly the decomposition form we want. ◻

Next we give an algorithm for computing above irreducible decomposition of a principal constructible set. It comes straightforwardly from the above proof.

Algorithm 1 Compute irreducible decomposition of a principal constructible set C

Input: Two subsets $E, N \subset k[U]$ such that $C = \mathbb{V}(E) \setminus \mathbb{V}(N) \neq \emptyset$.
Output: $\{(E_1, N_1), \ldots, (E_l, N_l)\}$ such that each $\mathbb{V}(E_i)$ is irreducible, $\mathbb{V}(E_i) \setminus \mathbb{V}(N_i)$ for $i = 1, \ldots, l$ are pairwise disjoint and $C = \bigcup_{i=1}^{l} \mathbb{V}(E_i) \setminus \mathbb{V}(N_i)$.

1: Compute the prime decomposition of $\sqrt{\langle E \rangle}$, i.e., $\sqrt{\langle E \rangle} = \bigcap_{i=1}^{l} \langle E_i \rangle$ where each subset $E_i \subset k[U]$ generates the prime ideal $\langle E_i \rangle$ for $i = 1, \ldots, l$.
2: Let $N_1 = N, N_2 = N \times E_1, \ldots,$ and $N_l = N \times \prod_{i=1}^{l-1} E_i$ for $i = 1, \ldots, l$.
3: **return** $\{(E_1, N_1), \ldots, (E_l, N_l)\}$.

Remark 3.4. *Computing radical ideal and prime decomposition can be implemented by the classical algorithm in [2, Chapter 8]. For other newer and more efficient methods, see [1, 12, 22].*

3.3 Complete in irreducible branches

For each principal constructible set $\mathbb{V}(E) \setminus \mathbb{V}(N)$ with $\mathbb{V}(E)$ irreducible, in Step 3 and Step 4, our goal is to construct a matrix $\mathbf{M} \in k[U][X]^{m\times m}$ such that for almost every $\mathbf{a} \in \mathbb{V}(E) \setminus \mathbb{V}(N)$, the matrix $\mathbf{M}(\mathbf{a}, X)$ is a unimodular completion matrix of $\mathbf{f}(\mathbf{a}, X)$. The result is formalized in the following theorem.

Theorem 3.5. *Let $\mathbf{f}(U, X) \in k[U][X]^m$ be a polynomial vector, and let $C = \mathbb{V}(E) \setminus \mathbb{V}(N)$ be a nonempty principle constructible set with $\mathbb{V}(E)$ irreducible. Suppose that $\mathbf{f}(\mathbf{a}, X)$ is unimodular for any $\mathbf{a} \in C$. Then there exists a matrix $\mathbf{M} \in k[U][X]^{m\times m}$ and a polynomial $d \in k[U]$ such that:*

1. *$d \notin \langle E\rangle$;*
2. *For any $\mathbf{a} \in \mathbb{V}(E) \setminus \mathbb{V}(N)$, the matrix $\mathbf{M}(\mathbf{a}, X)$ is a unimodular completion matrix of $\mathbf{f}(\mathbf{a}, X)$ if and only if $d(\mathbf{a}) \neq 0$.*

The main idea to construct $\mathbf{M}$ is as follows. First, we construct the function field of $\mathbb{V}(E)$ and apply the Quillen-Suslin theorem in the polynomial ring over this function field to obtain a unimodular matrix $\mathbf{M}^{(0)}$ (over the new polynomial ring). Then, pulling $\mathbf{M}^{(0)}$ back to the original polynomial ring, we obtain the desired matrix $\mathbf{M}$.

To apply the Quillen-Suslin theorem in the polynomial ring over this function field, the following lemma is required.

Lemma 3.6. *Let $\mathbf{f}(U, X) \in k[U][X]^m$ be a polynomial vector, and let $C = \mathbb{V}(E) \setminus \mathbb{V}(N)$ be a nonempty principal constructible set with $\mathbb{V}(E)$ irreducible. Assume that $\mathbf{f}(\mathbf{a}, X)$ is unimodular for any $\mathbf{a} \in C$. Then $\pi(\mathbf{f})$ is unimodular over $\mathcal{K}[X]$, where $\mathcal{K} = k(\mathbb{V}(E))$ and π is the canonical map.*

Proof. Suppose that $E = \{g_1, \ldots, g_r\}, N = \{h_1, \ldots, h_s\}$. We claim that

$$\mathbb{V}\left(\langle f_1, \ldots, f_m, g_1, \ldots, g_r\rangle_{k[U,X]}\right) \setminus \mathbb{V}\left(\langle h_1, \ldots, h_s\rangle_{k[U,X]}\right) = \emptyset. \tag{1}$$

Otherwise, there exists $(\mathbf{a}, \mathrm{b}) \in \mathbb{C}^{s+n}$ such that

$$f_i(\mathbf{a}, \mathbf{b}) = 0, g_j(\mathbf{a}) = 0, i = 1, \ldots, m, j = 1, \ldots, r.$$

and $h_i(\mathbf{a}) \neq 0$ for some i. This implies that $\mathbf{a} \in \mathbb{V}(E)\setminus\mathbb{V}(N)$ and $\mathbf{b} \in \mathbb{V}(f_1(\mathbf{a}, X), \ldots, f_m(\mathbf{a}, X))$. This contradicts the fact that $\mathbf{f}(\mathbf{a}, X)$ is unimodular. According to (1), we have

$$\mathbb{V}\left(\langle f_1, \ldots, f_n, g_1, \ldots, g_r\rangle_{k[U,X]}\right) \subset \mathbb{V}\left(\langle h_1, \ldots, h_s\rangle_{k[U,X]}\right).$$

By Hilbert Nullstellensatz theorem, we have

$$\sqrt{\langle h_1, \ldots, h_s\rangle_{k[U,X]}} \subset \sqrt{\langle f_1, \ldots, f_n, g_1, \ldots, g_r\rangle_{k[U,X]}}. \tag{2}$$

Since $\mathbb{V}(E)\setminus\mathbb{V}(N) \neq \emptyset$, there exists $h \in k[U]$, such that $h \in \langle N\rangle$, but $h \notin \sqrt{\langle E\rangle}$. By (2), $h \in \sqrt{\langle f_1, \ldots, f_n, g_1, \ldots, g_r\rangle_{k[U,X]}}$, which implies that

$$h^r = q_1 f_1 + q_2 f_2 + \cdots + q_m f_m + g,$$

for some $r > 0, q_1, \ldots, q_m \in k[U][X]$ and $g \in \langle E\rangle_{k[U,X]}$. Applying the canonical map π on the both sides, we have

$$\pi\left(h^r\right) \in \langle \pi(f_1), \ldots, \pi(f_m)\rangle_{\mathcal{K}[X]}.$$

Since $h \notin \sqrt{\langle E\rangle}$, $h^r \notin \sqrt{\langle E\rangle}$. Hence $\pi(h^r) \neq 0$ is a unit in $\mathcal{K}$, which means that $\pi(f_1), \ldots, \pi(f_m)$ generate the unit ideal in $\mathcal{K}[X]$ as needed. □

Thanks to Lemma 3.6, we are able to apply Quillen-Suslin theorem to $(\pi(f_1), \ldots, \pi(f_m))$ over $\mathcal{K}[X]$ and get a unimodular matrix $\mathbf{M_0}$ with entries in $\mathcal{K}[X]$ and determinant is in $\mathcal{K}$. According to Lemma 3.6, we now prove Theorem 3.5.

Proof of Theorem 3.5. Let $\mathcal{R} = k[\mathbb{V}(E)]$ and $\mathcal{K} = k(\mathbb{V}(E))$. By Lemma 3.6, $\pi(\mathbf{f})$ is unimodular over $\mathcal{K}[X]$, and according to the Quillen-Suslin theorem, there exists a matrix $\mathbf{M}^{(0)} \in \mathcal{K}[X]^{m\times m}$ such that $\mathbf{M}^{(0)}$ is a unimodular complete matrix of $\pi(\mathbf{f})$. Multiplying each row of $\mathbf{M}^{(0)}$ by appropriate elements in $\mathcal{R} \setminus \{0\}$, we obtain a matrix $\mathbf{M}^{(1)} \in \mathcal{R}[X]^{m\times m}$, which is also a unimodular completion of $\pi(\mathbf{f})$. Because $\mathbf{M}^{(1)}$ is unimodular, it follows that $\det(\mathbf{M}^{(1)}) \in \mathcal{R} \setminus \{0\}$.

Let $\mathbf{M} \in k[U][X]^{m\times m}$ and $d \in k[U]$ such that $\pi(\mathbf{M}) = \mathbf{M}^{(1)}$ and $\pi(d) = \det(\mathbf{M}^{(1)})$. Then $d \notin \langle E\rangle$ as $\pi(d) \neq 0$. Since the first row of $\mathbf{M}^{(1)}$ is $\pi(\mathbf{f})$, it follows that for any $\mathbf{a} \in \mathbb{V}(E) \setminus \mathbb{V}(N)$, the first row of $\mathbf{M}(\mathbf{a}, X)$ is $f(\mathbf{a}, X)$. Furthermore,

$$\det(\mathbf{M}) \equiv d \mod \sqrt{\langle E\rangle}.$$

Therefore, for any $\mathbf{a} \in \mathbb{V}(E) \setminus \mathbb{V}(N)$,

$$\det(\mathbf{M}(\mathbf{a}, X)) = d(\mathbf{a}) \in L.$$

This implies that for $\mathbf{a} \in \mathbb{V}(E) \setminus \mathbb{V}(N)$, $\mathbf{M}(\mathbf{a}, X)$ is unimodular if and only if $d(\mathbf{a}) \neq 0$.

3.4 Reduce to smaller branches

For a principal constructible set $\mathbb{V}(E) \setminus \mathbb{V}(N)$ where $\mathbb{V}(E)$ is irreducible, according to the above process and Theorem 3.5, we get a parametric matrix $\mathbf{M}(U, X) \in k[U][X]^{m\times m}$ which is exactly the completion of $\mathbf{f}$ we want to compute in the principal constructible set $\mathbb{V}(E) \setminus \mathbb{V}(N \times \{d\})$. Moreover, by splitting

$$\mathbb{V}(E) \setminus \mathbb{V}(N) = (\mathbb{V}(E \cup \{d\}) \setminus \mathbb{V}(N)) \cup (\mathbb{V}(E) \setminus \mathbb{V}(N \times \{d\})), \tag{3}$$

we only need to complete in a strictly smaller parameter branch $\mathbb{V}(E \cup \{d\}) \setminus \mathbb{V}(N)$ if it's not empty by Theorem 3.5 (1).

Repeating Step 2-4 for principal constructible set $\mathbb{V}(E \cup \{d\}) \setminus \mathbb{V}(N)$ and by the Noetherian property of the polynomial ring $k[U]$, we can solve the desired parametric matrices at every parametric branches in finite steps.

4 Algorithm

With the detailed analysis in previous sections, we now give the algorithm for computing unimodular completion matrix system for a given parametric vector $\mathbf{f} = (f_1, \ldots, f_m) \in k[U][X]^m$. To our knowledge, there is currently no such algorithm for completing the unimodular row with parameters to unimodular matrices in different parametric branches.

Algorithm 2 Compute the unimodular completion matrix system

Input: $\mathbf{f}(U,X) = (f_1, \ldots, f_m) \in k[U][X]^m$.
Output: $\mathcal{M} = \{(A_0, \emptyset), (A_1, \mathbf{M}_1), \ldots, (A_l, \mathbf{M}_l)\}$ is a unimodular completion matrix system of $\mathbf{f}$.

1: Compute the minimal CGS of $\{f_1, \ldots, f_m\}$ using the algorithm in [10] and denote the union of all principal constructible sets whose corresponding minimal Gröbner basis is $\{1\}$ as C. Let $A_0 = L^s \setminus C$. If $C = \emptyset$, return $\mathcal{M} = \{(L^s, \emptyset)\}$.
2: For each separated principal constructible set appearing in C, compute its disjoint irreducible decomposition using Algorithm 1 and denote the whole union of all such irreducible branches by S;
3: Set $\mathcal{M} = \{(A_0, \emptyset)\}$;
4: **while** $S \neq \emptyset$ **do**
5:   Take $(E, N) \in S$ and delete it from S, set $\mathcal{R} = k[\mathbb{V}(E)]$ with the canonical map π and $\mathcal{K} = k(\mathbb{V}(E))$;
6:   View $\pi(f_1), \ldots, \pi(f_m)$ as elements in $\mathcal{K}[X]$, compute the unimodular matrix $\mathbf{M}^{(0)} \in \mathcal{K}[X]^{m\times m}$ with first row being $(\pi(f_1), \ldots, \pi(f_m))$ using Quillen-Suslin theorem;
7:   Compute the multiplication of denominators for all entries in i-th row of $\mathbf{M}^{(0)}$ and denote as r_i, for $i = 1, \ldots, m$;
8:   For each i, multiply $r_i(U)$ to i-th row of $\mathbf{M}^{(0)}$ and denote the new matrix as $\mathbf{M}^{(1)}$;
9:   Compute $\mathbf{M}(U,X) = \pi^{-1}(\mathbf{M}^{(1)})$ and $d(U) = \pi^{-1}(\det(\mathbf{M}^{(1)}))$;
10:   Let $\mathcal{M} = \mathcal{M} \cup \{(\mathbb{V}(E) \setminus \mathbb{V}(N \times \{d\}), \mathbf{M})\}$;
11:   **if** $\mathbb{V}(E \cup \{d\}) \setminus \mathbb{V}(N) \neq \emptyset$ **then**
12:     Compute its disjoint irreducible decompositions using Algorithm 1 and add the results to S;
13:   **end if**
14: **end while**
15: **return** $\mathcal{M}$.

Remark 4.1. *The set S consists of elements (E_i, N_i) for $i \in \Lambda$, where Λ is the index set. For each $i \in \Lambda$, $\langle E_i \rangle$ is a prime ideal and $\mathbb{V}(E_i) \setminus \mathbb{V}(N_i)$ for $i \in \Lambda$ are pairwise disjoint. After Step 2, we have $C = \bigcup_{i\in\Lambda} \mathbb{V}(E_i) \setminus \mathbb{V}(N_i)$ by Lemma 3.3.*

Theorem 4.2. *Algorithm 2 terminates within a finite number of steps.*

Proof. Using König's lemma, it suffices to show that: (1) in each step, the algorithm only creates finite branches; (2) each branch terminates after finite steps.

(1) follows from the fact that any radical ideal can be decomposed into a finite intersection of prime ideals, since Algorithm 2 only creates new branches when calling Algorithm 1.

As for (2), for convenience, we can suppose that there is only one term (E, N) in S after Step 2. In Step 9, we have $d \notin E$ and $\langle E \rangle \subsetneq \langle E \rangle + \langle d \rangle$ according to Theorem 3.5.

In this case, if $\mathbb{V}(E \cup \{d\}) \setminus \mathbb{V}(N) = \emptyset$, the "while" process is over and we finish. Otherwise, we add the decompositions of $\mathbb{V}(E \cup \{d\}) \setminus \mathbb{V}(N) \neq \emptyset$ into the set S, with the guarantee that each new term is induced by an irreducible subvariety of $\mathbb{V}(E \cup \{d\})$, which is properly contained in $\mathbb{V}(E)$. Then we start a new iteration. We remark that this whole process will induce an increasing chain of ideals

$$\langle E \rangle \subsetneq \langle E \rangle + \langle d \rangle \subsetneq \ldots$$

This process will terminate within finitely many iterations because of the Noetherian property of $k[U]$ and Theorem 3.5. ◻

Theorem 4.3. *Algorithm 2 works correctly. That is, $\mathcal{M}$ constructed from Algorithm 2 is a unimodular completion matrix system of $\mathbf{f}$.*

Proof. Obviously, $\bigcup_{i=0}^{l} A_i = L^s$. For $i = 0, \ldots, l$, each A_i is disjoint by the construction of A_0, the computation of minimal comprehensive Gröbner system, Lemma 3.3 and Formula 3.

By Lemma 3.2, $\mathbf{f}(\mathbf{a}, X)$ is not unimodular if and only if $\mathbf{a} \in A_0$, for any $\mathbf{a} \in L^s$.

By Theorem 3.5, we have that $\mathbf{M}_i(\mathbf{a}, X)$ is unimodular over $L[X]$ with the first row being $\mathbf{f}(\mathbf{a}, X)$ for any $\mathbf{a} \in A_i$, where $i = 1, \ldots, l$.

As a consequence, $\mathcal{M}$ is a unimodular completion matrix system of $\mathbf{f}$. ◻

5 Example

In this section we give an example to illustrate the steps of the above proposed algorithm.

Example 5.1. Let

$$\mathbf{f}(u_1, u_2, x_1, x_2) = (u_2x_2 + u_1 + x_1, x_1^2 + u_1x_2 + u_2, u_1x_1 + x_2) = (f_1, f_2, f_3) \in \mathbb{Q}[u_1, u_2][x_1, x_2]^3.$$

Firstly, we compute the minimal comprehensive Gröbner system of f_1, f_2, f_3 and obtain $\mathcal{G} = \{(\mathbb{C}^2 \setminus \mathbb{V}(h), \{1\}), (\mathbb{V}(h), \{x_2 + u_1^3 - u_1u_2^2 + u_2, x_1 - u_1^3u_2 + u_1u_2^3 - u_2^2 + u_1\})\}$, where

$$h = u_1^4u_2 - u_1^2u_2^3 - u_1^3 + 2u_1u_2^2 - u_1^2 - u_2.$$

This implies that for any $(a_1, a_2) \in \mathbb{C}^2$, $\mathbf{f}(a_1, a_2, x_1, x_2)$ is unimodular if and only if $h(a_1, a_2) \neq 0$. Let $\mathcal{M} = \{(\mathbb{V}(h), \emptyset)\}$.

Let $C = \mathbb{C}^2 \setminus \mathbb{V}(h)$. Let

$$\mathcal{K}_1 = \mathbb{Q}(\mathbb{C}^2) = \mathrm{Frac}(\mathbb{Q}[u_1, u_2]/\mathbb{I}(\mathbb{C}^2)) = \mathbb{Q}(u_1, u_2),$$

where $\mathrm{Frac}(A)$ denotes the fractional field of a domain A. By Quillen-Suslin algorithm, $\pi_1(\mathbf{f})$ can be completed to a unimodular matrix $\mathbf{M}_1$ over $\mathcal{K}_1[X]$, and $\mathbf{M}_1 =$

$$\begin{pmatrix} u_2x_2 + u_1 + x_1 & u_1x_2 + x_1^2 + u_2 & u_1x_1 + x_2 \\ (u_1u_2 - 1)^2 u_1 & u_1^4u_2 - u_1^3 + u_1u_2x_2 - u_1^2 - x_2 & 0 \\ 0 & m[3,2] & (u_1u_2 - 1)^2 u_1 \end{pmatrix}$$

where $m[3,2] = (u_1^2u_2^2 - 2u_1u_2 + 1)x_1 + (u_2 - u_1u_2^2)x_2 - u_1^3u_2 + u_1^2 + u_1$. The determinant of $\mathbf{M}_1$ is

$$d_1 = \det(\mathbf{M}_1) = u_1^2(u_1u_2 - 1)^2h.$$

Then for any $(a_1, a_2) \in \mathbb{C}^2 \setminus \mathbb{V}(u_1^2(u_1u_2 - 1)^2h^2) = \mathbb{C}^2 \setminus \mathbb{V}(u_1(u_1u_2 - 1)h)$, the matrix $\mathbf{M}_1(a_1, a_2, x_1, x_2)$ is a unimodular completion matrix of $\mathbf{f}(a_1, a_2, x_1, x_2)$. Let $\mathcal{M} = \mathcal{M} \cup \{(\mathbb{C}^2 \setminus \mathbb{V}(u_1(u_1u_2 - 1)h), \mathbf{M}_1)\}$.

Then we consider the branch $\mathbb{V}(E_1) \setminus \mathbb{V}(h) = \mathbb{V}(u_1(u_1u_2 - 1)h) \setminus \mathbb{V}(h)$. By Algorithm 1, $\mathbb{V}(E_1) \setminus \mathbb{V}(h)$ can be decomposed as

$$\mathbb{V}(E_1) \setminus \mathbb{V}(h) = \mathbb{V}(u_1) \setminus \mathbb{V}(h) \cup \mathbb{V}(u_1u_2 - 1) \setminus \mathbb{V}(u_1h).$$

Consider the principal constructible set $\mathbb{V}(u_1) \setminus \mathbb{V}(h)$. Let

$$\mathcal{K}_2 = \mathbb{Q}(\mathbb{V}(u_1)) = \mathrm{Frac}(\mathbb{Q}[u_1, u_2]/\langle u_1 \rangle).$$

By Quillen-Suslin algorithm, $\pi_2(\mathbf{f})$ can be completed to a unimodular matrix $\mathbf{M}_2^{(0)}$ over $\mathcal{K}_2[X]$, and

$$\mathbf{M}_2^{(0)} = \begin{pmatrix} [u_2]x_2 + x_1 & x_1^2 + [u_2] & x_2 \\ [1] & x_1 & [0] \\ [0] & -[u_2]x_1 & [1] \end{pmatrix},$$

whose determinant is $[-u_2]$. Let

$$\mathbf{M}_2 = \begin{pmatrix} u_2x_2 + x_1 & x_1^2 + u_2 & x_2 \\ 1 & x_1 & 0 \\ 0 & -u_2x_1 & 1 \end{pmatrix}$$

and $d_2 = \pi_2^{-1}(\det(\mathbf{M}_2^{(0)})) = -u_2$. Then for any $(a_1, a_2) \in \mathbb{V}(u_1) \setminus \mathbb{V}(hu_2) = \mathbb{V}(u_1) \setminus \mathbb{V}(u_2)$, $\mathbf{M}_2(a_1, a_2, x_1, x_2)$ is a unimodular completion matrix of $\mathbf{f}(a_1, a_2, x_1, x_2)$. Let $\mathcal{M} = \mathcal{M} \cup \{(\mathbb{V}(u_1) \setminus \mathbb{V}(u_2), \mathbf{M}_2)\}$. Then we consider $\mathbb{V}(u_1, u_2) \setminus \mathbb{V}(h)$. We can check that $\mathbb{V}(u_1, u_2) \setminus \mathbb{V}(h) = \emptyset$.

We consider the branch $\mathbb{V}(u_1u_2 - 1) \setminus \mathbb{V}(u_1h)$. Let

$$\mathcal{K}_3 = \mathbb{Q}(\mathbb{V}(u_1u_2 - 1)) = \mathrm{Frac}(\mathbb{Q}[u_1, u_2]/\langle u_1u_2 - 1\rangle).$$

By Quillen-Suslin algorithm, $\pi_3(\mathbf{f})$ can be completed to a unimodular matrix $\mathbf{M}_3^{(0)}$ over $\mathcal{K}_3[X]$, and

$$\mathbf{M}_3^{(0)} = \begin{pmatrix} [u_2]y + x + [u_1] & x^2 + [u_1]y + [u_2] & [u_1]x + y \\ [0] & [1] & [0] \\ [u_2^2] & [0] & [u_2] \end{pmatrix},$$

whose determinant is $[1]$. Let

$$\mathbf{M}_3 = \begin{pmatrix} u_2y + x + u_1 & x^2 + u_1y + u_2 & u_1x + y \\ 0 & 1 & 0 \\ u_2^2 & 0 & u_2 \end{pmatrix}$$

and $d_3 = \pi_3^{-1}(\det(\mathbf{M}_3^{(0)})) = 1$. Then for any $(a_1, a_2) \in \mathbb{V}(u_1u_2-1) \setminus \mathbb{V}(u_1h) = \mathbb{V}(u_1u_2-1)$, $\mathbf{M}_3(a_1, a_2, x_1, x_2)$ is a unimodular completion matrix of $\mathbf{f}(a_1, a_2, x_1, x_2)$. Let $\mathcal{M} = \mathcal{M} \cup \{(\mathbb{V}(u_1u_2 - 1), \mathbf{M}_3)\}$.

In summary, the unimodular completion matrix system of $\mathbf{f}$ is

$$\begin{aligned} \mathcal{M} = \{&(\mathbb{V}(h), \emptyset), (\mathbb{C}^2 \setminus \mathbb{V}(u_1(u_1u_2 - 1)h), \mathbf{M}_1), \\ &(\mathbb{V}(u_1) \setminus \mathbb{V}(u_2), \mathbf{M}_2), (\mathbb{V}(u_1u_2 - 1), \mathbf{M}_3)\}. \end{aligned}$$

6 Concluding remarks

In this paper, we propose an algorithm for completing parametric unimodular vectors to unimodular matrices. By means of comprehensive Gröbner systems, we first partition the entire parameter space into a finite number of constructible subsets and select the unimodular branches. The key is that based on the irreducible decomposition of constructible subsets, we construct the fractional fields and then apply the Quillen-Suslin theorem on the each constructed field to compute the completion matrix. By pulling the matrix back to the original polynomial ring and further iterations, we obtain the unimodular completion matrix system.

Nevertheless, the difficulties in implementing the algorithmic Quillen-Suslin theorem over function fields arise when we try to implement our algorithm in specific codes in computer algebra software like *Maple*. This is likely to be a practical problem we need to solve next. Since the detailed implementation related to the Quillen-Suslin theorem only involves some simple computable fields such as the rational field or finite fields, it seems to be a promising area worth studying.

Acknowledgments

We would like to thank Professor Ruyong Feng for helpful discussions. This research was supported by the National Key R&D Program of China under Grant No. 2020YFA0712300, and the National Natural Science Foundation of China under Grant Nos. 12171469 and 12201210.

References

[1] Toru Aoyama and Masayuki Noro. 2018. Modular algorithms for computing minimal associated primes and radicals of polynomial ideals. In *Proceedings of the 2018 ACM International Symposium on Symbolic and Algebraic Computation*. 31–38.
[2] Thomas Becker, Volker Weispfenning, and Heinz Kredel. 1993. *Gröbner Bases: A Computational Approach to Commutative Algebra*. Springer-Verlag. doi:10.1007/978-1-4612-0913-3
[3] Xuefeng Chen, Peng Li, Long Lin, and Dingkang Wang. 2006. Proving geometric theorems by partitioned-parametric Gröbner bases. In *Automated Deduction in Geometry: 5th International Workshop, ADG 2004, Gainesville, FL, USA, September 16-18, 2004. Revised Papers 5*. Springer, 34–43.
[4] Bruce R Donald, Deepak Kapur, and Joseph L Mundy. 1992. *Symbolic and numerical computation for artificial intelligence*. Academic Press Harcourt Jovanovich.
[5] Anna Fabianska and Alban Quadrat. 2007. *Applications of the Quillen-Suslin theorem to multidimensional systems theory*. Ph. D. Dissertation. INRIA.
[6] Ruyong Feng and Ziming Li. 2016. Computational Algebraic Geometry. Lecture notes. Accessed: http://www.mmrc.iss.ac.cn/~ryfeng/papers/Notes.pdf.
[7] Michael Kalkbrener. 1997. On the stability of Gröbner bases under specializations. *Journal of Symbolic Computation* 24, 1 (1997), 51–58.
[8] Deepak Kapur, Dong Lu, Michael Monagan, Yao Sun, and Dingkang Wang. 2018. An efficient algorithm for computing parametric multivariate polynomial GCD. In *Proceedings of the 2018 ACM International Symposium on Symbolic and Algebraic Computation*. 239–246.
[9] Deepak Kapur, Dong Lu, Michael Monagan, Yao Sun, and Dingkang Wang. 2021. Algorithms for computing greatest common divisors of parametric multivariate polynomials. *Journal of Symbolic Computation* 102 (2021), 3–20.
[10] Deepak Kapur, Yao Sun, and Dingkang Wang. 2010. A new algorithm for computing comprehensive Gröbner systems. In *Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation*. 29–36.
[11] Deepak Kapur, Yao Sun, and Dingkang Wang. 2013. An efficient algorithm for computing a comprehensive Gröbner system of a parametric polynomial system. *Journal of Symbolic Computation* 49 (2013), 27–44.
[12] Taro Kawazoe and Masayuki Noro. 2011. Algorithms for computing a primary ideal decomposition without producing intermediate redundant components. *Journal of Symbolic Computation* 46, 10 (2011), 1158–1172.
[13] Tsit-Yuen Lam. 2006. *Serre's problem on projective modules*. Springer.
[14] Serge Lang. 2012. *Algebra*. Vol. 211. Springer Science & Business Media.
[15] Alessandro Logar and Bernd Sturmfels. 1992. Algorithms for the Quillen-Suslin theorem. *Journal of Algebra* 145, 1 (1992), 231–239.
[16] Antonio Montes. 2002. A new algorithm for discussing Gröbner bases with parameters. *Journal of Symbolic Computation* 33, 2 (2002), 183–208.
[17] Antonio Montes and Tomás Recio. 2007. Automatic discovery of geometry theorems using minimal canonical comprehensive Gröbner systems. In *Automated Deduction in Geometry: 6th International Workshop, ADG 2006, Pontevedra, Spain, August 31-September 2, 2006. Revised Papers 6*. Springer, 113–138.
[18] Katsusuke Nabeshima. 2007. A speed-up of the algorithm for computing comprehensive Gröbner systems. In *Proceedings of the 2007 international symposium on Symbolic and Algebraic Computation*. 299–306.
[19] Daniel Quillen. 1976. Projective modules over polynomial rings. *Inventiones mathematicae* 36, 1 (1976), 167–171.
[20] Moshe Roitman. 1975. On Serre's problem on projective modules. *Proc. Amer. Math. Soc.* 50, 1 (1975), 45–52.
[21] Jean-Pierre Serre. 1955. Faisceaux algébriques cohérents. *Annals of Mathematics* (1955), 197–278.
[22] Takeshi Shimoyama and Kazuhiro Yokoyama. 1996. Localization and primary decomposition of polynomial ideals. *Journal of Symbolic Computation* 22, 3 (1996), 247–277.
[23] Andrei Suslin. 1976. Projective modules over polynomial rings are free. In *Soviet mathematics-Doklady*, Vol. 17. 1160–1165.
[24] Akira Suzuki and Yosuke Sato. 2006. A simple algorithm to compute comprehensive Gröbner bases using Gröbner bases. In *Proceedings of the 2006 international symposium on Symbolic and Algebraic Computation*. 326–331.
[25] Dingkang Wang, Hesong Wang, and Fanghui Xiao. 2020. An extended GCD algorithm for parametric univariate polynomials and application to parametric Smith normal form. In *Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation*. 442–449.

[26] Volker Weispfenning. 1992. Comprehensive gröbner bases. *Journal of symbolic computation* 14, 1 (1992), 1–29.

[27] Ihsen Yengui. 2015. Constructive commutative algebra. *Lecture Notes in Mathematics* 2138 (2015).

[28] D Youla and P Pickel. 1984. The Quillen-Suslin theorem and the structure of n-dimensional elementary polynomial matrices. *IEEE transactions on circuits and systems* 31, 6 (1984), 513–518.