An algorithm for computing comprehensive order basis systems of parametric polynomial matrices

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Abstract An algorithm for computing parametric order bases for univariate polynomial matrices with parameters is first presented in this paper. Starting from the non-parametric univariate polynomial matrix, our key idea is to construct a special module and module order. Then based on Gröbner basis theory for modules, we present that the order basis can be obtained by computing a minimal Gröbner basis for this module under this order. Further, we extend the definition of the order basis to the parametric polynomial matrix, and give the concept of comprehensive order basis systems. More importantly, the method based on Gröbner bases for modules can be naturally generalized to the parametric case by means of comprehensive Gröbner systems for modules. As a consequence, we design a new algorithm for computing comprehensive order basis systems. The proposed algorithm has been implemented on the computer algebra system Singular and Maple.

Keywords Order basis, comprehensive order basis system, Gröbner basis, comprehensive Gröbner system.

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1 Introduction

The concept of order basis can be traced back to the proof of the transcendence for e presented by Hermite [1] in 1873 . Let $\mathbf{F} \in \mathbb{K}[[x]]^{m\times n}$ be a matrix of power series and given a non-negative integer σ, we are concerned with the problem of finding a vector $\mathbf{p}$ such that

$$\mathbf{F} \cdot \mathbf{p} \equiv \mathbf{0} \mod x^{\sigma},$$

which is involved in the well-known Hermite-Padé approximations defined by Padé [2]. In fact, all vectors satisfying the above equation form a free module over $\mathbb{K}[x]$. The researchers focus on how to find a specific basis to represent the whole free module. Beckermann and Labahn [3] gave a type of minimal degree property, named reduced order basis, and they also used the order basis to construct the Padé related rational interpolation tables. Meanwhile, the order basis is widely used in the inversion of structured matrices, normal forms of polynomial matrices, matrix inversion, column reduction, determinant, and nullspace basis computation [4–7].

Many researchers are committed to developing efficient algorithms for computing order bases. In 1994, Beckermann and Labahn [8] proposed a rapid algorithm that transforms the matrix problem into a higher-order vector problem (refer to as the Power Hermite-Padé problem). In 2003, Giorgi [6] introduced a divide-and-conquer approach for computing order bases, which converts the high-dimensional matrix order problem into a lower-dimensional vector problem of higher orders. This method has proven to be highly effective when the matrix is nearly square. In 2006, Storjohann [9] presented a novel algorithm that effectively reverses the construction established by Beckermann and Labahn. They transform the low-dimensional order basis problem into a high-dimensional one with the reduced order. These works reach complexity bounds that are deemed satisfactory in the most interesting cases, and return a basis in a so-called shifted reduced form. Subsequently, in 2013 Zhou [10] extended Storjohann's transformation to address limitations in Storjohann's method without sacrificing efficiency. In recent years, significant advancements have been achieved in the rapid computation of Popov form, which is a more stringent canonical representation compared to order basis [11–13]. Their algorithms are designed to accommodate arbitrary shifts without compromising efficiency.

Several algorithms have been devised to compute fraction-free order bases. For example, Beckermann and Labahn [14, 15] presented the FFFG algorithm for computing matrix rational interpolants by means of the Mahler systems they defined. They showed that type 2 Mahler systems at normal indices generate all the solutions to a simultaneous Padé problem of a given type for any order, and those solutions constitute an order basis. The main purpose of using a fraction-free approach is to avoid coefficient growth.

Although there are many fast algorithms for computing order bases of univariate polynomial matrices, there is still a blank in the study of parametric order basis. In practice, we always encounter situations with parametric matrices. For example, Danik and Dmitriev [16] consider the stabilizing regulators for a family of nonlinear control systems with a small positive parameter. They used Padé representation with parameters and limited the parameters so that they could discuss different situations to solve this representation, which can be done by the algorithm in this paper without any limitations for parameters. The parametric order basis for univariate polynomial matrices with parameters is precisely the issue we are considering and paying attention to.

Given a matrix $\mathbf{F} \in \mathbb{K}[U][x]^{m\times n}$ with parameters $U = \{u_1, \cdots, u_k\}$ and variable x. We expect to find all the order bases of the matrix $\mathbf{F}$ after each parameter takes any value in the algebraic closure L of $\mathbb{K}$. This is different from [14, 15] that give fraction-free order bases over an integral domain. In addition, the above mentioned algorithms for computing order bases for non-parametric polynomial matrices can not be simply extended to the case with parameters. This paper focuses on finding finite sets $\mathcal{O} = \{(A_1, \mathbf{O}_1), \cdots, (A_l, \mathbf{O}_l)\}$ such that for all $a \in A_i \subset L^k$, after specializing U induced by a (i.e., U takes the value a, denoted by π_a), $\pi_a(\mathbf{O}_i)$ is an order basis of $\pi_a(\mathbf{F})$. $\{A_1, \cdots, A_l\}$ represents the partition of parameter space. In fact, for each branch A_i, $\mathbf{F}$ shares the same expression of order basis for U taking any values in A_i. This finite set is defined by us as a comprehensive order basis system which is parallel to the concept of comprehensive Gröbner system proposed by Weispfenning [17]. For the comprehensive Gröbner system, many improved algorithms have been proposed [18–24].

In this paper, we begin to present the idea from the non-parametric case and give the method by using Gröbner bases proposed by Buchberger [25] in 1976 to compute the order basis. For a given matrix $\mathbf{F} \in \mathbb{K}[x]^{m\times n}$ and a positive integer list $\overrightarrow{\sigma}$, we construct a special matrix $\mathbf{F}'$ with the form of

$$\mathbf{F}' = \begin{bmatrix} \mathbf{F} & x^{\overrightarrow{\sigma}} \\ \mathbf{I}_n & \mathbf{0}_{n\times m} \end{bmatrix} \in \mathbb{K}[x]^{(m+n)\times(m+n)}.$$

The idea is to calculate the minimal Gröbner basis for the module generated by column vectors of $\mathbf{F}'$ under a special block order. Then we can obtain an order basis directly from this minimal Gröbner basis. Note that Gröbner bases for modules have been applied to compute the univariate polynomial matrix forms (Hermite normal form, Popov normal form, shifted reduced forms, etc.). Further, we extend the definition of the order basis to the parametric polynomial matrix, and give an exact definition of comprehensive order basis systems. By means of comprehensive Gröbner systems for modules which presented by [26] as the generalization of comprehensive Gröbner systems for polynomial rings, the method based on Gröbner bases for modules can be naturally generalized to the parametric case. Therefore, we design an algorithm for computing comprehensive order basis systems. Moreover, we have implemented the algorithm on Singular and Maple, and provide two examples to illustrate the steps of the algorithm.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and definitions, including the order basis, Gröbner basis, and comprehensive Gröbner system. In the last of Section 2, we give a strict definition of a comprehensive order basis system. The main results are presented in Section 3. We prove that by computing a minimal Gröbner basis for constructed special modules with respect to a specific block order, an order basis directly from the minimal Gröbner basis can be obtained. Then we further extend the idea to the parametric case. In Section 4, we propose the algorithm for computing comprehensive order basis systems based on the main theorem. Finally, we end with some concluding remarks in Section 5.

2 Preliminaries

In this section, we will introduce some basic notations and definitions for further discussion in this paper.

Let $\mathbb{K}$ be a field, $\mathbb{K}[x]$ be the polynomial ring with variable x, $\mathbb{K}[U][x]$ be the parametric polynomial ring with parameters $U = \{u_1, \cdots, u_k\}$ and univariate x, $\mathbf{F} \in \mathbb{K}[x]^{m\times n}$ (or $\mathbf{F} \in \mathbb{K}[U][x]^{m\times n}$) be a $m \times n$ matrix with entries in $\mathbb{K}[x]$ (or $\mathbb{K}[U][x]$). Let $\overrightarrow{\sigma} = [\sigma_1, \cdots, \sigma_m]$ be a non-negative integer list, and $\overrightarrow{s} = [s_1, \cdots, s_n] \in \mathbb{Z}^n$ be an integer list.

A subset of $\mathbb{K}[x]^n$ is called a module over $\mathbb{K}[x]$, if this subset is closed under addition and scalar multiplication by elements of $\mathbb{K}[x]$. For a finite set of vectors $\mathbf{f}_1, \cdots, \mathbf{f}_s \in \mathbb{K}[x]^n$, we consider the set of all polynomial vectors in $\mathbb{K}[x]^n$ which can be written as a $\mathbb{K}[x]$-linear combination of these vectors:

$$M = \{h_1\mathbf{f}_1 + \cdots + h_s\mathbf{f}_s \in \mathbb{K}[x]^n : h_i \in \mathbb{K}[x] \text{ for } i = 1, \cdots, s\}.$$

Then M is a submodule of $\mathbb{K}[x]^n$ generated by $\{\mathbf{f}_1, \cdots, \mathbf{f}_s\}$ and denoted by $\langle \mathbf{f}_1, \cdots, \mathbf{f}_s\rangle$. The set $\{\mathbf{f}_1, \cdots, \mathbf{f}_s\}$ is called a generating set of M. However, the generating set of M is not necessarily linearly independent. If $\{\mathbf{f}_1, \cdots, \mathbf{f}_s\}$ is $\mathbb{K}[x]$-linearly independent, $\{\mathbf{f}_1, \cdots, \mathbf{f}_s\}$ is a basis of M. Moreover, if a module M has a basis, then M is called a free module. Since $\mathbb{K}[x]$ is a principal ideal domain, any submodule of $\mathbb{K}[x]^n$ is free.

First, we introduce the concept of order bases.

Definition 2.1 A *column degree* for a column vector $\mathbf{p} = (p_1, \cdots, p_n)^T \in \mathbb{K}[x]^{n\times 1}$, denoted by cdeg $\mathbf{p}$, is just the maximum of the degrees of its elements, that is,

$$\text{cdeg } \mathbf{p} = \max_{1\le i\le n} \deg p_i.$$

Given an integer list $\overrightarrow{s} = [s_1, \cdots, s_n] \in \mathbb{Z}^n$(we always call $\overrightarrow{s}$ a *shift*), a *shifted column degree* of $\mathbf{p}$ with respect to a shift $\overrightarrow{s}$ (denoted by $\text{cdeg}_{\overrightarrow{s}}\ \mathbf{p}$), or simply the $\overrightarrow{s}$*-column degree* of $\mathbf{p}$ is

$$\text{cdeg}_{\overrightarrow{s}}\ \mathbf{p} = \max_{1\le i\le n}\{\deg p_i + s_i\} = \deg(x^{\overrightarrow{s}} \cdot \mathbf{p}),$$

where

$$x^{\overrightarrow{s}} = \text{diag}([x^{s_1}, \cdots, x^{s_n}]) = \begin{bmatrix} x^{s_1} & & \\ & \ddots & \\ & & x^{s_n} \end{bmatrix}.$$

For a matrix $\mathbf{P} \in \mathbb{K}[x]^{m\times n}$, we use cdeg $\mathbf{P}$ and $\text{cdeg}_{\overrightarrow{s}}\ \mathbf{P}$ for the lists of its column degrees and shifted $\overrightarrow{s}$-column degrees for all its column vectors.

Next, we give a comparable order for cdeg $\mathbf{P}$ and $\text{cdeg}_{\overrightarrow{s}}\ \mathbf{P}$. We can rearrange the list cdeg $\mathbf{P} = [a_1, \cdots, a_n]$ by the entries of the list from small to large. When comparing the two lists, cdeg $\mathbf{P}_1$ and cdeg $\mathbf{P}_2$, we first need to rearrange them and then compare the ordered lists $\overrightarrow{a}$ and $\overrightarrow{b}$. As for two pre-ordered integer lists $\overrightarrow{a} = [a_1, \cdots, a_n]$ and $\overrightarrow{b} = [b_1, \cdots, b_n]$, we compare $\overrightarrow{a}$ and $\overrightarrow{b}$ by following rules:

$$\begin{cases} \overrightarrow{a} \geq \overrightarrow{b}, & \text{if } a_i \geq b_i, \forall i \in [1, \cdots, n]; \\ \overrightarrow{a} \leq \overrightarrow{b}, & \text{if } a_i \leq b_i, \forall i \in [1, \cdots, n]; \\ \overrightarrow{a} > \overrightarrow{b}, & \text{if } a_i \geq b_i, \forall i \in [1, \cdots, n], \text{and } a_j > b_j \text{ for at least one } j \in [1, \cdots, n]; \\ \overrightarrow{a} < \overrightarrow{b}, & \text{if } a_i \leq b_i, \forall i \in [1, \cdots, n], \text{and } a_j < b_j \text{ for at least one } j \in [1, \cdots, n]. \end{cases}$$

Note that not all integer lists are comparable by the above rules. For example, the lists $[1, 4]$ and $[2, 3]$ can not be comparable. Moreover, cdeg $\mathbf{P}_1 \geq$ cdeg $\mathbf{P}_2$ if and only if the corresponding ordered lists $\overrightarrow{a} \geq \overrightarrow{b}$.

Definition 2.2 A matrix $\mathbf{P} \in \mathbb{K}[x]^{m\times n}$ is said to be *column reduced* if cdeg $\mathbf{P} \leq$ cdeg $\mathbf{PU}$ for any unimodular matrix $\mathbf{U}$. Note that this includes the comparability for cdeg $\mathbf{P}$ and cdeg $\mathbf{PU}$. More generally, for a shift $\overrightarrow{s} \in \mathbb{Z}^n$, a matrix $\mathbf{P} \in \mathbb{K}[x]^{m\times n}$ is said to be $\overrightarrow{s}$*-column reduced* if $\text{cdeg}_{\overrightarrow{s}}\ \mathbf{P} \leq \text{cdeg}_{\overrightarrow{s}}\ \mathbf{PU}$ for any unimodular matrix $\mathbf{U}$.

Lemma 2.3 *A matrix* $\mathbf{P} \in \mathbb{K}[x]^{m\times n}$ *with no zero columns is column reduced if and only if* lcoeff $\mathbf{P}$ *has full column rank. The* lcoeff $\mathbf{P}$ *is the leading column coefficient matrix of* $\mathbf{P}$, *defined as*

$$\begin{aligned} \text{lcoeff } \mathbf{P} &= [\text{lcoeff } \mathbf{p_1}, \cdots, \text{lcoeff } \mathbf{p_n}] \\ &= [\text{coeff}(\mathbf{p_1}, \text{cdeg } \mathbf{p_1}), \cdots, \text{coeff } (\mathbf{p_n}, \text{cdeg } \mathbf{p_n})], \end{aligned}$$

where $\text{coeff}(\mathbf{p_i}, \text{cdeg } \mathbf{p_i}) = \begin{bmatrix} \text{coeff}(p_{1i}, \text{cdeg}(\mathbf{p_i})) \\ \vdots \\ \text{coeff}(p_{mi}, \text{cdeg}(\mathbf{p_i})) \end{bmatrix}$ for $\mathbf{p_i} = \begin{bmatrix} p_{1i} \\ \vdots \\ p_{mi} \end{bmatrix} \in \mathbb{K}[x]^{m\times 1}$.

Definition 2.4 A column vector $\mathbf{p} \in \mathbb{K}[x]^{n\times 1}$ has order $(\mathbf{F}, \overrightarrow{\sigma})$ if $\mathbf{F} \cdot \mathbf{p} \equiv \mathbf{0} \mod x^{\overrightarrow{\sigma}}$, which means the following matrix equation holds for some $\mathbf{r} \in \mathbb{K}[x]^{m\times 1}$.

$$\mathbf{F} \cdot \mathbf{p} = x^{\overrightarrow{\sigma}} \mathbf{r} = \begin{bmatrix} x^{\sigma_1} & & \\ & \ddots & \\ & & x^{\sigma_m} \end{bmatrix} \mathbf{r}.$$

Particularly, if $\overrightarrow{\sigma} = [\sigma, \cdots, \sigma]$, we say $\mathbf{p}$ has order $(\mathbf{F}, \sigma)$. All vectors which have order $(\mathbf{F}, \overrightarrow{\sigma})$ form a free $\mathbb{K}[x]$-module, denoted by $\langle(\mathbf{F}, \overrightarrow{\sigma})\rangle$.

Definition 2.5 A polynomial matrix $\mathbf{P}$ is called an *order basis* of $\mathbf{F}$ with respect to the order $\overrightarrow{\sigma}$ and shift $\overrightarrow{s}$, denoted by $(\mathbf{F}, \overrightarrow{\sigma}, \overrightarrow{s})$-basis, if the column vectors of $\mathbf{P}$ is the basis of module $\langle(\mathbf{F}, \overrightarrow{\sigma})\rangle$, and $\mathbf{P}$ has minimal $\overrightarrow{s}$-column degree. Or precisely, the following three conditions hold:

1) Each column vector of $\mathbf{P}$ has $(\mathbf{F}, \overrightarrow{\sigma})$ order;

2) $\mathbf{P}$ is a nonsingular matrix of dimension n, and is $\overrightarrow{s}$-column reduced;

3) $\forall \mathbf{q} \in \langle(\mathbf{F}, \overrightarrow{\sigma})\rangle$, $\mathbf{q}$ can be $\mathbb{K}[x]$-linearly expressed as a combination of column vectors of $\mathbf{P}$.

Remark 2.6 For the definition of order bases in [10], $\mathbf{F} \in \mathbb{K}[[x]]^{m\times n}$ is a matrix of formal power series. But for computing a $(\mathbf{F}, \sigma, \overrightarrow{s})$-basis with input matrix $\mathbf{F} \in \mathbb{K}[[x]]^{m\times n}$, shift $\overrightarrow{s}$, and order σ one can view $\mathbf{F}$ as a polynomial matrix with degree $\sigma - 1$, as higher order terms are not needed in the computation. Then for convenience, throughout this paper we consider $\mathbf{F} \in \mathbb{K}[x]^{m\times n}$.

Now we introduce Gröbner bases and comprehensive Gröbner systems for modules.

Let $\succ$ be a monomial order on $\mathbb{K}[x]$, and $\succ_{T_n}$ be a module order by extending $\succ$ to $\mathbb{K}[x]^n$. By convention, $\mathbf{e_1} \succ_{T_n} \mathbf{e_2} \succ_{T_n} \cdots \succ_{T_n} \mathbf{e_n}$, where $\mathbf{e_1}, \ldots, \mathbf{e_n}$ are unit vectors in $\mathbb{K}[x]^n$. For $\mathbf{p} \in \mathbb{K}[x]^n$, the leading term, leading coefficient and leading monomial of $\mathbf{p}$ with respect to $\succ_{T_n}$ are denoted by $\mathrm{LT}(\mathbf{p})$, $\mathrm{LC}(\mathbf{p})$, $\mathrm{LM}(\mathbf{p})$. We say $\mathbf{p} \succ_{T_n} \mathbf{q}$ if $\mathrm{LM}(\mathbf{p}) \succ_{T_n} \mathrm{LM}(\mathbf{q})$ or $\mathrm{LM}(\mathbf{p}) = \mathrm{LM}(\mathbf{q})$ and $(\mathbf{p} - \mathrm{LT}(\mathbf{p})) \succ_{T_n} (\mathbf{q} - \mathrm{LT}(\mathbf{q}))$.

The definition of Gröbner bases for modules is as follows.

Definition 2.7 Let M be a submodule of $\mathbb{K}[x]^n$, and $\succ_{T_n}$ be a monomial order on $\mathbb{K}[x]^n$.

1) We denote $\langle \mathrm{LT}(M)\rangle$ the monomial submodule generated by the leading term of all $\mathbf{p} \in M$ with respect to $\succ_{T_n}$.

2) A finite set $G = \{\mathbf{g_1}, \ldots, \mathbf{g_s}\} \subset M$ is called a Gröbner basis of M, if $\langle \mathrm{LT}(M)\rangle = \langle \mathrm{LT}(\mathbf{g_1}), \ldots, \mathrm{LT}(\mathbf{g_s})\rangle$.

3) A Gröbner basis G of M is said to be **minimal**, if $\mathrm{LM}(\mathbf{g}) \notin \langle \mathrm{LM}(G \setminus \{\mathbf{g}\})\rangle$ for all $\mathbf{g} \in G$.

Next, we introduce some definitions for parametric univariate polynomials.

Let L be the algebraic closure of $\mathbb{K}$, a **specialization** of R is a homomorphism $\pi : \mathbb{K}[U] \to L$. In this paper, we consider the specializations induced by the elements in L^k, which means for an element $a \in L^k$, there is a homomorphism defined as $\pi_a : f \to f(a)$ for a polynomial $f \in \mathbb{K}[U]$. Every specialization $\pi : \mathbb{K}[U] \to L$ extends canonically to a homomorphism $\pi : \mathbb{K}[U][x] \to L[x]$ or $\mathbb{K}[U][x]^{m\times n} \to L[x]^{m\times n}$ by applying π coefficient-wise. We denote the variety defined by $F \subset \mathbb{K}[U]$ as $V(F) = \{a \in L^k \mid f(a) = 0, \forall f \in F\} \subset L^k$. We can use the following definition to group parametric specializations, called a parametric constraint.

Definition 2.8 For $E, N \subset \mathbb{K}[U]$, we call a pair (E, N) a **parametric constraint** and the set $A = V(E) \setminus V(N)$ an **algebraically constructible set** defined by (E, N).

For parametric systems, by the upper description, we review the definitions of comprehensive Gröbner systems and minimal comprehensive Gröbner systems for modules.

Definition 2.9 Let $F \subset \mathbb{K}[U][x]^n$, $A_1, \ldots, A_l$ be algebraically constructible subsets of L^k and $G_1, \ldots, G_s$ be subsets of $\mathbb{K}[U][x]^n$, and S be a subset of L^k such that $S \subset A_1 \cup \cdots \cup A_l$. A finite set $\mathcal{G} = \{(A_1, G_1), \cdots, (A_l, G_l)\}$ is called a **comprehensive Gröbner system**(CGS) on S for F if $\pi_a(G_i)$ is a Gröbner basis of the submodule $\langle \pi_a(F)\rangle \subset L[x]^n$. Each pair (A_i, G_i) is called a branch of $\mathcal{G}$.

Definition 2.10 A comprehensive Gröbner system $\mathcal{G} = \{(A_1, G_1), \cdots, (A_l, G_l)\}$ on S for $F \subset \mathbf{K}[U][x]^n$ is said to be **minimal** if for each $i = 1, \ldots, l$, the following properties holds.

1) $A_i \neq \emptyset$, and furthermore, for each $i, j = 1, \ldots, l$, $A_i \cap A_j = \emptyset$ whenever $i \neq j$;

2) $\pi_a(G_i)$ is a minimal Gröbner basis of $\langle \pi_a(F) \rangle \subset L[x]^n$ for all $a \in A_i$;

3) for each $\mathbf{g} \in G_i, \pi_a(\mathrm{LC}_x(\mathbf{g})) \neq 0$ for all $a \in A_i$, where $\mathrm{LC}_x(\mathbf{g})$ denotes the leading coefficient of $\mathbf{g}$ with respect to the variable x under the order $\succ_{T_n}$.

Finally, we define the comprehensive order basis system, which is analogical to the comprehensive Gröbner system.

Definition 2.11 Let $\mathbf{F} \in \mathbf{K}[U][x]^{m\times n}$, $\overrightarrow{\sigma}$ and $\overrightarrow{s}$ be integer lists. Let $A_1, \cdots, A_l$ be subsets of L^k and $\mathbf{O}_1, \cdots, \mathbf{O}_l$ be polynomial matrices in $\mathbb{K}[U][x]^{n\times n}$, and S be a subset of L^k such that $S \subset A_1 \cup \cdots \cup A_l$. A finite set $\mathcal{O} = \{(A_1, \mathbf{O}_1), \cdots, (A_l, \mathbf{O}_l)\}$ is called a **comprehensive order basis system** on S for $\mathbf{F}$ with respect to $\overrightarrow{\sigma}$ and $\overrightarrow{s}$ if $\pi_a(\mathbf{O}_i)$ is a $(\pi_a(\mathbf{F}), \overrightarrow{\sigma}, \overrightarrow{s})$-basis for $a \in A_i$ and $i = 1, \cdots, l$. Each $(A_i, \mathbf{O}_i)$ is called a branch of $\mathcal{O}$.

3 Order Basis and Comprehensive Order Basis System

In this section, we study the order basis based on Gröbner bases for modules. That is, we can construct a special module, and compute a minimal Gröbner basis of this module to obtain an order basis directly. Then, we will extend the method to parametric cases for computing comprehensive order basis systems.

3.1 Order basis for non-parametric matrices

First, we introduce a block term order.

Definition 3.1 Let $\mathbb{K}$ be a field, and $\mathbb{K}[x]$ be a polynomial ring. Let $\mathbf{e_1}, \cdots, \mathbf{e_{m+n}}$ be the unit vectors of $\mathbb{K}[x]^{m+n}$, the symbol **deg** be the normal degree on $\mathbb{K}[x]$ about variable x. For two monomials $x^\alpha \mathbf{e_i}$ and $x^\beta \mathbf{e_j}$ on $\mathbb{K}[x]^{m+n}$, we define a block order $>_{T_{m|n}}$ on $\mathbb{K}[x]^{m+n}$ satisfying:

1) $x^\alpha \mathbf{e_i} >_{T_{m|n}} x^\beta \mathbf{e_j}$ for $1 \leq i \leq m$ and $m+1 \leq j \leq m+n$;

2) $x^\alpha \mathbf{e_i} >_{T_{m|n}} x^\beta \mathbf{e_j}$ for $1 \leq i, j \leq m$, if $i < j$, or $i = j$ and $\deg(x^\alpha) > \deg(x^\beta)$;

3) $x^\alpha \mathbf{e_i} >_{T_{m|n}} x^\beta \mathbf{e_j}$ for $m+1 \leq i, j \leq m+n$, if $\deg(x^\alpha) > \deg(x^\beta)$, or $\deg(x^\alpha) = \deg(x^\beta)$ and $i < j$.

The block order $>_{T_{m|n}}$ is a term order on module $\mathbb{K}[x]^{m+n}$, and we can describe the main theorem now.

Theorem 3.2 *Let $\mathbf{F} \in \mathbb{K}[x]^{m\times n}$ be a polynomial matrix of size $m \times n$, and $\overrightarrow{\sigma}$ be an integer list. Consider a new polynomial matrix $\mathbf{F}'$ with size of $(m+n)\times(m+n)$ as follows:*

$$\mathbf{F}' = \begin{bmatrix} \mathbf{F} & x^{\overrightarrow{\sigma}} \\ \mathbf{I}_n & \mathbf{0}_{n\times m} \end{bmatrix} \in \mathbb{K}[x]^{(m+n)\times(m+n)}, \quad \text{where } \mathbf{I}_n \text{ is the } n\times n \text{ identity matrix.}$$

Let M be a module generated by the column vectors of $\mathbf{F}'$. Assume $G = \{\mathbf{g}_1, \cdots, \mathbf{g}_s\}$ is a minimal Gröbner basis of M with respect to $>_{T_{m|n}}, g_{ij} \in \mathbb{K}[x]$ and $\mathbf{g_j} = \sum_{i=1}^{m+n} g_{ij}\mathbf{e_i}$ with $\mathrm{LM}(\mathbf{g}_i) >_{T_{m|n}} \mathrm{LM}(\mathbf{g}_j)$ for $1 \le i, j \le s$. Then $\widehat{\mathbf{G}}$ is a $(\mathbf{F}, \overrightarrow{\sigma}, \overrightarrow{\mathbf{0}})$ basis, where

$$\widehat{\mathbf{G}} = \begin{bmatrix} g_{m+1,m+1} & \cdots & g_{m+1,m+n} \\ \vdots & \ddots & \vdots \\ g_{m+n,m+1} & \cdots & g_{m+n,m+n} \end{bmatrix}_{n\times n}.$$

Proof Let $\mathbf{G} = [g_{ij}] \in K[x]^{(m+n)\times s}$ corresponding to the Gröbner basis G. We first claim $s = m+n$. On the one hand, it is easy to conclude that the column vectors of $\mathbf{F}'$ are $\mathbb{K}[x]$-linear independent, so the minimal Gröbner basis G has at least $m+n$ elements, which means $s \ge m+n$. On the other hand, for any $\mathbf{g_i}, \mathbf{g_j} \in G$, $i \ne j$, suppose $\mathrm{LM}(\mathbf{g}_i) = x^a\mathbf{e}_s$ and $\mathrm{LM}(\mathbf{g}_j) = x^b\mathbf{e}_t$ are the leading monomials of $\mathbf{g}_i$ and $\mathbf{g}_j$. Then we claim that $s \ne t$. Otherwise, we have $\mathrm{LM}(\mathbf{g}_j) \in \langle \mathrm{LM}(\mathbf{g}_i)\rangle$ or $\mathrm{LM}(\mathbf{g}_j) \in \langle \mathrm{LM}(\mathbf{g}_i)\rangle$. This is inconsistent with the minimal property of G. Therefore, each element in G corresponds to a distinct unit vector $\mathbf{e_i}$. Since there are at most $m+n$ unit vectors in $\mathbb{K}[x]^{m+n}$, so we have $s \le m+n$.

Now, we are ready to prove that $\widehat{\mathbf{G}}$ satisfies the following three properties.

1) Any column vector in $\widehat{\mathbf{G}}$ has $(\mathbf{F}, \overrightarrow{\sigma})$-order.

   We claim $g_{ij} = 0$ for any $i \le m,\ j \ge m+1$. We have proved that there is a one-to-one correspondence between elements (leading terms) in G and unit vectors $\mathbf{e}$ of $\mathbb{K}[x]^{m+n}$. Since the block order $>_{T_{m|n}}$ compare the position firstly by items 1) and 2) of Definition 3.1, then $\mathbf{g_1}, \cdots, \mathbf{g_m}$ correspond to $\mathbf{e_1}, \cdots, \mathbf{e_m}$. Assume that there exist $i, j \in \mathbb{Z}^+, i \le m, j \ge m+1$, such that $g_{ij} \ne 0$, According to Definition 3.1, $\mathrm{LM}(\mathbf{g_j}) = x^a\mathbf{e}_i$. That is, $(\mathbf{g_j})$ is corresponding to $\mathbf{e}_i$, which contradicts that $(\mathbf{g}_i)$ corresponds to $\mathbf{e}_i$ since $i \le m$. Therefore, we have

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{0}_{m\times n} \\ \mathbf{G}_2 & \widehat{\mathbf{G}} \end{bmatrix},$$

   where $\mathbf{G}_1 \in \mathbf{K}[x]^{m\times n}$ and $\mathbf{G}_2 \in \mathbf{K}[x]^{n\times n}$. Since $G \subset M$, the elements in G can be $\mathbf{K}[x]$-linear expressed by the column vectors of $\mathbf{F}'$. Thus, there exists a matrix $\mathbf{C} \in \mathbb{K}[x]^{(m+n)\times(m+n)}$ such that

$$\mathbf{F}'\mathbf{C} = \mathbf{G}. \tag{1}$$

   That is

$$\begin{bmatrix} \mathbf{F} & x^{\overrightarrow{\sigma}} \\ \mathbf{I}_n & \mathbf{0}_{n\times m} \end{bmatrix} \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_3 & \mathbf{C}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{0}_{m\times n} \\ \mathbf{G}_2 & \widehat{\mathbf{G}} \end{bmatrix}, \tag{2}$$

   where $\mathbf{C}_1 \in \mathbb{K}[x]^{n\times m}$, $\mathbf{C}_2 \in \mathbb{K}[x]^{n\times n}$, $\mathbf{C}_3 \in \mathbb{K}[x]^{m\times m}$, and $\mathbf{C}_4 \in \mathbb{K}[x]^{m\times n}$. It followed by

$$\mathbf{F}\cdot\mathbf{C}_2 + x^{\overrightarrow{\sigma}}\mathbf{C}_4 = \mathbf{0}_{m\times n} \text{ and } \mathbf{I}_n\cdot\mathbf{C}_2 + \mathbf{0}_{n\times m}\cdot\mathbf{C}_4 = \widehat{\mathbf{G}}, \tag{3}$$

   which means $\mathbf{F}\cdot\widehat{\mathbf{G}} = x^{\overrightarrow{\sigma}}\cdot(-\mathbf{C}_4)$. So any column vector of $\widehat{\mathbf{G}}$ has $(\mathbf{F}, \overrightarrow{\sigma})$-order.

2) $\widehat{\mathbf{G}}$ is a nonsingular matrix and $\overrightarrow{s}$-column reduced.

   So far, we know that the leading term of $\{\mathbf{g_{m+1}}, \cdots, \mathbf{g_{m+n}}\}$ are corresponding to distinct $\mathbf{e}_{m+i}, 1 \leq i \leq n$. According to Definition 3.1, the leading terms of column vectors in $\widehat{\mathbf{G}}$ correspond to the leading terms of $\{\mathbf{g_{m+1}}, \cdots, \mathbf{g_{m+n}}\}$, then the leading terms of column vectors in $\widehat{\mathbf{G}}$ are in different positions. It means that $\widehat{\mathbf{G}}$ is a nonsingular matrix, and the leading coefficient matrix of $\widehat{\mathbf{G}}$ is a triangle matrix after suitable column exchanges. According to Lemma 2.3, $\widehat{\mathbf{G}}$ is $\overrightarrow{s}$-column reduced.

3) For any $\mathbf{q} \in \langle(\mathbf{F}, \overrightarrow{\sigma})\rangle$, $\mathbf{q}$ can be $\mathbb{K}$[x]-linear expressed by the column vectors of $\widehat{\mathbf{G}}$.

   Since $\mathbf{q} \in \langle(\mathbf{F}, \overrightarrow{\sigma})\rangle$, there exists $\mathbf{r} \in \mathbb{K}[x]^{m\times 1}$ such that $\mathbf{Fq} = x^{\overrightarrow{\sigma}}\mathbf{r}$. Then the following equation holds:

$$\begin{bmatrix} \mathbf{F}_{m\times n} \\ \mathbf{0}_{n\times n} \end{bmatrix} \mathbf{q} = \begin{bmatrix} \mathbf{F}_{m\times n}\mathbf{q} \\ \mathbf{0}_{n\times 1} \end{bmatrix} = \begin{bmatrix} x^{\overrightarrow{\sigma}}\mathbf{r} \\ \mathbf{0}_{n\times 1} \end{bmatrix} = \begin{bmatrix} x^{\overrightarrow{\sigma}} \\ \mathbf{0}_{n\times m} \end{bmatrix} \mathbf{r}. \tag{4}$$

   From above, we have

$$\begin{bmatrix} \mathbf{0}_{m\times 1} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{m\times n} \\ \mathbf{I}_{n\times n} \end{bmatrix} \mathbf{q} - \begin{bmatrix} \mathbf{F}_{m\times n} \\ \mathbf{0}_{n\times n} \end{bmatrix} \mathbf{q} = \begin{bmatrix} \mathbf{F}_{m\times n} \\ \mathbf{I}_{n\times n} \end{bmatrix} \mathbf{q} - \begin{bmatrix} x^{\overrightarrow{\sigma}} \\ \mathbf{0}_{n\times m} \end{bmatrix} \mathbf{r} \in M. \tag{5}$$

   Since G is the minimal Gröbner basis of M, $\begin{bmatrix} \mathbf{0}_{m\times 1} \\ \mathbf{q} \end{bmatrix} \in M$ can be $\mathbb{K}[x]$-linear expressed by G. Suppose $\begin{bmatrix} \mathbf{0}_{m\times 1} \\ \mathbf{q} \end{bmatrix} = \sum_{i=1}^{m+n} \beta_i \mathbf{g_i}$, $\beta_i \in \mathbb{K}[x]$. According to the proof of item 1), $g_{ij} = 0$ for $i \leq m, j \geq m+1$, then

$$\sum_{i=1}^{m} \beta_i \mathbf{g_i} = \begin{bmatrix} \mathbf{0}_{m\times 1} \\ \mathbf{q} \end{bmatrix} - \sum_{i=m+1}^{n} \beta_i \mathbf{g_i} = \begin{bmatrix} \mathbf{0}_{m\times 1} \\ \mathbf{v} \end{bmatrix} \tag{6}$$

   with $\mathbf{v} \in \mathbb{K}[x]^{n\times 1}$. By the above analyses, we have $\mathrm{LM}(\mathbf{g_i}) = x^{a_i}\mathbf{e}_i$ for $i = 1, \ldots, m$. Therefore, it is easy to obtain $\beta_i = 0$ for $i = 1, \ldots, m$ from Equation (6), which implies that

$$\begin{bmatrix} \mathbf{0}_{m\times 1} \\ \mathbf{q} \end{bmatrix} = \sum_{i=m}^{m+n} \beta_i \mathbf{g_i}.$$

   In other words, $\mathbf{q}$ can be $\mathbb{K}[x]$-linear expressed as a combination of the column vectors in $\widehat{\mathbf{G}}$.

According to Definition 2.5, we have proved that $\widehat{\mathbf{G}}$ is a $(\mathbf{F}, \overrightarrow{\sigma}, \overrightarrow{\mathbf{0}})$ basis.

In Theorem 3.2, we assume that $\overrightarrow{s} = [0, \cdots, 0]$. Now we consider the shift $\overrightarrow{s} = [s_1, \cdots, s_n]$ with at least one $s_i \neq 0$. Similar to Definition 3.1, we define a new block order $>_{T_{m|n}, \overrightarrow{s}}$ as follows.

Definition 3.3 Let $\mathbf{e_1}, \cdots, \mathbf{e_{m+n}}$ be the unit vectors of $\mathbb{K}[x]^{m+n}$. For two monomials $x^\alpha \mathbf{e_i}$ and $x^\beta \mathbf{e_j}$ on $\mathbb{K}[x]^{m+n}$, we define a block order $>_{T_{m|n}, \overrightarrow{s}}$ with respect to the integer list $\overrightarrow{s}$ satisfying:

1) $x^\alpha \mathbf{e_i} >_{T_{m|n}, \overrightarrow{s}} x^\beta \mathbf{e_j}$ for $1 \le i \le m$ and $m+1 \le j \le m+n$;

2) $x^\alpha \mathbf{e_i} >_{T_{m|n}, \overrightarrow{s}} x^\beta \mathbf{e_j}$ for $1 \le i, j \le m$, if $i < j$, or $i = j$ and $\deg(x^\alpha) > \deg(x^\beta)$;

3) $x^\alpha \mathbf{e_i} >_{T_{m|n}, \overrightarrow{s}} x^\beta \mathbf{e_j}$ for $m+1 \le i, j \le m+n$, if $\deg(x^\alpha) + s_{i-m} > \deg(x^\beta) + s_{j-m}$, or $\deg(x^\alpha) + s_{i-m} = \deg(x^\beta) + s_{j-m}$ and $i < j$.

We give the following examples to illustrate the block order $>_{T_{m|n}, \overrightarrow{s}}$.

Example 3.4 Let $m = 2, n = 3$, and $\overrightarrow{s} = [2, 4, 1]$. Then we have following facts:

1. $x^2 \mathbf{e_2} >_{T_{m|n}, \overrightarrow{s}} x^4 \mathbf{e_3}$ by Rule 1.
2. $x^2 \mathbf{e_1} >_{T_{m|n}, \overrightarrow{s}} x^4 \mathbf{e_2}$ by Rule 2).
3. $x^3 \mathbf{e_4} >_{T_{m|n}, \overrightarrow{s}} x^4 \mathbf{e_3}$ by Rule 3), because $\deg(x^3) + 4 > \deg(x^4) + 2$.

Theorem 3.5 *Theorem 3.2 holds for the block order* $>_{T_{m|n}, \overrightarrow{s}}$ *with* $\overrightarrow{s} \neq \overrightarrow{0}$.

Proof First of all, the reduction process of calculating Gröbner bases can be well defined under the new block order $>_{T_{m|n}, \overrightarrow{s}}$. This is because the reduction happens on two terms whose leading terms correspond to the same unit vector $\mathbf{e_i}$ for modules M. When adding the same integer s_i, it does not affect the usual reduction process.

From the proof of Theorem 3.2, the items 1) and 3) hold since they are not relevant to the changes of module orders. For the item 2), although the new order changes the leading terms and reduction when calculating the Gröbner basis, the shift $\overrightarrow{s}$ does not affect the one-to-one correspondence between $\{\mathbf{g_i}\}$ and $\{\mathbf{e_i}\}$ for $i = 1, \cdots, m+n$. Thus, the main result still holds.

Since Theorem 3.2 holds for the new block order $>_{T_{m|n}, \overrightarrow{s}}$ with $\overrightarrow{s} \neq \overrightarrow{0}$, we can use the same way to compute a $(\mathbf{F}, \overrightarrow{\sigma}, \overrightarrow{s})$-basis with $\overrightarrow{s} \neq \overrightarrow{0}$. Based on Theorem 3.2 and 3.5, we can design an algorithm to compute the order basis of $(\mathbf{F}, \overrightarrow{\sigma}, \overrightarrow{s})$. That is, we only need to construct the module M by inputting polynomial matrices $\mathbf{F}$ and then compute a minimal Gröbner basis for M with respect to $>_{T_{m|n}}$.

3.2 Comprehensive order basis system for parametric matrices

For parametric univariate polynomial matrices, we can generalize the above non-parametric method to the parametric case for computing comprehensive order basis systems by means of the minimal CGSs for modules.

Theorem 3.6 *Let* $\mathbf{F} \in \mathbb{K}[U][x]^{m\times n}$ *be a polynomial matrix of size* $m \times n$ *with parameters* $U = \{u_1, \cdots, u_k\}$ *and variable* x, S *be a subset of* L^k, $\overrightarrow{\sigma}$ *and* $\overrightarrow{s}$ *be integer lists. Consider a new parameter polynomial matrix* $\mathbf{F}'$ *with size of* $(m+n) \times (m+n)$ *as follows:*

$$\mathbf{F}' = \begin{bmatrix} \mathbf{F} & x^{\overrightarrow{\sigma}} \\ \mathbf{I}_n & \mathbf{0}_{n\times m} \end{bmatrix} \in \mathbb{K}[U][x]^{(m+n)\times(m+n)}.$$

Let M be the module generated by the column vectors of $\mathbf{F}'$. Assume $\mathfrak{O} = \{(A_1, O_1), \ldots, (A_l, O_l)\}$ is a minimal comprehensive Gröbner system of M on S with respect to the order $>_{T_{m|n}, \overrightarrow{s}}$. For each branch (A_i, O_i), we can get a matrix $\widehat{\mathbf{O}}_{\mathbf{i}}$ by using the method in Theorem 3.2. Then the set $\widehat{\mathfrak{O}} = \{(A_1, \widehat{\mathbf{O}}_{\mathbf{1}}), \ldots, (A_l, \widehat{\mathbf{O}}_{\mathbf{l}})\}$ is a comprehensive order basis system on S for $\mathbf{F}$ with respect to $\overrightarrow{\sigma}$ and $\overrightarrow{s}$.

Proof Since $\mathfrak{O}$ is a minimal comprehensive Gröbner system of M, in each branch $\pi_a(O_i)$ is a minimal Gröbner basis of $\pi_a(\mathbf{M})$ for any $a \in A_i$. Besides, there is no element in O_i specializing to $\mathbf{0}$ because the leading coefficients of all elements in O_i are non-zero under specialization. Thus, it is easy to derive the results that the correspond matrix $\pi_a(\widehat{\mathbf{O}}_{\mathbf{i}})$ is a $(\pi_a(\mathbf{F}), \overrightarrow{\sigma}, \overrightarrow{s})$-basis for all $a \in A_i$ and $i = 1, \ldots, l$ from Theorem 3.2 and 3.5.

4 Algorithm and Example

We now give the main algorithm for computing a comprehensive order basis system. Theorem 3.6 ensures that this algorithm works correctly. The algorithm also terminates, since the algorithm for computing minimal comprehensive Gröbner system terminates.

Algorithm 1 Computing comprehensive order basis systems

Require: integer list $\overrightarrow{s}$, non-negative integer list $\overrightarrow{\sigma}$, and $(E, N, \mathbf{F})$: $E, N \subset \mathbb{K}[U]$and $\mathbf{F} \in \mathbb{K}[U][x]^{m\times n}$.

Ensure: a finite set $\{(E_i, N_i, \widehat{\mathbf{O}}_i)_{i=1}^{l}\}$ such that $\{(V(E_i) \setminus V(N_i), \widehat{\mathbf{O}}_i)_{i=1}^{l}\}$ is a comprehensive order basis system for $\mathbf{F}$ on $V(E) \setminus V(N)$ with respect to $\overrightarrow{\sigma}$ and $\overrightarrow{s}$.

1) Construct $\mathbf{F}'$ defined in Theorem 3.6 and the module block order $>_{T_{m|n}, \overrightarrow{s}}$.
2) Compute a minimal CGS $\{(A_i, O_i)_{i=1}^{l}\}$ for the module M generated by column vectors of $\mathbf{F}'$ on $V(E) \setminus V(N)$ with respect to $>_{T_{m|n}, \overrightarrow{s}}$, and get $\{(E_i, N_i, \mathbf{O}_{\mathbf{i}})_{i=1}^{l}\}$ where $A_i = V(E_i) \setminus V(N_i)$ and $\mathbf{O}_{\mathbf{i}}$ is the corresponding matrix to the parametric Gröbner basis O_i.
3) Let $\widehat{\mathbf{O}}_i$ be the bottom right $n \times n$ submatrix of $\mathbf{O}_{\mathbf{i}}$ for $i = 1, \cdots, l$.
4) Return $\{(E_i, N_i, \widehat{\mathbf{O}}_i), i = 1, \cdots, l\}$.

For the computation of CGSs for modules, there exists an algorithm given by [26] which is based on the results proposed by [23]. In this paper, we extend the KSW algorithm proposed in [18] for computing CGSs over polynomial rings to the case of modules and then compute CGSs for modules since the KSW algorithm generates fewer branches and is the most efficient algorithm so far.

Below is a brief explanation about algorithm implementation.

There is currently no implementation version of CGS algorithms for modules under the block order $>_{T_{m|n}, \overrightarrow{s}}$ on Singular and Maple. During the algorithm implementation process, we should transform the modules into a specific polynomial ring. A general way can be found in Chapter 5 of [27], which is adding extra position variables p_i to expand the original module, and removing the polynomial including two and above degree of position variables in final Gröbner basis. The remainder polynomials can be transformed to the Gröbner basis for the original module. The difficulty is how to define the proper term order on the expanded polynomial ring. In our implementation, we use the weighted degree lexicographic order about x and position variables p_i, which is defined as

$$\mathbf{wdeg}(x^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n},[s_0,\cdots,s_n]) = \alpha_0 s_0 + \cdots + \alpha_n s_n.$$

Note that the weighted degree lexicographic order is a term order on the expanded polynomial ring, so Gröbner bases can be calculated. Also the remainder polynomials are all linear about position variables, i.e., $\sum_{i=1}^{n}\alpha_i = 1$. Let $s_0 = 1$, assume $\alpha_i \neq 0$,

$$\mathbf{wdeg}(x^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n},[s_0,\cdots,s_n]) = \alpha_0 + s_i = \deg(x^{\alpha}) + s_i.$$

This is consistent to the block order $>_{T_{m|n},\vec{s}}$.

The proposed algorithms have been implemented on both Maple and Singular, the codes and examples are available at http://www.mmrc.iss.ac.cn/ dwang/software.html.

Here we use two simple examples to illustrate the steps.

Example 4.1 (Example 3.21 in [10].) Let $\mathbb{K} = Z_2$, $U = \emptyset$, $\vec{s} = [0,1,2,3,3,3]$, $\vec{\sigma} = [8,4,4]$, and

$$\mathbf{F} = \begin{bmatrix} 0 & x^8 & x^6+x^9 & x^4+x^6+x^9 & x^6+x^8+x^9+x^{10} & x^5+x^8 \\ 0 & 0 & x^5 & x^4+x^6 & x^4+x^6 & x^5+x^6 \\ 0 & x^4 & x^5 & x^5 & x^4+x^5+x^6 & x^4 \end{bmatrix}.$$

Now we use the algorithm to compute the $(\mathbf{F},\vec{\sigma},\vec{s})$-basis.

Step 1) By Theorem 3.6, we construct $\mathbf{F}'$ as following:

$$\mathbf{F}' = \begin{bmatrix} 0 & x^8 & x^6+x^9 & x^4+x^6+x^9 & x^6+x^8+x^9+x^{10} & x^5+x^8 & x^8 & 0 & 0 \\ 0 & 0 & x^5 & x^4+x^6 & x^4+x^6 & x^5+x^6 & 0 & x^4 & 0 \\ 0 & x^4 & x^5 & x^5 & x^4+x^5+x^6 & x^4 & 0 & 0 & x^4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and the corresponding order is $>_{T_{3|6},[0,1,2,3,3,3]}$ (under Definition 3.3).

Step 2) Since $U = \emptyset$, then $E = \emptyset$, $N = \mathbb{K}$. Now we compute the minimal CGS for the module M generated by column vectors of $\mathbf{F}'$ on $V(E) \setminus V(N)$ with respect to $>_{T_{3|6},[0,1,2,3,3,3]}$

and get $(E_1, N_1, \mathbf{O_1})$, where $E_1 = E, N_1 = N$ and

$$\mathbf{O_1} = \begin{bmatrix} x^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & x & x^2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & x & 1 & 0 \end{bmatrix}.$$

Step 3) The corresponding matrix (the bottom right $n \times n$ submatrix of $\mathbf{O_1}$) is

$$\widehat{\mathbf{O}}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & x & x^2 \\ 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 1 & 0 \end{bmatrix}.$$

Step 4) We obtain a system $\{(E_1 = \emptyset, N_1 = \mathbb{K}, \widehat{\mathbf{O}}_1)\}$.

Thus, the comprehensive order basis system for $\mathbf{F}$ is $\{(V(\emptyset) \setminus V(\mathbb{K}), \widehat{\mathbf{O}}_1)\}$. This result is the same as [10] after exchanging some columns.

This is an example without parameters, the following example is with extra comprehensive variables.

Example 4.2 Let $\mathbb{K} = \mathbb{Q}$, $U = \{u_1, u_2\}$, $\overrightarrow{s} = [1, 2]$, $\overrightarrow{\sigma} = [4, 4]$, and

$$\mathbf{F} = \begin{bmatrix} x^2 + u_1 & x^2 + 1 \\ x + u_2 & x + u_1 \end{bmatrix}.$$

Now we use the algorithm to compute a comprehensive order basis system for $\mathbf{F}$ with respect to $\overrightarrow{\sigma}$ and $\overrightarrow{s}$.

Step 1) By Theorem 3.6, we construct $\mathbf{F}'$ as following:

$$\mathbf{F}' = \begin{bmatrix} x^2 + u_1 & x^2 + 1 & x^4 & 0 \\ x + u_2 & x + u_1 & 0 & x^4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and the corresponding order is $>_{T_{2|2},[1,2]}$ (under Definition 3.3).

Step 2) Since $U = [u_1, u_2]$, and $E = \emptyset$, $N = \mathbb{K}$. Now we compute the minimal CGS for the module M generated by column vectors of $\mathbf{F}'$ on $V(E) \setminus V(N)$ with respect to $>_{T_{2|2},[1,2]}$. And there are three branches with $\{(E_1, N_1, \mathbf{O_1}), (E_2, N_2, \mathbf{O_2}), (E_3, N_3, \mathbf{O_3})\}$, where $E_1 = \emptyset, N_1 = \{u_1^2 - u_2\}, E_2 = \{u_1^2 - u_2\}, N_2 = \{u_1 - 1\}, E_3 = \{u_1 - 1, u_2 - 1\}, N_3 = \{1\}$,

$$\mathbf{O}_1 = \begin{bmatrix} 34 & 0 & 0 & 0 \\ g_{21} & u_1^8 - 4u_1^6u_2 + 6u_1^4u_2^2 - 4u_1^2u_2^3 + u_2^4 & 0 & 0 \\ g_{31} & g_{32} & 0 & x^4 \\ g_{41} & g_{42} & x^4 & 0 \end{bmatrix},$$

$$\mathbf{O}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -x^3 - x^2 + x + u_1 & u_1x - x & 0 & 0 \\ -u_1x^2 - x & -u_2x^2 - x^2 - u_1x - 1 & 0 & x^3 \\ -u_1x^3 + x^3 + u_2x^2 - x^2 + u_1x + 1 & u_1x^3 - u_2x^3 + u_1u_2x^2 + x^2 + u_2x + u_1 & x^4 & -u_1x^3 \end{bmatrix},$$

$$\mathbf{O}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -x^3 - x^2 + x + 1 & x^4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -x^2 + 1 & 0 & x^4 & -1 \end{bmatrix}.$$

Step 3) The corresponding matrices (the bottom right $n \times n$ submatrices of $\mathbf{O_1}, \mathbf{O_2}, \mathbf{O_3}$) are

$$\widehat{\mathbf{O}}_1 = \begin{bmatrix} 0 & x^4 \\ x^4 & 0 \end{bmatrix}, \widehat{\mathbf{O}}_2 = \begin{bmatrix} 0 & x^3 \\ x^4 & -u_1x^3 \end{bmatrix}, \widehat{\mathbf{O}}_3 = \begin{bmatrix} 0 & 1 \\ x^4 & -1 \end{bmatrix}.$$

Step 4) We obtain a system $\{(E_1 = \emptyset, N_1 = \{u_1^2 - u_2\}, \widehat{\mathbf{O}}_1), (E_2 = \{u_1^2 - u_2\}, N_2 = \{u_1 - 1\}, \widehat{\mathbf{O}}_2), (E_3 = \{u_1 - 1, u_2 - 1\}, N_3 = \{1\}, \widehat{\mathbf{O}}_3)\}$

Thus, the comprehensive order basis system for $\mathbf{F}$ with respect to $\overrightarrow{\sigma} = [4, 4]$ and $\overrightarrow{s} = [1, 2]$ is $\{(V(E_i) \setminus V(N_i), \widehat{\mathbf{O}}_i), i = 1, 2, 3\}$.

5 Concluding Remarks

In this paper we prove that the order basis can be obtained by computing a minimal Gröbner basis for a constructed special module M with respect to a specific block order $>_{T_{m|n},\overrightarrow{s}}$. Moreover, we first give the concept of comprehensive order basis systems for parametric polynomial matrices. Based on comprehensive Gröbner systems for modules, we obtained an algorithm for computing comprehensive order basis systems. We believe our algorithm can be further improved and the comprehensive order basis system has more applications for the unimodular completion or determinant computing.

References

[1] Hermite C, *Sur la fonction exponentielle*, Gauthier-Villars, 1873.

[2] Padé H, Sur la représentation approchée d'une fonction par des fractions rationnelles, *Annales scientifiques de l'Ecole normale supérieure*, 1892, **9**: 3-93.

[3] Beckermann B and Labahn G, Recursiveness in matrix rational interpolation problems, *Journal of Computational and Applied Mathematics*, 1997, **77**: 5-34.

[4] Labahn G, Inversion components of block Hankel-like matrices, *Linear Algebra and its Applications*, 1992, **177**: 7-48.

[5] Beckermann B, Labahn G and Villard G, Shifted normal forms of polynomial matrices, *Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation*, 1999, 189-196.

[6] Giorgi P, Jeannerod C-P and Villard G, On the complexity of polynomial matrix computations, *Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation*, 2003,135-142.

[7] Storjohann A and Villard G, Computing the rank and a small nullspace basis of a polynomial matrix, *Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation*, 2005, 309-316.

[8] Beckermann B and Labahn G, A uniform approach for the fast computation of matrix-type Padé approximants, *SIAM Journal on Matrix Analysis and Applications*,1994, **15**: 804-823.

[9] Storjohann A, Notes on computing minimal approximant bases, *In Challenges in Symbolic Computation Software. Dagstuhl Seminar Proceedings*, 2006, **6271**: 1-6.

[10] Zhou W, Fast order basis and kernel basis computation and related problems, *University of Waterloo*, 2013, 1-76.

[11] Jeannerod C-P, Neiger V and Villard G, Fast computation of approximant bases in canonical form, *Journal of Symbolic Computation*, 2020, **98**: 192-224.

[12] Neiger V, Bases of relations in one or several variables: fast algorithms and applications, *École Normale Supérieure de Lyon-University of Waterloo*, 2016.

[13] Jeannerod C-P, Neiger V, Schost, É and Villard G, Fast computation of minimal interpolation bases in Popov form for arbitrary shifts, *Proceedings of the 2016 International Symposium on Symbolic and Algebraic Computation*, 2016, 295-302.

[14] Beckermann B, Labahn G. Fraction-free computation of matrix rational interpolants and matrix GCDs, *SIAM Journal on Matrix Analysis and Applications*, 2000, **22(1)**: 114-144.

[15] Beckermann B, Labahn G. Fraction-free computation of simultaneous padé approximants, *Proceedings of the 2009 International Symposium on Symbolic and Algebraic Computation*, 2009, 15-22.

[16] Danik Y, Dmitriev M. Symbolic Padé representation of stabilizing regulators for a class of nonlinear control systems with a parameter, *Procedia Computer Science*, 2021, **186**: 154-160.

[17] Weispfenning V, Comprehensive Gröbner bases, *Journal of Symbolic Computation*, 1992, **14**: 1-29.

[18] Kapur D, Sun Y and Wang D K, An efficient algorithm for computing a comprehensive Gröbner system of a parametric polynomial system, *Journal of Symbolic Computation*, 2013,**49**: 27-44.

[19] Montes A, A new algorithm for discussing Gröbner bases with parameters, *Journal of Symbolic Computation*, 2002, **33**: 183-208.

[20] Suzuki A, Comprehensive Grobner Bases via ACGB, *The 10th Internatinal Conference on Applications of Computer Algebra*, 2004, 65–73.

[21] Manubens M and Montes A, Improving the DISPGB algorithm using the discriminant ideal, *Journal of Symbolic Computation*, 2006, **41**: 1245-1263.

[22] Wibmer M, Gröbner bases for families of affine or projective schemes,*Journal of Symbolic Computation*, 2007, **42**: 803-834.

[23] Suzuki A and Sato Y, A simple algorithm to compute comprehensive Gröbner bases using Gröbner bases, *Proceedings of the 2006 International Symposium on Symbolic and Algebraic Computation*, 2006, 326-331.

[24] Kapur D, Comprehensive Gröbner basis theory for a parametric polynomial ideal and the associated completion algorithm, *Journal of Systems Science and Complexity*, 2017, **30**: 196-233.

[25] Buchberger B, A theoretical basis for the reduction of polynomials to canonical forms, *ACM SIGSAM Bulletin*, 1976, **10**: 19-29.

[26] Nabeshima K, On the computation of parametric Gröbner bases for modules and syzygies, *Japan Journal of Industrial and Applied Mathematics*, 2010, **27(2)**: 217-238.

[27] Cox D, Little J and O'Shea D, *Using Algebraic Geometry*, Springer, 2005.

6 Appendix

In Example 4.2,

$g_{21} = -34x^3 - 34x^2 + (17u_1 - 25u_2 + 38 + 4u_2^2)x + (8u_1^7u_2 + 6u_1^7 + 24u_1^6u_2 - 24u_1^5u_2^2 - 16u_1^6 - 22u_1^5u_2 - 72u_1^4u_2^2 + 24u_1^3u_2^3 + 9u_1^5 + 40u_1^4u_2 + 26u_1^3u_2^2 + 72u_1^2u_2^3 - 8u_1u_2^4 + 13u_1^4 - 14u_1^3u_2 - 32u_1^2u_2^2 - 10u_1u_2^3 - 24u_2^4 - 21u_1^3 - 22u_1^2u_2 + 5u_1u_2^2 + 8u_2^3 - 4u_1^2 + 21u_1u_2 + 9u_2^2 + 34u_1 + 4u_2)$,

$g_{31} = (8u_1^4u_2 + 6u_1^4 + 8u_1^2u_2^2 - 34u_1^3 - 46u_1^2u_2 + 16u_1u_2^2 + 75u_1^2 + 30u_1u_2 - 24u_2^2 - 80u_1 + 8u_2 + 33)x^3 + (-8u_1^5u_2 - 6u_1^5 - 16u_1^4u_2 + 8u_1^3u_2^2 + 22u_1^4 + 26u_1^3u_2 + 16u_1^2u_2^2 - 31u_1^3 - 26u_1^2u_2 - 24u_1u_2^2 - 13u_1 - 24u_2 + 24u_1^2 + 48u_1u_2 + 4)x^2 + (8u_1^4u_2 + 6u_1^4 + 16u_1^3u_2 - 8u_1^2u_2^2 - 22u_1^3 - 34u_1^2u_2 - 16u_1u_2^2 + 25u_1^2 + 18u_1u_2 + 24u_2^2 + 4u_1 - 4u_2 - 34)x + (-8u_1^5u_2 - 6u_1^5 - 24u_1^4u_2 + 16u_1^3u_2^2 + 16u_1^4 + 16u_1^3u_2 + 48u_1^2u_2^2 - 8u_1u_2^3 - 9u_1^3 - 24u_1^2u_2 - 10u_1u_2^2 - 24u_2^3 - 13u_1^2 + 5u_1u_2 + 8u_2^2 + 21u_1 + 9u_2 + 4)$,

$g_{32} = (u_1^5 - 3u_1^4 + 3u_1^3 + 2u_1^2u_2 - u_1u_2^2 - 3u_1^2 - 2u_1u_2 + u_2^2 + 3u_1 - 1)x^3 + (-u_1^6 + u_1^5 + 2u_1^4u_2 - u_1^4 - 2u_1^3u_2 - u_1^2u_2^2 + 2u_1^3 + u_1^2u_2 + u_1u_2^2 - u_1^2 - 2u_1u_2 + u_2)x^2 + (u_1^5 - u_1^4 - 2u_1^3u_2 + 2u_1^2u_2 + u_1u_2^2 - u_2^2)x + (-u_1^6 + 3u_1^4u_2 - 3u_1^2u_2^2 + u_2^3)$,

$g_{41} = (8u_1^4u_2 - 16u_1^3u_2^2 + 6u_1^4 - 4u_1^3u_2 - 24u_1^2u_2^2 - 28u_1^3 + 22u_1^2u_2 + 64u_1u_2^2 + 59u_1^2 - 30u_1u_2 - 24u_2^2 - 71u_1 + 4u_2 + 34)x^3 + (8u_1^4u_2^2 + 14u_1^4u_2 + 16u_1^3u_2^2 - 8u_1^2u_2^3 + 6u_1^4 - 14u_1^3u_2 - 34u_1^2u_2^2 - 16u_1u_2^3 - 28u_1^3 - 19u_1^2u_2 + 18u_1u_2^2 + 24u_2^3 + 47u_1^2 + 28u_1u_2 - 8u_2^2 - 21u_1 - 9u_2 - 38)x^2 + (-8u_1^5u_2 - 6u_1^5 - 16u_1^4u_2 + 8u_1^3u_2^2 + 22u_1^4 + 34u_1^3u_2 + 16u_1^2u_2^2 - 25u_1^3 - 18u_1^2u_2 - 24u_1u_2^2 - 4u_1^2 + 4u_1u_2 + 34u_1)x + (8u_1^6u_2 + 6u_1^6 + 24u_1^5u_2 - 16u_1^4u_2^2 - 16u_1^5 - 16u_1^4u_2 - 48u_1^3u_2^2 + 8u_1^2u_2^3 + 9u_1^4 + 24u_1^3u_2 + 10u_1^2u_2^2 + 24u_1u_2^3 + 13u_1^3 - 5u_1^2u_2 - 8u_1u_2^2 - 21u_1^2 - 9u_1u_2 - 4u_1 + 34)$,

$g_{42} = (u_1^5 - 2u_1^4u_2 - 2u_1^4 + 2u_1^3u_2 + 2u_1^2u_2^2 + 3u_1^3 - 3u_1u_2^2 - 3u_1^2 + u_2^2 + u_1)x^3 + (u_1^5u_2 + u_1^5 - u_1^4u_2 - 2u_1^3u_2^2 - 2u_1^4 - u_1^3u_2 + 2u_1^2u_2^2 + u_1u_2^3 + u_1^3 + 2u_1^2u_2 - u_2^3 - u_1u_2)x^2 + (-u_1^6 + u_1^5 + 2u_1^4u_2 - 2u_1^3u_2 - u_1^2u_2^2 + u_1u_2^2)x + (u_1^7 + 3u_1^3u_2^2 - u_1u_2^3 - 3u_1^5u_2)$.