Theory and Algorithms for Bivariate Polynomial Matrix Factorizations

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Abstract

Based on the primitive factorization theorem, this paper presents an improved algorithm for computing free bases of syzygy modules of bivariate polynomial matrices, which additionally enables efficient computation of $\boldsymbol{\mu}$-bases for rational parametric surfaces. Experimental results show that the new algorithm outperforms two existing algorithms in terms of computational efficiency. Furthermore, by leveraging this algorithm, we generalize the general matrix factorization theory of full-rank bivariate polynomial matrices to the rank-deficient case for the first time.

Keywords: Bivariate polynomial matrices, Primitive factorization theorem, Syzygy modules, Free bases, General matrix factorization

Mathematics Subject Classification: 68W30 , 15B33

1 Introduction

An essential research topic in symbolic computation and multidimensional systems theory concerns factorizations of multivariate polynomial matrices, which find extensive applications across circuits, signal processing, computer aided geometric design, and various engineering disciplines (see, e.g., [1–4]). Since the 1970s, when engineers and mathematicians first applied algebraic analysis methods [5–7] to this subject, the research methods have gradually evolved: from Gröbner basis [8], a classical approach for handling polynomial-related problems in symbolic computation, to module theory [9] in commutative algebra. Moreover, with the rapid development of computational software such as Maple and Singular, significant progress has been made in this field (see [10] and references therein).

Since univariate polynomial rings are Euclidean domains, univariate polynomial matrices can be factorized through the application of the Euclidean division algorithm. Huang and Chen [3] used univariate polynomial matrix factorization to compute μ-bases of rational parametric curves. When the number of variables is greater than one, the above Euclidean division algorithm fails due to the fact that any multivariate polynomial ring is no longer a principal ideal domain.

In 1977, during their investigation of two-dimensional systems theory, Morf et al. [6] proposed the classical primitive factorization theorem; however, this theorem is subject to a critical restriction – the necessity for the underlying number field to be algebraically closed. In 1982, Guiver and Bose [5] eliminated the aforementioned constraint by leveraging the technique of modulo univariate irreducible polynomials. Consequently, the primitive factorization algorithm developed under this theorem has emerged as a critical tool for fully addressing the general matrix factorization problem of full-rank bivariate polynomial matrices. Furthermore, Deng et al. [4] developed an efficient algorithm for computing μ-bases of rational parametric surfaces by means of the primitive factorization theorem. However, the method proposed in [5] cannot be straightforwardly generalized to cases where the number of variables is greater than two, rendering the general matrix factorization problem for multivariate polynomial matrices an open challenge.

In 1979, Youla and Gnavi [7], while investigating the structural properties of multidimensional systems, classified polynomial matrix factorizations into three distinct categories based on their inherent characteristics: zero prime factorization, minor prime factorization, and factor prime factorization. In 2001, Lin and Bose [11] conjectured that any full-rank multivariate polynomial matrix whose maximal reduced minors generate the unit ideal admits a zero prime factorization. Subsequently, this conjecture was independently resolved by Pommaret [12], Srinivas [13], Wang and Feng [14], and Liu et al. [15] using different approaches. In 2005, Wang and Kwong [16] first established that the necessary and sufficient condition for the existence of a minor prime factorization of any full-rank multivariate polynomial matrix is the freeness of a specific quotient module. In 2007, Wang [17] introduced the concept of regularity and proposed the necessary and sufficient condition for the existence of factor prime factorizations under such a regularity constraint. Subsequently, Liu and Wang characterized the equivalent condition for regularity [18], and addressed the uniqueness of general matrix factorizations for full-rank multivariate polynomial matrices [19]. In recent years, Guan et al. [20, 21] and Lu et al. [22, 23] have respectively advanced the factor prime factorization and minor prime factorization problems for rank-deficient multivariate polynomial matrices, and have made some new progress.

In this paper, we revisit the factorization problem of bivariate polynomial matrices. Building upon the primitive factorization theorem, we explore how to compute free bases of syzygy modules more efficiently, while further explore the general matrix factorization theory for the rank-deficient case. This study aims to extend existing theoretical foundations and enhance practical implementations in symbolic computation.

The rest of the paper is organized as follows. We introduce some related concepts and the primitive factorization theorem in Section 2. In Section 3, we present an improved algorithm for computing free bases of syzygy modules, and the experimental data show the efficiency of this algorithm. In Section 4, we establish the general matrix factorization theory for rank-deficient bivariate polynomial matrices. We conclude the paper in Section 5.

2 Preliminaries

Throughout the paper, we assume without loss of generality that $r \leq l \leq m$, where r, l, m are three positive integers. In addition, we use "w.r.t." to represent "with respect to".

Let $\mathbb{K}$ be a field, $\mathbb{K}[s,t]$ be the bivariate polynomial ring in the variables s, t over $\mathbb{K}$, and $\mathbb{K}(s)[t]$ be the univariate polynomial ring in the variable t with coefficients in $\mathbb{K}(s)$, where $\mathbb{K}(s)$ is the fraction field of $\mathbb{K}[s]$. Given $f \in \mathbb{K}[s,t]$, we use $\deg_s(f)$ to denote the degree of f w.r.t. s. We use $\mathbb{K}[s,t]^{l\times m}$ to represent the set of $l \times m$ matrices with entries in $\mathbb{K}[s,t]$. Let $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$ with rank r, we use $d_r(\mathbf{F})$ to denote the greatest common divisor (GCD) of all the $r \times r$ minors of $\mathbf{F}$.

2.1 Basic notions

In linear algebra, a matrix $\mathbf{U} \in \mathbb{K}^{l\times l}$ is called invertible if $\det(\mathbf{U}) \neq 0$. In symbolic computation, there exists an analogous concept as follows.

Definition 1. *Let $\mathcal{R}$ be a commutative ring and $\mathbf{U} \in \mathcal{R}^{l\times l}$. Then $\mathbf{U}$ is said to be unimodular if $\det(\mathbf{U})$ is a unit in $\mathcal{R}$.*

In Definition 1, if $\mathcal{R} = \mathbb{K}[s,t]$, then $\det(\mathbf{U}) \in \mathbb{K} \setminus \{0\}$; if $\mathcal{R} = \mathbb{K}(s)[t]$, then $\det(\mathbf{U}) \in \mathbb{K}(s) \setminus \{0\}$.

Every vector space over $\mathbb{K}$ in linear algebra possesses a basis. However, modules over polynomial rings do not necessarily admit bases in general. A simple example is supplied by the module $M = \langle \vec{u}_1, \vec{u}_2 \rangle \subset \mathbb{K}[s,t]^{1\times 2}$, where $\vec{u}_1 = (s,s)$ and $\vec{u}_2 = (t,t)$. Since $\vec{u}_1$ and $\vec{u}_2$ are not $\mathbb{K}[s,t]$-linearly independent, the generating set $\{\vec{u}_1, \vec{u}_2\}$ is not a basis of M. Furthermore, it is straightforward to verify that M cannot be generated by any single vector $\vec{u} \in \mathbb{K}[s,t]^{1\times 2}$. Consequently, M does not admit a basis. If a module has a basis, then it is given a special name.

Definition 2. *Let M be a module over a commutative ring $\mathcal{R}$. M is said to be a free module if M has a basis (that is, a generating set that is $\mathcal{R}$-linearly independent). In particular, this basis is called a free basis.*

Definition 3. *Given* $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$*, the set*

$$\mathrm{Syz}(\mathbf{F}) = \left\{ \vec{u} \in \mathbb{K}[s,t]^{m\times 1} \mid \mathbf{F}\vec{u} = \vec{0} \right\}$$

is a $\mathbb{K}[s,t]$*-module, which is called a syzygy module of* $\mathbf{F}$*.*

Let $A \in \mathbb{K}^{l\times m}$ and $Y = (y_1, \dots, y_m)^{\mathrm{T}} \in \mathbb{K}^{m\times 1}$. A well-known conclusion in linear algebra is that the rank of the kernel space for $AY = \vec{0}$ is $m - \mathrm{rank}(A)$. This result also applies to the case of bivariate polynomial matrices.

Lemma 1 ([24])**.** *Let* $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$ *with rank* r*. Then the rank of* $\mathrm{Syz}(\mathbf{F})$ *is* $m - r$*.*

Although certain modules are not free, Chen et al. [25] demonstrated in their investigation of the computation of μ-bases for rational parametric surfaces that the syzygy module of the homogeneous expression form of an arbitrary rational parametric surface must be a free module. Subsequently, Liu and Wang [26] extended this result to arbitrary full-rank bivariate polynomial matrices. Based on the conclusion in [26], we derive the following corollary.

Corollary 2. *Let* $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$ *with rank* r*. Then* $\mathrm{Syz}(\mathbf{F})$ *is a free module of rank* $m - r$*.*

Proof. According to Lemma 1, the rank of $\mathrm{Syz}(\mathbf{F})$ is $m - r$. Let $\mathbf{F}_1 \in \mathbb{K}[s,t]^{r\times m}$ be an arbitrary full row rank submatrix of $\mathbf{F}$. There exists a full column rank matrix $\mathbf{G} \in \mathbb{K}(s,t)^{l\times r}$ such that $\mathbf{F} = \mathbf{G}\mathbf{F}_1$. On the one hand, it is easy to check that $\mathrm{Syz}(\mathbf{F}_1) \subseteq \mathrm{Syz}(\mathbf{F})$. On the other hand, if $\vec{u} \in \mathrm{Syz}(\mathbf{F})$, then

$$\mathbf{F}\vec{u} = (\mathbf{G}\mathbf{F}_1)\vec{u} = \mathbf{G}(\mathbf{F}_1\vec{u}) = \vec{0}.$$

Since $\mathbf{G}$ has full column rank, $\mathbf{F}_1\vec{u} = \vec{0}$. This implies that $\vec{u} \in \mathrm{Syz}(\mathbf{F}_1)$ and $\mathrm{Syz}(\mathbf{F}) \subseteq \mathrm{Syz}(\mathbf{F}_1)$. Thus, $\mathrm{Syz}(\mathbf{F}) = \mathrm{Syz}(\mathbf{F}_1)$. It follows from the proof of Lemma 4.1 in [26] that $\mathrm{Syz}(\mathbf{F}_1)$ is a free module. Therefore, $\mathrm{Syz}(\mathbf{F})$ is a free module of rank $m - r$. □

The following concept, first proposed in [7], is central to multidimensional systems.

Definition 4 ([7])**.** *Let* $\mathbf{A} \in \mathbb{K}[s,t]^{r\times m}$ *be of full row rank. If* $d_r(\mathbf{A})$ *is a nonzero constant in* $\mathbb{K}$*, then* $\mathbf{A}$ *is said to be a minor left prime (MLP) matrix.*

Let $\mathbf{B} \in \mathbb{K}[s,t]^{m\times r}$ be of full column rank. Similarly, $\mathbf{B}$ can be defined as a minor right prime (MRP) matrix.

A general matrix factorization of a bivariate polynomial matrix is now formulated as follows.

Definition 5. *Let* $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$ *with rank* r*, and* $f \in \mathbb{K}[s,t]$ *be an arbitrary divisor of* $d_r(\mathbf{F})$*.* $\mathbf{F}$ *is said to admit a general matrix factorization w.r.t.* f *if* $\mathbf{F}$ *can be factorized as*

$$\mathbf{F} = \mathbf{G}_0\mathbf{F}_0,$$

where $\mathbf{G}_0 \in \mathbb{K}[s,t]^{l\times r}$ *satisfies* $d_r(\mathbf{G}_0) = f$*, and* $\mathbf{F}_0 \in \mathbb{K}[s,t]^{r\times m}$*.*

In particular, $\mathbf{F}$ in Definition 5 is said to admit an MLP factorization if $\mathbf{F}_0$ is an MLP matrix. At this point, it follows directly from the Binet-Cauchy formula that $d_r(\mathbf{G}_0) = d_r(\mathbf{F})$.

2.2 Primitive factorization theorem

Prior to presenting the primitive factorization theorem, we introduce the following concept.

Definition 6. *Let $f \in \mathbb{K}[s,t]$, we consider f as a polynomial in t with coefficients in $\mathbb{K}[s]$. We write f in the form*

$$f = \sum_{i=0}^{N} a_i(s) \cdot t^i,$$

where $a_i(s) \in \mathbb{K}[s]$ for $i = 0, \dots, N$. Then the **content** *of f w.r.t. t is the GCD of $a_0(s), \dots, a_N(s)$.*

Analogously, the content of f w.r.t. s can be defined.

Theorem 3 (Primitive Factorization Theorem, [5])**.** *Let $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$ be of full row rank, and $g \in \mathbb{K}[s]$ be the content of $d_l(\mathbf{F})$ w.r.t. t. Then there are $\mathbf{G}_0 \in \mathbb{K}[s,t]^{l\times l}$ and $\mathbf{F}_0 \in \mathbb{K}[s,t]^{l\times m}$ such that*

$$\mathbf{F} = \mathbf{G}_0\mathbf{F}_0 \text{ and } \det(\mathbf{G}_0) = g.$$

To prove the correctness of Theorem 3, Guiver and Bose [5] presented a constructive proof based on the following calculation procedure, which is termed the primitive factorization algorithm.

Assume that $p \in \mathbb{K}[s]$ is an irreducible divisor of g. Let $R_p = \mathbb{K}[s]/(p)$. Then R_p is a field and $R_p[t]$ is a Euclidean domain. We consider the following homomorphism

$$\begin{array}{cccc} \phi_p : & \mathbb{K}[s,t] & \to & R_p[t] \\ & \sum_{i=0}^{N} a_i(s) \cdot t^i & \mapsto & \sum_{i=0}^{N} \overline{a_i(s)} \cdot t^i, \end{array}$$

where $\overline{a_i(s)} \equiv a_i(s) \pmod p \in R_p$ for $i = 0, \dots, N$. This homomorphism can extend canonically to $\phi_p : \mathbb{K}[s,t]^{l\times m} \to R_p[t]^{l\times m}$ by applying ϕ_p entry-wise. Let $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$, we use $\overline{\mathbf{F}}$ to denote $\phi_p(\mathbf{F})$ in $R_p[t]^{l\times m}$.

Since $p \mid d_l(\mathbf{F})$, we can transform $\overline{\mathbf{F}} \in R_p[t]^{l\times m}$ into the following form

$$\overline{\mathbf{F}}_1 = \begin{pmatrix} * & \cdots & \cdots & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & \cdots & \cdots & * \\ \overline{0} & \cdots & \cdots & \cdots & \overline{0} \end{pmatrix}$$

exclusively via elementary transformations of the first kind (row interchange) and third kind (adding a multiple of one row to another) over $R_p[t]$. For each entry of every elementary matrix, we take its representative element in $\mathbb{K}[s,t]$ with s-degree less than $\deg_s(p)$. Through these operations, each elementary matrix in $R_p[t]^{l\times l}$ corresponds to an elementary matrix in $\mathbb{K}[s,t]^{l\times l}$. Let $\mathbf{U}$ denote the product of all such elementary matrices in $\mathbb{K}[s,t]^{l\times l}$. Then $\mathbf{U}$ is unimodular, and there exists $\mathbf{H} \in \mathbb{K}[s,t]^{l\times m}$ satisfying

$$\mathbf{U}\mathbf{F} = \operatorname{diag}\{1, \dots, 1, p\} \cdot \mathbf{H}.$$

By iterating this computational procedure until g is extracted from $d_l(\mathbf{F})$, we obtain $\mathbf{F} = \mathbf{G}_0\mathbf{F}_0$ with $\det(\mathbf{G}_0) = g$.

Remark 1. *The primary computational procedure of the above primitive factorization algorithm closely parallels the Gaussian elimination method in linear algebra. This algorithm cannot be directly extended to multivariate cases, which fundamentally stems from the fact that* $(\mathbb{K}[x_1, \ldots, x_{n-1}]/(p))[x_n]$ *no longer constitutes a Euclidean domain, where* $x_1, \ldots, x_n$ *are variables,* $n > 2$ *and* $p \in \mathbb{K}[x_1, \ldots, x_{n-1}]$ *is irreducible.*

3 Improved Algorithm for Computing Free Bases of Syzygy Modules

Lin [27] derived a special case result applicable to the bivariate setting while studying the properties of syzygy modules for multivariate polynomial matrices.

Proposition 4 (Proposition 7 in [27])**.** *Let* $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$ *be of full row rank. There exists an MRP matrix* $\mathbf{W} \in \mathbb{K}[s,t]^{m\times r}$ *such that* $\mathbf{FW} = \mathbf{0}$*, where* $r = m - l$*. Furthermore, the column vectors of* $\mathbf{W}$ *constitute a free basis for* $\mathrm{Syz}(\mathbf{F})$*.*

Given a rational parametric surface in homogeneous form $P(s,t) = (a,b,c,d)$, where $a,b,c,d \in \mathbb{K}[s,t]$ are jointly coprime. In 2005, Chen et al. [25] proved the equivalence between the μ-basis of P and the free basis of $\mathrm{Syz}(P)$. Subsequently, Deng et al. [4] first presented an algorithm for computing a μ-basis of P based on Proposition 4, by using the primitive factorization algorithm and the GCD extraction algorithm (see Theorem 6.2 in [6]). In the following, we present an improved algorithm for computing free bases of bivariate polynomial matrices.

Algorithm 1 computing free bases of syzygy modules of bivariate polynomial matrices

Require: a matrix $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$ with rank r.

Ensure: a free basis of $\mathrm{Syz}(\mathbf{F})$.

1: let $\mathbf{F}_1 \in \mathbb{K}[s,t]^{r\times m}$ be an arbitrary full row rank submatrix of $\mathbf{F}$;

2: compute a unimodular matrix $\mathbf{V}_1 \in \mathbb{K}(s)[t]^{m\times m}$ such that $\mathbf{F}_1\mathbf{V}_1$ is a lower triangular matrix;

3: convert the last $m-r$ columns of $\mathbf{V}_1$ to $\mathbf{W} \in \mathbb{K}[s,t]^{m\times(m-r)}$ by clearing denominators;

4: compute an MRP factorization $\mathbf{W} = \mathbf{W}_1\mathbf{G}$ of $\mathbf{W}$ by using the primitive factorization algorithm;

5: **return** $\mathbf{W}_1$.

We present a detailed explanation of the calculation process for Step 2 in Algorithm 1:

First, we consider $\mathbf{F}_1$ a univariate polynomial matrix in $\mathbb{K}(s)[t]^{r\times m}$. Let

$$\vec{f_1} = (f_{11}, \ldots, f_{1m})$$

be the first row of $\mathbf{F}_1$, and $g_1 \in \mathbb{K}(s)[t]$ be the GCD of $f_{11}, \ldots, f_{1m}$. Note that $\mathbb{K}(s)[t]$ is a Euclidean domain, there is a unimodular matrix $\mathbf{U}_1 \in \mathbb{K}(s)[t]^{m\times m}$ such that

$$\vec{f_1}\mathbf{U}_1 = (g_1, 0, \ldots, 0).$$

It follows that

$$\mathbf{F}_1\mathbf{U}_1 = \begin{pmatrix} g_1 & \vec{0} \\ \vec{u}_1 & \mathbf{F}_2 \end{pmatrix},$$

where $\vec{u}_1 \in \mathbb{K}(s)[t]^{(r-1)\times 1}$ and $\mathbf{F}_2 \in \mathbb{K}(s)[t]^{(r-1)\times(m-1)}$.

Second, let $\vec{f_2}$ be the first row of $\mathbf{F}_2$. There exists a unimodular matrix $\mathbf{U}_2 \in \mathbb{K}(s)[t]^{(m-1)\times(m-1)}$ such that

$$\vec{f_2}\mathbf{U}_2 = (g_2, 0, \ldots, 0),$$

where $g_2 \in \mathbb{K}(s)[t]$ be the GCD of the entries in $\vec{f_2}$. Thus,

$$\mathbf{F}_2\mathbf{U}_2 = \begin{pmatrix} g_2 & \vec{0} \\ \vec{u}_2 & \mathbf{F}_3 \end{pmatrix},$$

where $\vec{u}_2 \in \mathbb{K}(s)[t]^{(r-2)\times 1}$ and $\mathbf{F}_3 \in \mathbb{K}(s)[t]^{(r-2)\times(m-2)}$.

Third, let $\tilde{\mathbf{U}}_2 = \begin{pmatrix} 1 & \\ & \mathbf{U}_2 \end{pmatrix} \in \mathbb{K}(s)[t]^{m\times m}$. Then

$$\mathbf{F}\mathbf{U}_1\tilde{\mathbf{U}}_2 = \begin{pmatrix} g_1 & 0 & \vec{0} \\ v_{21} & g_2 & \vec{0} \\ \vec{u}_{31} & \vec{u}_2 & \mathbf{F}_3 \end{pmatrix},$$

where $v_{21} \in \mathbb{K}(s)[t]$ and $\vec{u}_{31} \in \mathbb{K}(s)[t]^{(r-2)\times 1}$, satisfying $\vec{u}_1 = \begin{pmatrix} v_{21} \\ \vec{u}_{31} \end{pmatrix}$.

Finally, repeat the above process, and we can obtain a unimodular matrix $\mathbf{V}_1 \in \mathbb{K}(s)[t]^{m\times m}$ such that

$$\mathbf{F}_1\mathbf{V}_1 = \begin{pmatrix} g_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ v_{21} & g_2 & 0 & & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & & & \vdots \\ v_{r1} & v_{r2} & \cdots & g_r & 0 & \cdots & 0 \end{pmatrix}. \quad (1)$$

Therefore, $\mathbf{F}_1\mathbf{V}_1$ is a lower triangular matrix.

Theorem 5. *Algorithm 1 outputs as specified within a finite number of steps.*

Proof. Correctness. Based on the proof process of Corollary 2, $\mathrm{Syz}(\mathbf{F}) = \mathrm{Syz}(\mathbf{F}_1)$. In addition, from the explanation of Step 2 provided above, there is a unimodular matrix $\mathbf{V}_1 \in \mathbb{K}(s)[t]^{m\times m}$ such that $\mathbf{F}_1\mathbf{V}_1$ is a lower triangular matrix.

Let $\mathbf{V}_2 \in \mathbb{K}(s)[t]^{m\times(m-r)}$ be the matrix formed by the last $m-r$ column vectors of $\mathbf{V}_1$. It follows from Equation (1) that

$$\mathbf{F}_1\mathbf{V}_2 = \mathbf{0}.$$

Let $g_i \in \mathbb{K}[s]$ be the least common multiple of all the denominators in the i-th column of $\mathbf{V}_1$, where $i = 1,\ldots,m$. Let $\mathbf{W} = \mathbf{V}_2 \cdot \mathrm{diag}\{g_{r+1},\ldots,g_m\}$. Then $\mathbf{W} \in \mathbb{K}[s,t]^{m\times(m-r)}$, and we have

$$\mathbf{F}_1\mathbf{W} = \mathbf{0}.$$

Next, we prove $d_{m-r}(\mathbf{W}) \in \mathbb{K}[s]$.

Let $\mathbf{U} = \mathbf{V}_1 \cdot \mathrm{diag}\{g_1,\ldots,g_m\}$. Then $\mathbf{U} \in \mathbb{K}[s,t]^{m\times m}$. By the fact that $\mathbf{V}_1$ is a unimodular matrix over $\mathbb{K}(s)[t]$, we have $\det(\mathbf{V}_1) \in \mathbb{K}(s)$. This implies that

$$\det(\mathbf{U}) \in \mathbb{K}[s].$$

Let $a_1,\ldots,a_\beta \in \mathbb{K}[s,t]$ be all the $(m-r)\times(m-r)$ minors of $\mathbf{W}$. According to the Laplace expansion formula, it follows from $\mathbf{W}$ being the submatrix of $\mathbf{U}$ that

$$\det(\mathbf{U}) = a_1b_1 + \cdots + a_\beta b_\beta,$$

where $b_1,\ldots,b_\beta \in \mathbb{K}[s,t]$. As $d_{m-r}(\mathbf{W}) = \gcd(a_1,\ldots,a_\beta)$, we get

$$d_{m-r}(\mathbf{W}) \mid \det(\mathbf{U}) \text{ and } d_{m-r}(\mathbf{W}) \in \mathbb{K}[s].$$

Based on the primitive factorization theorem, there exist $\mathbf{W}_1 \in \mathbb{K}[s,t]^{m\times(m-r)}$ and $\mathbf{G} \in \mathbb{K}[s,t]^{(m-r)\times(m-r)}$ such that

$$\mathbf{W} = \mathbf{W}_1\mathbf{G} \text{ and } \det(\mathbf{G}) = d_{m-r}(\mathbf{W}).$$

Based on the Binet-Cauchy formula, we have $d_{m-r}(\mathbf{W}_1) = 1$. This implies that $\mathbf{W}_1$ is an MRP matrix. In addition, it follows from $\mathbf{F}_1\mathbf{W} = \mathbf{0}$ and $\det(\mathbf{G}) \neq 0$ that

$$\mathbf{F}_1\mathbf{W}_1 = \mathbf{0}.$$

Based on Proposition 4, the columns of $\mathbf{W}_1$ is a free basis of $\mathrm{Syz}(\mathbf{F}_1)$.

Termination. The key techniques of Step 2 and the primitive factorization algorithm are the Gaussian elimination method, with the only distinction lying in the different univariate polynomial rings where Euclidean division is performed. Therefore, Algorithm 1 terminates within a finite number of steps. □

Now, we use an example to illustrate the effectiveness of Algorithm 1.

Example 1 (see Example 2, [4])**.** *The Steiner surface $P(s,t) = (a,b,c,d)$ defined by*

$$\begin{cases} a(s,t) = 2st, \\ b(s,t) = 2t, \\ c(s,t) = 2s, \\ d(s,t) = s^2 + t^2 + 1. \end{cases}$$

Since the GCD of a,b,c,d in $K(s)[t]$ is $2s$, there exits a unimodular matrix $\mathbf{V}_1 \in K(s)[t]^{4\times 4}$ such that

$$P\mathbf{V}_1 = (2st, 2t, 2s, s^2+t^2+1) \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -\frac{t}{s} & -t & -\frac{s^2+t^2+1}{2s} \\ 0 & 0 & 0 & 1 \end{pmatrix} = (2s, 0, 0, 0),$$

where $\det(\mathbf{V}_1) = -1 \in \mathbb{K}(s)$. Let $\mathbf{V}_2 \in \mathbb{K}(s)[t]^{4\times 3}$ be the submatrix composed of the last 3 columns of $\mathbf{V}_1$. Then the least common multiples of denominators in each column of $\mathbf{V}_2$ are s, 1 and $2s$. Let $\mathbf{W} = \mathbf{V}_2 \cdot \operatorname{diag}\{s, 1, 2s\}$. Then $d_3(\mathbf{W}) = s$. We use the primitive factorization algorithm to factorize $\mathbf{W}$ and obtain an MRP factorization:

$$\mathbf{W} = \mathbf{W}_1\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -st & t^2+1 \\ -t & -s^2-1 & st \\ 0 & 2s & -2t \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ t & 0 & t^2+1 \\ s & 0 & st \end{pmatrix},$$

where $\mathbf{W}_1$ is MRP. Therefore, the columns of $\mathbf{W}_1$ is a free basis of $\operatorname{Syz}(P)$, i.e., a μ-basis of the Steiner surface $P(s,t)$.

Remark 2. *Compared with Example 2 in [4], the calculation process in Example 1 is more concise and clear.*

We have implemented Algorithm 1 and the algorithm (hereinafter referred to as the DCS Algorithm) proposed in [4] in the computer algebra system Maple. In addition, Fabiańska and Quadrat [28] designed another algorithm (hereinafter referred to as the FQ Algorithm) for computing free bases by utilizing the famous Quillen-Suslin theorem and implemented it in Maple. We randomly generated some bivariate polynomial matrices to verify the computational efficiency of the above three programs, and the results are presented in the following Table 1.

We explain the setup for our experiments so that the timings reported here can be reproduced independently.

1. The timings in Table 1 were obtained using a personal laptop equipped with an Intel Core i5 1.4 GHz processor, Intel Iris Plus 645 1536 MB Graphics, and 8GB of memory.
2. To get reliable timings, particularly when the computing time is minimal compared to the clock resolution, we executed each program multiple times on the same input and calculated the average of the computation times. Additionally, if the

Table 1 Comparisons of the computational costs of three algorithms (sec)

Example	Algorithm 1	DCS Algorithm	FQ Algorithm
P_1	0.279	0.303	1.226
P_2	0.203	0.21	>3600
$\mathbf{F}_1$	0.849	2.577	>3600
$\mathbf{F}_2$	56.853	102.484	>3600
$\mathbf{F}_3$	210.685	>3600	>3600
$\mathbf{F}_4$	20.307	109.149	>3600
$\mathbf{F}_5$	14.437	52.564	>3600
$\mathbf{F}_6$	193.978	649.538	>3600
$\mathbf{F}_7$	305.639	357.033	>3600
$\mathbf{F}_8$	9.94	54.029	>3600

computation time for executing a program on a given input exceeded one hour, we terminated the execution.

3. P_1 and P_2 in Table 1 are two rational parametric surfaces, which are respectively from Examples 3 and 4 in [4]. $\mathbf{F}_1, \ldots, \mathbf{F}_8$ are bivariate polynomial matrices generated randomly according to the following rules. First, the sizes of these matrices range from 2×4 to 6×10; second, the total degrees of all polynomials in these matrices do not exceed 5; third, the number of monomials in each polynomial is controlled within 8. Please refer to Appendix A for specific details.
4. The codes and examples are available on the web:
   http://www.mmrc.iss.ac.cn/~dwang/software.html.

As is evident from Table 1, our algorithm performs better than the DCS Algorithm and the FQ Algorithm. Through a rigorous comparative analysis of the computational procedures of the three algorithms, it is demonstrated that the DCS Algorithm necessitates an additional invocation of the primitive factorization algorithm compared to Algorithm 1, whereas the FQ Algorithm involves more intricate free resolution computations and the construction of unimodular matrices. Consequently, both algorithms exhibit inferior computational efficiency relative to Algorithm 1.

4 Factorization Theory for Rank-deficient Bivariate Polynomial Matrices

In 1982, Guiver and Bose [5] employed the primitive factorization algorithm to completely resolve the general factorization problem for full-rank bivariate polynomial matrices, and derived the following result.

Lemma 6 (Theorem 3 in [5]). *Let $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$ be of full row rank, and $f \in \mathbb{K}[s,t]$ be an arbitrary divisor of $d_l(\mathbf{F})$. Then there are $\mathbf{G}_0 \in \mathbb{K}[s,t]^{l\times l}$ and $\mathbf{F}_0 \in \mathbb{K}[s,t]^{l\times m}$ such that*

$$\mathbf{F} = \mathbf{G}_0\mathbf{F}_0 \text{ and } \det(\mathbf{G}_0) = f.$$

To generalize Lemma 6 to the rank-deficient case, we first introduce the MLP factorization theory for rank-deficient matrices developed in [23].

Lemma 7 (Theorem 3.1 in [23])**.** *Let $\mathcal{R}$ be a polynomial ring, $\mathbf{F} \in \mathcal{R}^{l\times m}$ with rank r, and $\mathbf{F}_1 \in \mathcal{R}^{r\times m}$ be an arbitrary full row rank submatrix of $\mathbf{F}$. Then $\mathbf{F}$ has an MLP factorization if and only if $\rho(\mathbf{F}_1) : d_r(\mathbf{F}_1)$ is a free module of rank r.*

In Lemma 7, $\rho(\mathbf{F}_1)$ denotes the submodule in $\mathcal{R}^{1\times m}$ generated by the rows of $\mathbf{F}_1$, and

$$\rho(\mathbf{F}_1) : d_r(\mathbf{F}_1) = \left\{\vec{u} \in \mathcal{R}^{1\times m} \mid d_r(\mathbf{F}_1)\cdot\vec{u} \in \rho(\mathbf{F}_1)\right\}$$

is called the quotient module of $\rho(\mathbf{F}_1)$ w.r.t. $d_r(\mathbf{F}_1)$. Let

$$\mathbf{H} = \left(\mathbf{F}_1^{\mathrm{T}}, -d_r(\mathbf{F}_1)\cdot\mathbf{I}_m\right) \in \mathcal{R}^{m\times(r+m)},$$

where $\mathbf{I}_m$ is the $m\times m$ identity matrix. Wang and Kwong [16] established a one-to-one correspondence between $\rho(\mathbf{F}_1) : d_r(\mathbf{F}_1)$ and $\mathrm{Syz}(\mathbf{H})$. That is, $\rho(\mathbf{F}_1) : d_r(\mathbf{F}_1)$ is free if and only if $\mathrm{Syz}(\mathbf{H})$ is free. If $\mathcal{R} = \mathbb{K}[s,t]$, then we have the following result.

Theorem 8 (MLP factorization)**.** *Let $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$ with rank r. Then $\mathbf{F}$ admits an MLP factorization.*

Proof. Let $\mathbf{F}_1 \in \mathbb{K}[s,t]^{r\times m}$ be an arbitrary full row rank submatrix of $\mathbf{F}$, and set $\mathbf{H} = (\mathbf{F}_1^{\mathrm{T}}, -d_r(\mathbf{F}_1)\cdot\mathbf{I}_m) \in \mathbb{K}[s,t]^{m\times(r+m)}$. It is obvious that $\mathrm{rank}(\mathbf{H}) = m$. According to Corollary 2, $\mathrm{Syz}(\mathbf{H})$ is a free module of rank r. That is, $\rho(\mathbf{F}_1) : d_r(\mathbf{F}_1)$ is a free module of rank r. It follows from Lemma 7 that $\mathbf{F}$ admits an MLP factorization. □

Now, we present the following general factorization theory for the rank-deficient case.

Theorem 9 (general factorization theory)**.** *Let $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$ with rank r, and $f \in \mathbb{K}[s,t]$ be an arbitrary divisor of $d_r(\mathbf{F})$. Then there are $\mathbf{G}_0 \in \mathbb{K}[s,t]^{l\times r}$ and $\mathbf{F}_0 \in \mathbb{K}[s,t]^{r\times m}$ such that $\mathbf{F} = \mathbf{G}_0\mathbf{F}_0$ and $d_r(\mathbf{G}_0) = f$.*

Proof. Based on Theorem 8, $\mathbf{F}$ admits an MLP factorization:

$$\mathbf{F} = \mathbf{G}_{01}\mathbf{F}_{01},$$

where $\mathbf{G}_{01} \in \mathbb{K}[s,t]^{l\times r}$ with $d_r(\mathbf{G}_{01}) = d_r(\mathbf{F})$, and $\mathbf{F}_{01} \in \mathbb{K}[s,t]^{r\times m}$ is an MLP matrix. Since $\mathbf{G}_{01}$ is a full column rank matrix, by Lemma 6 there exist $\mathbf{G}_0 \in \mathbb{K}[s,t]^{l\times r}$ and $\mathbf{G}_1 \in \mathbb{K}[s,t]^{r\times r}$ such that

$$\mathbf{G}_{01} = \mathbf{G}_0\mathbf{G}_1 \text{ and } \det(\mathbf{G}_1) = \frac{d_r(\mathbf{F})}{f}.$$

It follows from the Binet-Cauchy formula that $d_r(\mathbf{G}_0) = f$. Let $\mathbf{F}_0 = \mathbf{G}_1\mathbf{F}_{01}$. Then $\mathbf{F} = \mathbf{G}_0\mathbf{F}_0$ is a general matrix factorization of $\mathbf{F}$ w.r.t. f. □

Building upon Theorem 9, we propose an algorithm for computing the general matrix factorization of $\mathbf{F}$ w.r.t. f.

Theorem 10. *Algorithm 2 outputs as specified within a finite number of steps.*

Algorithm 2 computing general matrix factorizations of bivariate polynomial matrices

Require: a matrix $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$ with rank r, and an arbitrary divisor f of $d_r(\mathbf{F})$.
Ensure: a general matrix factorization of $\mathbf{F}$ w.r.t. f.

1: let $\mathbf{F}_1 \in \mathbb{K}[s,t]^{r\times m}$ be an arbitrary full row rank submatrix of $\mathbf{F}$;
2: let $\mathbf{H} = (\mathbf{F}_1^{\mathrm{T}}, -d_r(\mathbf{F}_1)\cdot \mathbf{I}_m)$, and use Algorithm 1 to compute an MRP matrix $\mathbf{M} \in \mathbb{K}[s,t]^{(r+m)\times r}$ such that $\mathbf{HM} = \mathbf{0}$;
3: let $\mathbf{F}_{01}$ be the transpose of the matrix formed by the last m rows of $\mathbf{M}$, and compute the right inverse $\mathbf{F}_{01}^{-1}$ of $\mathbf{F}_{01}$;
4: let $\mathbf{G}_{01} = \mathbf{F}\mathbf{F}_{01}^{-1}$, and use Lemma 6 to compute $\mathbf{G}_0 \in \mathbb{K}[s,t]^{l\times r}$ and $\mathbf{G}_1 \in \mathbb{K}[s,t]^{r\times r}$ such that $\mathbf{G}_{01} = \mathbf{G}_0\mathbf{G}_1$ and $\det(\mathbf{G}_1) = \frac{d_r(\mathbf{F})}{f}$;
5: **return** $\mathbf{G}_0$ and $\mathbf{G}_1\mathbf{F}_{01}$.

Proof. Correctness. It follows from Proposition 4 that the columns of $\mathbf{M}$ constitute a free basis of $\mathrm{Syz}(\mathbf{H})$. It is easy to see that

$$\mathbf{H}\cdot \begin{pmatrix} d_r(\mathbf{F}_1)\cdot \mathbf{I}_r \\ \mathbf{F}_1^{\mathrm{T}} \end{pmatrix} = \mathbf{0}.$$

This implies that each column of the matrix $\begin{pmatrix} d_r(\mathbf{F}_1)\cdot \mathbf{I}_r \\ \mathbf{F}_1^{\mathrm{T}} \end{pmatrix}$ belongs to $\mathrm{Syz}(\mathbf{H})$. It follows that

$$\rho\left((d_r(\mathbf{F}_1)\cdot \mathbf{I}_r, \mathbf{F}_1)\right) \subseteq \rho(\mathbf{M}^{\mathrm{T}}). \tag{2}$$

Let $\mathbf{M}^{\mathrm{T}} = (\mathbf{F}_{00}, \mathbf{F}_{01})$, where $\mathbf{F}_{00} \in \mathbb{K}[s,t]^{r\times r}$ and $\mathbf{F}_{01} \in \mathbb{K}[s,t]^{r\times m}$. According to Equation (2), there exists a bivariate polynomial matrix $\mathbf{G}' \in \mathbb{K}[s,t]^{r\times r}$ such that

$$(d_r(\mathbf{F}_1)\cdot \mathbf{I}_r, \mathbf{F}_1) = \mathbf{G}'\cdot(\mathbf{F}_{00}, \mathbf{F}_{01}). \tag{3}$$

It is easy to verify that $d_r((d_r(\mathbf{F}_1)\cdot\mathbf{I}_r, \mathbf{F}_1)) = d_r(\mathbf{F}_1)$. By the fact that $\mathbf{M}^{\mathrm{T}} = (\mathbf{F}_{00}, \mathbf{F}_{01})$ is an MLP matrix, we have $\det(\mathbf{G}') = d_r(\mathbf{F}_1)$ using the Binet-Cauchy formula. Therefore, we have an MLP factorization of $\mathbf{F}_1$:

$$\mathbf{F}_1 = \mathbf{G}'\mathbf{F}_{01}, \tag{4}$$

where $\mathbf{F}_{01}$ is an MLP matrix. According to the proof of Theorem 3.1 in [23], we get

$$\rho(\mathbf{F}) \subseteq \rho(\mathbf{F}_{01}). \tag{5}$$

It follows from Equation (5) that $\mathbf{F}$ admits an MLP factorization:

$$\mathbf{F} = \mathbf{G}_{01}\mathbf{F}_{01}, \tag{6}$$

where $\mathbf{G}_{01} \in \mathbb{K}[s,t]^{l\times r}$ and $d_r(\mathbf{G}_{01}) = d_r(\mathbf{F})$. As $\mathbf{F}_{01}$ is a full row rank matrix, we compute the generalized right inverse $\mathbf{F}_{01}^{-1} \in \mathbb{K}(s,t)^{m\times r}$ of $\mathbf{F}_{01}$ over $\mathbb{K}(s,t)$ such that

$$\mathbf{F}_{01}\mathbf{F}_{01}^{-1} = \mathbf{I}_r.$$

Then, $\mathbf{G}_{01} = \mathbf{F}\mathbf{F}_{01}^{-1}$. Based on Lemma 6, there exist two polynomial matrices $\mathbf{G}_0 \in \mathbb{K}[s,t]^{l\times r}$ and $\mathbf{G}_1 \in \mathbb{K}[s,t]^{r\times r}$ such that

$$\mathbf{G}_{01} = \mathbf{G}_0\mathbf{G}_1 \text{ and } \det(\mathbf{G}_1) = \frac{d_r(\mathbf{G}_{01})}{f} = \frac{d_r(\mathbf{F})}{f}.$$

By the Binet-Cauchy formula, we have $d_r(\mathbf{G}_0) = f$. Let $\mathbf{F}_0 = \mathbf{G}_1\mathbf{F}_{01}$. Combining Equation (6), we obtain a general matrix factorization of $\mathbf{F}$ w.r.t. f:

$$\mathbf{F} = \mathbf{G}_0\mathbf{F}_0 \text{ and } d_r(\mathbf{G}_0) = f.$$

Termination. The termination of Algorithm 2 rely on Algorithm 1 and Lemma 6. Therefore, Algorithm 1 terminates within a finite number of steps. □

5 Concluding Remarks

This paper first investigates the computational problem of free bases for syzygy modules of bivariate polynomial matrices. The key idea is to treat $\mathbf{F} \in \mathbb{K}[s,t]^{l\times m}$ as a matrix over $\mathbb{K}(s)[t]$. By performing elementary transformations over $\mathbb{K}(s)[t]$, we can obtain a unimodular matrix, a subset of whose columns lies in $\mathrm{Syz}(\mathbf{F})$. By clearing denominators to get a bivariate polynomial matrix over $\mathbb{K}[s,t]$, it is ensured that the GCD of its maximal minors is a univariate polynomial in $\mathbb{K}[s]$. This critical property enables the direct application of the primitive factorization algorithm to compute a free basis for the syzygy module of $\mathbf{F}$ over $\mathbb{K}[s,t]$. We propose an improved algorithm (Algorithm 1), and experimental results demonstrate that Algorithm 1 outperforms two existing methods in terms of computational efficiency. In addition, we study general matrix factorizations under rank-deficient conditions, develop a general matrix factorization theory (Theorem 9) for this case, and present an algorithm (Algorithm 2) by leveraging Algorithm 1. Consequently, we have completely solved the general matrix factorization problem for rank-deficient bivariate polynomial matrices. We anticipate that the theory and algorithms proposed in this paper can promote the development of related fields such as multidimensional systems and computer aided geometric design.

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Appendix A Examples in Table 1

Examples listed in the first column of Table 1 are as follows. Given the large size of some matrices, for the sake of convenience, we represent them in their transposed forms.

(1) Given a bi-quadratic surface $P_1(s,t) := (a,b,c,d)$ defined by

$$\begin{cases} a(s,t) = t^2 + st + 2s^2 - 2s^2t, \\ b(s,t) = t^2 + 2st + st^2 + 2s^2 - s^2t + 2s^2t^2, \\ c(s,t) = -t^2 + st + 2st^2 + 2s^2 - s^2t - 2s^2t^2, \\ d(s,t) = 2st - 2st^2 - 2s^2t - s^2t^2. \end{cases}$$

(2) Given a rational parametric surface $P_2(s,t) := (a,b,c,d)$ defined by

$$\begin{cases} a(s,t) = -3s^2t^2 + 5s^2t - 5t^2 - 4st + 5, \\ b(s,t) = -3s^2t^2 + 3s^2t + s^2 + st^2 - s - 2t^2 - 5st + 1, \\ c(s,t) = -5s^2t^2 + 6s^2t + 2st - t^2 - t - 5, \\ d(s,t) = -4s^2t^2 + 3s^2t - st + 6t^2 - t + 1. \end{cases}$$

(3) $\mathbf{F}_1 \in \mathbb{Q}[s,t]^{2\times 4}$ and the transposed matrix of $\mathbf{F}_1$ is

$$\mathbf{F}_1^{\mathrm{T}} = \begin{pmatrix} -3s^3 + 3s^2t + 2 & 3s^2 - 3t^2 - s \\ s^2 + 3st - s & st^2 - 2t^3 - 3s - 3t \\ -2s^2t - t^3 - 2s^2 + 3t^2 & 3s^2 - 2t^2 - t - 3 \\ s^3 + 2s^2 - 2st - t & 2t^3 - 2t + 3 \end{pmatrix}.$$

(4) $\mathbf{F}_2 \in \mathbb{Q}[s,t]^{2\times 4}$ and the transposed matrix of $\mathbf{F}_2$ is

$$\mathbf{F}_2^{\mathrm{T}} = \begin{pmatrix} -t^5-s^4+6s^2t-6st^2-4st+5t^2-2s+5 & -2st^4-2t^5+4s^4+st^3+6s^3-s^2+2t^2+2 \\ -3s^5-5s^4t+5st^4+2st^2+5s^2+5 & -4s^4t-3s^2t^3+4s^2t^2-4s^3+4st^2-2t^2-3t+1 \\ -2s^2t^3+2t^5-2t^4-s^3-5s^2t-5t^3-1 & -s^4+6s^2t^2-5st^3-5s^2t-2st^2+5t^3+t^2-2 \\ -2st^4+5t^5-3s^4+3st^3+2s^3+st-6t^2-3 & 4s^4t+4st^4-5s^3t+t^4-4s^2t-5st^2-2t^3 \end{pmatrix}.$$

(5) $\mathbf{F}_3 \in \mathbb{Q}[s,t]^{3\times 5}$ and the transposed matrix of $\mathbf{F}_3$ is

$$\mathbf{F}_3^{\mathrm{T}} = \begin{pmatrix} -s^4+3st^2-t^3-s^2+4st & 2s^4+4s^2t^2+1 & -2s^4+2st^3+5t^4-5st^2 \\ -2s^2t^2+st^2-5t^3-3s^2+5s & 4s^4+2s^2t+4s^2-5t & s^4-5s^2t-4t^3 \\ -5t^4+5st^2+3t^3+st & s^3-2s^2t+st^2+t^3-3s & -2s^2t^2-3t^4-2t^3-2st \\ 4s^3t+5t^4+5st-3s-3 & -5s^4+5s^2t^2+t^4-2st & -4s^3-2st-3t^2+s+2 \\ 2s^4-4s^3t-3st^2+4s-t & -4t^4-4t^3+s^2+2st+4 & -4s^4-3s^2t^2-4s^2t-4s^2+3t \end{pmatrix}.$$

(6) $\mathbf{F}_4 \in \mathbb{Q}[s,t]^{3\times 5}$ and the transposed matrix of $\mathbf{F}_4$ is

$$\mathbf{F}_4^{\mathrm{T}} = \begin{pmatrix} -4st^2+5t^3+3s-5t-4 & 5s^3-3s^2+5st-5s & -3st^2-2s^2-st+3t^2+t \\ -3s^3-5s^2t-2st^2-s-4 & 2s^3-3st^2-5st+s+3 & -4st^2+5t^3-3t^2+s \\ -2st^2+2s^2-t^2-2 & -3s^2t+2t^3-2t^2-2s+5t & -5s^3+4s^2t+4s^2+4 \\ 3s^3-4s^2t-3st^2-3s^2-t & 2s^3+3t^3-s^2-5t+3 & -st^2-3t^2+2s-4 \\ -s^3-2st^2-5s^2+t^2 & -5t^3+3s^2-5t^2-3t-3 & -5s^3-s^2t+2st-2 \end{pmatrix}.$$

(7) $\mathbf{F}_5 \in \mathbb{Q}[s,t]^{3\times 8}$ and the transposed matrix of $\mathbf{F}_5$ is

$$\mathbf{F}_5^{\mathrm{T}} = \begin{pmatrix} -st-s+2t & -3t^2+3s-1 & s^2-2t^2-t \\ 3s^2+3st-3t & s^2+st-3t^2 & 2s^2+3st-3s \\ s^2-2t-3 & 2st+t^2+2s & -3t^2-2 \\ -s^2+2t^2 & -s^2+2t^2-2 & 1+3s+3t \\ 2+3s & 2+2s-t & -2st+3t^2+t \\ 3st-t+2 & st+2t^2+3t & -3st-2t+3 \\ -3s^2-s+1 & -3t^2-3t & -t^2-2t \\ -3t^2-2s-3t & -2s^2-2st-3 & st-t \end{pmatrix}.$$

(8) $\mathbf{F}_6 \in \mathbb{Q}[s,t]^{4\times 6}$ and the transposed matrix of $\mathbf{F}_6$ is

$$\mathbf{F}_6^{\mathrm{T}} = \begin{pmatrix} -3s^3+4st^2+4s^2-4st & 3s^2t+4st^2-s^2+2st & -4s^2t-st^2+2t^3-2t & -3t^2+s+3t \\ -4s^3-2s^2t-2s^2+t^2 & s^2-2st+4t & -s^3-3s^2t+2st+s & 3t^3-s^2-3t^2+2t \\ -2s^2t+4t & -2st^2+2s^2-t^2-2 & -4s^2t+st+s & s^2t-3t^2-4s \\ -3s^2+st+t^2-2s & 2t^3+4t+2 & -4t^3-s^2-t^2-4s & -3s^2t-2s^2+3st+s \\ -s^2t-3t^3-2t^2+2 & -s^2+st-t^2+3s & 3s^3+2t^2-3t+1 & t^3+3t \\ 3s^2t+s^2-2s+3 & -2s^2t+4t^3-4s^2-1 & -4s^2t+st^2+t^3-4st & 2s^3-3s^2t-st^2-4st \end{pmatrix}.$$

(9) $\mathbf{F}_7 \in \mathbb{Q}[s,t]^{4\times 8}$ and the transposed matrix of $\mathbf{F}_7$ is

$$\mathbf{F}_7^{\mathrm{T}} = \begin{pmatrix} -2t^2+t+2 & -3t^2+3t+2 & 3st+3s-2t & st+2s-1 \\ 2st+3t^2 & 3st & st+3s-3 & -s^2-3t^2+2t \\ 3s^2-3st+1 & s^2-t^2-2t & 2-2t & 2s^2-t^2-3s \\ 3s^2+3t^2+2t & st+t^2-s & -s^2-3st+t^2 & -2s+t \\ -3st+s+t & 2t^2-2s & -s^2+3st+t & -2s^2+3s \\ -st+3t^2 & s^2-3st-3 & 3s^2-3st-3t^2 & 3t^2+2t \\ -st-3s & 3st+s+t & 2s^2-3s-3 & -s^2+2 \\ s-t & -s-3t & -s^2-3st+2 & -3-2s+3t \end{pmatrix}.$$

(10) $\mathbf{F}_8 \in \mathbb{Q}[s,t]^{6\times 10}$ and the transposed matrix of $\mathbf{F}_8$ is

$$\mathbf{F}_8^{\mathrm{T}} = \begin{pmatrix} 3s+t+2 & -2s-2t-3 & -s+3t+3 & t+1 & s+t-3 & -s-1 \\ 2t+1 & 2s+2t-3 & -s-3 & s-2t & 2s & -s+3t-2 \\ 2s-3 & 2s+1 & -s+3t-1 & -2s+2t+1 & 3s+t+2 & 3s+3t+1 \\ 3s+t+3 & -s+t+2 & -3s-t+2 & 3s-3t & -2s-t-1 & -2s+2t \\ 3s-t+1 & -3s-2 & 3t+1 & s+2t+2 & -s-2 & -3s+3t+3 \\ -3s-1 & s-2t-3 & 2s-t+1 & -s+t+1 & -s-t-1 & 2s-2t-2 \\ -s-1 & 3t-3 & 3s+3t-2 & 3s-2t+2 & 2s-t+1 & -3s+2 \\ -s+t-2 & -2s+3t+3 & -2s+1 & -3s-t+1 & 2s-t-3 & -3s+3t-2 \\ -s-2t-1 & s+3t-3 & 2s+3t-1 & s+3t-2 & s+2 & 2s-t+2 \\ -2t+1 & -2s & t-3 & 2s-2 & 2s+3 & 3s-2t \end{pmatrix}.$$