Complete Numerical Isolation of Real Roots in Zero-dimensional Triangular Systems

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Abstract

We present a complete numerical algorithm of isolating all the real zeros of a zero-dimensional triangular polynomial system $F_n \subseteq \mathbb{Z}[x_1, \ldots, x_n]$. Our system F_n is general, with no further assumptions. In particular, our algorithm successfully treats multiple zeros directly in such systems. A key idea is to introduce evaluation bounds and sleeve bounds. We also present a much more efficient algorithm for zero-dimensional triangular systems without multiple roots. We implemented our algorithms and promising experimental results are shown.

Key words: Triangular system, real zero isolation, sleeve bound, evaluation bound

1. Introduction

Many problems in the computational sciences and engineering can be reduced to the solving of polynomial equations. There are two basic approaches to solving such polynomial systems – numerically or algebraically. Usually, the numerical methods have no global guarantees of correctness. Algebraic methods for solving polynomial systems include Gröbner bases, characteristic sets, CAD, and resultants (1; 2; 3; 5; 11; 13; 14; 18). One general idea in polynomial equation solving is to reduce the original system into a triangular system. Zero-dimensional polynomial systems are among the most important cases to solve. This paper considers zero-dimensional triangular systems only.

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A zero-dimensional triangular system has the form $F_n = \{f_1, \ldots, f_n\}$, where each $f_i \in \mathbb{Z}[x_1, \ldots, x_i]$ $(i = 1, \ldots, n)$ and x_i is a variable that occurs in f_i. We are interested in real zeros $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ of F_n. The standard idea here is to first solve for $f_1(x_1) = 0$, and for each solution $x_1 = \xi_1$ of f_1, we find the solutions $x_2 = \xi_2$ of $f_2(\xi_1, x_2) = 0$, etc. The problem is reduced to solving univariate polynomials of the form $f_i(\xi_1, \ldots, \xi_{i-1}, x_i) = 0$. Such polynomials have algebraic number coefficients. We could isolate roots of such polynomials by using standard root isolation algorithms such as the Sturm sequence method, but using algebraic number arithmetic. But even for small n $(n = 2, 3)$, such algorithms are quite slow. The numerical approach is to replace the ξ_i's by approximations, and thus reduce the problem to isolating roots of numerical polynomials. The challenge is how to guarantee completeness of such numerical algorithms.

We will provide a numerical algorithm that solves such triangular systems in the following precise sense: given an n-dimensional box $R = J_1 \times \cdots \times J_n \subseteq \mathbb{R}^n$ where J_i are intervals, and any precision $\epsilon > 0$, it will isolate the real zeros of F_n in R to precision ϵ. Our algorithm is ***complete*** in the sense that there are no additional requirements on F_n; previous algorithms are incomplete in that they fail for certain F_n's.

Our solution places no restriction on F_n. The reason why we consider general zero-dimensional triangular systems is that the triangular systems derived in cylinder algebraic decomposition or topology determination (6) are generally with multiple roots and even non regular (for definition see (3)).

Many algorithms that seek to provide "exact numerical" solution assume computation over the rational numbers $\mathbb{Q}$. But this is much less efficient than using dyadic numbers: let $\mathbb{D} := \mathbb{Z}[\frac{1}{2}] = \{m2^n : m, n \in \mathbb{Z}\}$ denote the set of dyadic numbers (or bigfloats)(22). Most current fast algorithms for bigfloats can be derived from Brent's work (4). In the following, we use the symbol $\mathbb{F}$ to denote either $\mathbb{D}$ or $\mathbb{Q}$. We use intervals to isolate real numbers: let $\square\mathbb{F}$ denote the set of intervals of the form $[a, b]$ where $a \leq b \in \mathbb{F}$.

Given a polynomial $f \in \mathbb{R}[x]$ and an interval $I = [a, b] \in \square\mathbb{F}$, we construct two polynomials $f^u, f^d \in \mathbb{F}[x]$, called ***sleeve functions***, such that

$$f^u(x) > f(x) > f^d(x), \forall x \in I.$$

We call (I, f^u, f^d) a ***sleeve*** of f over I. Let upper bounds on $SB_I(f^u, f^d)$ be called ***sleeve bounds***. Note that the coefficients of $f^u f^d$ are in $\mathbb{F}$, but f have real coefficients which can be arbitrarily approximated. Based on the sleeve of f, we describe two algorithms, one for general zero-dimensional triangular system and the other for such systems with only simple roots.

The key idea for general triangular systems is the introduction of ***evaluation bounds***. For a polynomial $f \in \mathbb{R}[x]$ and a subset $I \subseteq \mathbb{R}$, let

$$EB_I(f) := \min\{|f(z)| : z \in \mathtt{Zero}_I(f') \cup \{a, b\} \setminus \mathtt{Zero}_I(f)\}. \quad (1)$$

Lower bounds on $EB_I(f)$ are called ***evaluation bounds***. If the following ***sleeve-evaluation inequality***

$$SB_I(f^u, f^d) < EB_I(f) \quad (2)$$

holds, we show that the isolating intervals of $f^u f^d$ can be used to define isolating intervals of f. The algorithm provided in this paper proceeds by computing a sleeve composed of dyadic polynomials, isolating the roots of this sleeve using a classical algorithm, and recover actual information about the roots of the system from the roots of the sleeve. The use of evaluation bounds appears to be new. It is the ability to compute lower estimates on $EB_I(f)$ that allows us to detect zeros of even multiplicities.

For general zero dimensional triangular systems without multiple roots, we introduce a much more efficient method without computing the evaluation bound. The basic idea is that when f is square free, $\frac{\partial f}{\partial x}$ has no root in a neighborhood of each real root of f. Based on this, we give a criterion to use the roots of the sleeve function to isolate the roots of the triangular system. Experiments show that the algorithm can be used to isolate the roots for fairly large triangular systems efficiently.

As a consequence of the above analysis, isolating the real roots of f is reduced to real root isolation for the sleeve functions f^d and f^u. Univariate root isolation is a well-developed subject in its own right. In our implementation, we use the method in (15).

The idea of using a sleeve to solve equations was used in (16) and (12). In particular, Lu et al (12) proposed an algorithm to isolate the real roots of triangular systems. Their method could solve many problems in practice, but it is incomplete as it fails in the presence of multiple zeros. Collins et al (8) considered the problem with interval arithmetic methods and Descartes' method using floating point computation. Again, they pointed out that if a real coefficient is implicitly zero, the method will fail. Xia and Yang (19) consider real root isolation of a semi-algebraic set. They ultimately considered the regular and square-free triangular systems. They mentioned that their method will fail in some cases. They later revised their method to work (20) for regular and square-free triangular systems. Eigenwillig et al considered root-isolation for real polynomials with bitstream coefficients (9). Their algorithm requires f to be square free. Our evaluation bound is similar to the curve separation bound in (23). Interesting work on general polynomial systems was done by Hong and Stahl (10).

In Section 2, we describe the basic technique of using sleeves and evaluation bounds. In Section 3, we give methods to compute evaluation bounds, to compute sleeves and sleeve bounds for a triangular system. In Section 4, we present the root isolation algorithm for triangular systems. In Section 5, we present an algorithm for triangular systems without multiple roots. We conclude the paper in Section 6.

2. Root Isolation for Real Univariate Polynomials

We give a framework for isolating the real roots of a univariate polynomial equation with real coefficients.

2.1. Evaluation and Sleeve Bounds

In this section, we fix f, f^u, f^d to be C^1 functions, and $I \in \square\mathbb{F}$. For any real function f, let $\mathtt{Zero}_I(f)$ denote the set of distinct real zeros of f in I. If $I = \mathbb{R}$, then we simply write $\mathtt{Zero}(f)$. If $\#(\mathtt{Zero}_I(f)) = 1$, we call I an ***isolating interval*** of f. Sometimes, we need to count the zeros up to the parity (i.e., evenness or oddness) of their multiplicity. Call a zero $\xi \in \mathtt{Zero}(f)$ an ***even*** (resp., ***odd***) ***zero*** if its multiplicity is even (resp., odd). Define the ***multiset*** $\mathtt{ZERO}_I(f)$ whose underlying set is $\mathtt{Zero}_I(f)$ and where the multiplicity of $\xi \in \mathtt{ZERO}_I(f)$ is 1 (resp., 2) if ξ is an odd (resp., even) zero of f.

To avoid special treatment near the endpoints of an interval (see (7)), we assume

$$f^u(a)f^d(a) > 0, \quad f^u(b)f^d(b) > 0. \tag{3}$$

We say that the sleeve (I, f^u, f^d) is ***faithful*** for f if (3) and (2) are both satisfied.

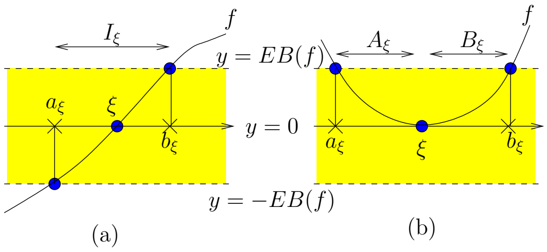

Fig. 1. Neighborhood of ξ: $I_\xi = A_\xi \cup \{\xi\} \cup B_\xi$.

Intuitively, f is nicely behaved if we restrict f to a neighborhood of a zero ξ where $|f| < EB(f)$. This is illustrated in Figure 1.

Given f and I, define the polynomials $\widehat{f}(x) := f(x) - EB_I(f)$, $\overline{f}(x) := f(x) + EB_I(f)$. If $\xi \in \mathtt{Zero}_I(f)$, we define the points a_ξ, b_ξ and the open interval A_ξ, B_ξ, I_ξ (see Figure 1):

$$\begin{aligned} a_\xi &:= \max\{\{a\} \cup \mathtt{Zero}(\widehat{f} \cdot \overline{f}) \cap (-\infty, \xi)\},\ A_\xi := (a_\xi, \xi), \\ b_\xi &:= \min\{\{b\} \cup \mathtt{Zero}(\widehat{f} \cdot \overline{f}) \cap (\xi, +\infty)\},\ \ B_\xi := (\xi, b_\xi),\ \ I_\xi := (a_\xi, b_\xi). \end{aligned} \tag{4}$$

Basic properties of these intervals are captured below.

Lemma 1. Let (I, f^u, f^d) be a faithful sleeve for f. For all $\xi, \zeta \in \mathtt{Zero}_I(f)$, we have:
(i) If $\xi \neq \zeta$ then I_ξ and I_ζ are disjoint.
(ii) $\mathtt{Zero}_I(f^u f^d) \subseteq \bigcup_\xi I_\xi$.
(iii-a) $A_\xi \cap \mathtt{Zero}(f^u)$ is empty iff $A_\xi \cap \mathtt{Zero}(f^d)$ is non-empty.
(iii-b) $B_\xi \cap \mathtt{Zero}(f^u)$ is empty iff $B_\xi \cap \mathtt{Zero}(f^d)$ is non-empty.
(iv) The derivative f' has a constant sign in A_ξ or B_ξ for any $\xi \in \mathtt{Zero}_I(f)$.

Proof. (i) Suppose $\xi < \zeta$ are consecutive zeros of $\mathtt{Zero}_I(f)$. Then either f is positive on (ξ, ζ) or f is negative on (ξ, ζ). Without loss of generality, f is positive on (ξ, ζ). Then the multiset $\mathtt{ZERO}_I(\widehat{f}) = \mathtt{ZERO}(f - EB_I(f))$ has at least two zeros (they may have the same value) in (ξ, ζ). This proves $b_\xi \le a_\zeta$ and so I_ξ and I_ζ are disjoint.
(ii) Let $z \in \mathtt{Zero}_I(f^u f^d)$. Then (2) implies that $|f(z)| < EB_I(f)$. By the definition of evaluation bound, this also means that $f'(z) \neq 0$. Thus there are two cases: either $f(z)f'(z) > 0$ or $f(z)f'(z) < 0$. First, suppose $f(z)f'(z) > 0$. Then there is a unique largest $\xi \in \mathtt{Zero}(f)$ that is less than z, and there is a unique smallest $b_\xi \in \mathtt{Zero}(\widehat{f})$ that is greater than z. This proves that $z \in (\xi, b_\xi)$. Similarly, if $f(z)f'(z) < 0$, we will see that $z \in (a_\xi, \xi)$ for some $\xi \in \mathtt{Zero}_I(f)$.
(iii-a) Either $f(a_\xi) > 0$ or $f(a_\xi) < 0$. If $f(a_\xi) > 0$ then (2) implies $f^d(a_\xi) > 0$. But $f^d(\xi) < 0$, and hence $A_\xi \cap \mathtt{Zero}(f^d)$ is non-empty. Now, since f^u is positive over A_ξ, we conclude that $A_\xi \cap \mathtt{Zero}(f^u)$ is empty. The other case, $f(a_\xi) < 0$ will similarly imply that $A_\xi \cap \mathtt{Zero}(f^d)$ is empty and $A_\xi \cap \mathtt{Zero}(f^u)$ is non-empty.
(iii-b) This is similar to (iii-a).
(iv) It is obvious. Otherwise, we assume there exists an $s \in A_\xi$ such that $f'(s) = 0$. We derive a contradiction from the definitions of a_ξ (see Figure 1), where $A_\xi = (a_\xi, \xi)$. □

If $s, t \in \mathtt{Zero}_I(f^u f^d)$ such that $s < t$ and $(s, t) \cap \mathtt{Zero}_I(f^u f^d)$ is empty, then we call (s, t) a ***sleeve interval*** of (I, f^u, f^d). From Lemma 1(iii), we have

Corollary 2. Each zero of $\mathtt{Zero}_I(f)$ is isolated by some sleeve interval of (I, f^u, f^d).

Lemma 3. Let (I, f^u, f^d) be a faithful sleeve. For all $\xi \in \mathtt{Zero}_I(f)$, the multiset $\mathtt{ZERO}_{B_\xi}(f^u \cdot f^d)$ has odd size. Similarly, the multiset $\mathtt{ZERO}_{A_\xi}(f^u \cdot f^d)$ has odd size. As a consequence, the multiset $\mathtt{ZERO}_{I_\xi}(f^u f^d)$ has even size.

Proof. We just prove the result for the multiset $\mathtt{ZERO}_{B_\xi}(f^u \cdot f^d)$. Without loss of generality, let $f(b_\xi) > 0$ (the case $f(b_\xi) < 0$ is similar). By the sleeve-evaluation inequality, $f^d(b_\xi) > 0$. Note that when $b_\xi = b$, the inequality is also true since (I, f^u, f^d) is faithful. But $f^d(\xi) < 0$. Hence f^d has an odd number of zeros (counting multiplicities) in the interval $B_\xi = (\xi, b_\xi)$. Moreover, $f^u > f$ implies f^u has no zeros in B_ξ. □

It follows from the preceding lemma that for each zero ξ of f, the multiset $\mathtt{ZERO}_{I_\xi}(f^u f^d)$ has even size. Hence the multiset $\mathtt{ZERO}_I(f^u f^d)$ has even size, say $2m$. So we may denote the sorted list of zeros of $\mathtt{ZERO}_I(f^u f^d)$ by

$$(t_0, t_1, \ldots, t_{2m-1}) \tag{5}$$

where $t_0 \le t_1 \le \cdots \le t_{2m-1}$. Intervals of the form $J_i := [t_{2i}, t_{2i+1}]$ where $t_{2i} < t_{2i+1}$ are called ***candidate intervals*** of the sleeve. We immediately obtain:

Corollary 4. Each $\xi \in \mathtt{Zero}_I(f)$ is contained in some candidate interval of a faithful sleeve (I, f^u, f^d).

Which of these candidate intervals actually contain zeros of f? To do this, we classify a candidate interval $[t_{2j}, t_{2j+1}]$ in (5) into two types:

$$\left.\begin{array}{ll}\text{(Odd):} & t_{2j} \in \mathtt{Zero}(f^d) \text{ iff } t_{2j+1} \in \mathtt{Zero}(f^u) \\ \text{(Even):} & t_{2j} \in \mathtt{Zero}(f^d) \text{ iff } t_{2j+1} \in \mathtt{Zero}(f^d)\end{array}\right\} \tag{6}$$

We call a candidate interval J an ***odd*** or ***even candidate interval*** if it satisfies (6)(Odd) or (6)(Even). We now treat the easy case of deciding which candidate intervals are isolating intervals of f:

Lemma 5 (Odd Zero). Let J be a candidate interval. The following are equivalent:
(i) J is an odd candidate interval.
(ii) J contains a unique zero ξ of f. Moreover ξ is an odd zero of f.

Proof. Let $J = [t, t']$.
(i) implies (ii): Without loss of generality, let $f^u(t) = 0$ and $f^d(t') = 0$. Thus, $f(t) < 0$ and $f(t') > 0$. Thus f has an odd zero in J. By Corollary 2, we know that candidate intervals contain at most one distinct zero.
(ii) implies (i): Since ξ is an odd zero, we see that f must be monotone over J. Without loss of generality, assume f is increasing. This implies $f^d(t) < 0$ and hence $f^u(t) = 0$. Similarly, $f^u(t') > 0$ and hence $f^d(t') = 0$. Hence J is an odd candidate. □

Isolating even zeros is more subtle and will be dealt with in the next section.

2.2. Monotonicity Property

We will exploit a special property of (I, f^u, f^d) for f:

$$\frac{\partial f^u}{\partial x} \ge \frac{\partial f}{\partial x} \ge \frac{\partial f^d}{\partial x} \qquad \text{holds in } I \tag{7}$$

We call this the ***monotonicity property***. In this subsection, we assume (7) and the faithfulness of the sleeve. We now strengthen one half of Lemma 3 above.

Lemma 6. For any $\xi \in \mathtt{Zero}_I(f)$, there is a unique zero of odd multiplicity of $f^u \cdot f^d$ in $A_\xi = (a_\xi, \xi)$.

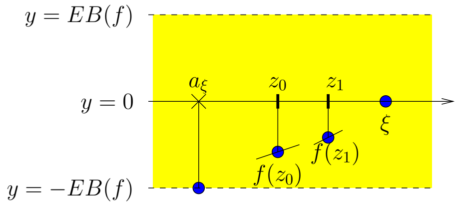

Fig. 2. A_ξ has a unique zero of $f^u \cdot f^d$: CASE of $f^u(z_0) = f^u(z_1) = 0$.

Proof. Alternatively, this lemma says that the multiset $\mathtt{ZERO}_{A_\xi}(f^u f^d)$ has size 1. By way of contradiction, suppose $z_0 \le z_1$ are two zeros of $f^u f^d$ in $A_\xi = (a_\xi, \xi)$. We allow the possibility that $z_0 = z_1$. From Lemma 1(iii), we know that either $z_0, z_1 \in \mathtt{ZERO}(f^u)$ or $z_0, z_1 \in \mathtt{ZERO}(f^d)$. There are two cases:

(A) z_0, z_1 are roots of f^u. See Figure 2. By Rolle's theorem, there exists $z \in [z_0, z_1]$ s.t. $\frac{\partial f^u}{\partial x}(z) = 0$. Therefore, there exist $z^- < z < z^+$ that are arbitrarily close to z s.t.

$$\frac{\partial f^u}{\partial x}(z^-) \cdot \frac{\partial f^u}{\partial x}(z^+) < 0. \tag{8}$$

On the other hand, note that $f(z_j) < f^u(z_j) = 0$ for $j = 0, 1$. Since $f(\xi) = 0$, and $z_j < \xi$, which means that (z_j, ξ) contains a point z with $f'(z) > 0$. But f' has constant sign in A_ξ form Lemma 1 (iv), and so this sign of f' is positive. Then by monotonicity (7),

$$\frac{\partial f^u}{\partial x}(z^-) \ge f'(z^-) > 0, \quad \text{and} \quad \frac{\partial f^u}{\partial x}(z^+) \ge f'(z^+) > 0. \tag{9}$$

Now we see that (8) and (9) are contradictory.

(B) z_0, z_1 are roots of f^d. We similarly derive a contradiction. □

Corollary 7. If t_{2j} is an even zero of $f^u f^d$, then $[t_{2j}, t_{2j+1}]$ contains no zero of f.

If t_{2j} is an even zero we have either $t_{2j} = t_{2j+1}$ or $t_{2j} = t_{2j-1}$. But for the former case, (t_{2j}, t_{2j+1}) clearly has no zeros of f. The next result is a consequence of monotonicity and faithfulness:

Lemma 8. The interval $J_0 = [t_0, t_1]$ is a candidate interval and it isolates a zero of f.

In Lemma 5, we showed that (6)(Odd) holds iff J_j isolates an odd zero of f. The next result shows what condition must be added to (6)(Even) in order to to characterize the isolation of even zeros.

Lemma 9 (Even Zero). Let $J_j = [t_{2j}, t_{2j+1}]$ $(j > 0)$ be an even candidate interval. Then J_j isolates an even zero ξ of f iff
(i) $f^d(t_{2j}) = 0$ and $\frac{\partial f^u}{\partial x}$ has real zero in (t_{2j-1}, t_{2j+1}), or
(ii) $f^u(t_{2j}) = 0$ and $\frac{\partial f^d}{\partial x}$ has real zero in (t_{2j-1}, t_{2j+1}).

Proof. Note that since $j > 0$, then t_{2j-1} is a zero of f^d iff t_{2j} is a zero of f^d. Let t_{2j} be a zero of f^d. So $f^d(t_{2j+1}) = 0$ and $t_{2j+1} \in B_\xi$ for a zero ξ of f. This means $\frac{\partial f}{\partial x}$ is

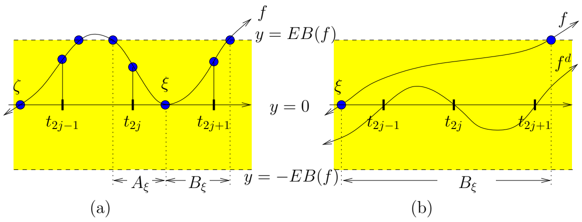

Fig. 3. Detection of even zero when $t_{2j}, t_{2j+1} \in \mathtt{Zero}_I(f^d)$: (a) even zero, (b) no zero

positive in (ξ, t_{2j+1}). There are two cases: (a) $t_{2j} < \xi < t_{2j+1}$ or (b) $\xi < t_{2j} < t_{2j+1}$. If (a), then $t_{2j-1} \in B_\zeta$ for some zero ζ of f and $\zeta \neq \xi$(see Figure 3(a)). By (2), we have $0 < f^u(t_{2j-1}) < EB(f)$, $0 < f^u(t_{2j}) < EB(f)$. Since $t_{2j-1} \in B_\zeta, t_{2j} \in A_\xi$ and $\zeta \neq \xi$, there exists a point $\eta \in (t_{2j-1}, t_{2j})$ such that $f(\eta) \geq EB(f)$. So $f^u(\eta) > EB(f)$. That means there is an extremum point of f^u in (t_{2j-1}, t_{2j}). That is, there exists a zero of $\frac{\partial f^u}{\partial x}$ in $(t_{2j-1}, t_{2j}) \subset (t_{2j-1}, t_{2j+1})$. If (b), then $\frac{\partial f^u}{\partial x}(x) > 0$ for all $x \in (t_{2j-1}, t_{2j+1})$ since $\frac{\partial f}{\partial x}$ has constant sign in B_ξ (see Figure 3(b)). We finish the proof. □

2.3. *Effective Root Isolation of f*

So far, we have been treating the roots t_j of $f^u f^d$ exactly. But in our algorithms, we only have numbers in $\mathbb{F}$. We now want to replace t_j by their isolating intervals $[a_j, b_j]$. As usual, we assume that (I, f^u, f^d) is faithful and satisfies the monotonicity property (7). Let $\mathtt{ZERO}_I(f^u f^d)$ be the sorted list given in (5), and $[a_i, b_i]$ an isolating interval of t_i, where any two distinct intervals $[a_i, b_i]$ and $[a_j, b_j]$ are disjoint. Let

$$SL_{f,I} = ([a_0, b_0], [a_1, b_1], \ldots, [a_{2m-1}, b_{2m-1}]) \tag{10}$$

be the isolating intervals for roots of $f^u f^d$ in $\mathtt{ZERO}_I(f^u f^d)$. Assume that $[a_i, b_i] = [a_j, b_j]$ iff $t_i = t_j$. Note that $t_i = t_j$ implies $|i - j| \leq 1$. Let $K_i := [a_{2i}, b_{2i+1}]$.

By Corollary 7, J_i is not an isolating interval if t_{2i} is an even zero. Hence, we call K_i an ***effective candidate*** iff $t_{2i} < t_{2i+1}$ and t_{2i} is an odd zero. Thus, K_i contains the candidate interval $J_i = [t_{2i}, t_{2i+1}]$. Furthermore, K_i is called an ***effective even candidate*** (resp., ***effective odd candidate***) if J_i is an even (resp., odd) candidate interval (cf. (6)).

Our next theorem characterizes when K_i is an isolating interval of f. This is the "effective version" of Lemma 5 and Lemma 9. But before this theorem, we provide a useful partial criterion:

Lemma 10. Let $K_i = [a_{2i}, b_{2i+1}]$ be an effective even candidate. Then K_i isolates an even zero provided one of the following conditions hold:
(E')^d: $t_{2i} \in \mathtt{Zero}(f^d)$ and $\frac{\partial f^u}{\partial x}$ is negative at a_{2i} or b_{2i},
(E')^u: $t_{2i} \in \mathtt{Zero}(f^u)$ and $\frac{\partial f^d}{\partial x}$ is positive at a_{2i} or b_{2i}.

Proof. Say t_{2i} is a zero of f^d. We have $t_{2i+1} \in B_\xi$ for some $\xi \in \mathtt{Zero}(f)$, and $f' = \frac{\partial f}{\partial x}$ is positive at t_{2i+1}. There are just two cases: either (a) t_{2i} is in A_ξ, or (b) t_{2i} is in B_ξ. If (a) holds, then ξ is an even zero in $[c, t_{2i+1}]$ (where $c = a_{2i}$ or b_{2i}), and our lemma is true. So assume (b) and (E')^d. From (E')^d and the monotonicity (7), we know that f' is negative at c. If $c = b_{2i}$ then we get a contradiction since (b) implies f' is positive over $B_\xi \supseteq [t_{2i}, t_{2i+1}] \supseteq [c, t_{2i+1}]$. If $c = a_{2i}$, the argument is more subtle. We know that $\xi \in [t_{2j}, t_{2j+1}]$ for some $j < i$ and $t_{2j+1} < t_{2i}$ (for t_{2i} is an odd zero). Moreover, f' has constant sign in $B_\xi \supseteq [t_{2j+1}, t_{2i+1}] \supseteq [c, t_{2i+1}]$. Again this yields a contradiction. □

For the even effective candidates, we shall need a ***constant sign property***:

$$\left.\begin{array}{l}\text{Let } t_{2j}, t_{2j+1} (j \geq 1) \text{ all be real zeros of } f^u \text{ or } f^d.\\ \quad\text{If } t_{2j}, t_{2j+1} \in \texttt{Zero}(f^d) \text{ then } \frac{\partial f^u}{\partial x} \text{ is positive in } [a_{2j-1}, b_{2j-1}] \text{ and } [a_{2j+1}, b_{2j+1}].\\ \quad\text{If } t_{2j}, t_{2j+1} \in \texttt{Zero}(f^u) \text{ then } \frac{\partial f^d}{\partial x} \text{ is negative in } [a_{2j-1}, b_{2j-1}] \text{ and } [a_{2j+1}, b_{2j+1}].\end{array}\right\} \quad (11)$$

Note that $t_{2j-1} \in B_\zeta, t_{2j+1} \in B_\xi$ for some $\zeta, \xi \in \texttt{Zero}_I(f)$. And we know $\frac{\partial f^u}{\partial x}(x) > 0$ ($\frac{\partial f^d}{\partial x}(x) < 0$) for all $x \in B_\eta (\eta = \xi, \zeta)$ when $t \in \texttt{Zero}_I(f^d)(t = t_{2j-1}, t_{2j+1})$ ($t \in \texttt{Zero}_I(f^u)$). So the constant sign can be reached. We strengthen this to a necessary and sufficient criterion:

THEOREM 11 (Effective Isolation Criteria). Let $K_i = [a_{2i}, b_{2i+1}]$ be an effective candidate. If K_i is an even effective candidate, further assume that constant sign property holds. Then K_i is an isolating interval of f iff one of the following conditions hold:
(O) K_i is an effective odd candidate.
(E): K_i is an effective even candidate and, $i = 0$ or $i > 0$ and $\frac{\partial f^u}{\partial x}$ (resp., $\frac{\partial f^d}{\partial x}$) has some zero in $[b_{2i-1}, b_{2i+1}]$ if f^d (resp., f^u) has two distinct zeros in K_i.

Proof. As a preliminary remark, we note that K_i contains at most one zero of f.

($\Leftarrow$) We first show that (O) or (E) implies that K_i is an isolating interval. Suppose (O) holds. We may assume that f^u has a zero in $[a_{2i}, b_{2i}]$ and f^d has a zero in $[a_{2i+1}, b_{2i+1}]$. Thus $[a_{2i}, b_{2i+1}]$ contains a candidate interval $J_i = [t_{2i}, t_{2i+1}]$ satisfying the conditions of Lemma 5, and J_i has an odd zero of f. Suppose (E) holds. Without loss of generality, assume f^u has two distinct zeros in K_i. If $i = 0$, then clearly, K_i has a zero of f. Otherwise, these zeros must be t_{2i} and t_{2i+1}. By assumption, $\frac{\partial f^d}{\partial x}$ has some zero in $[b_{2i-1}, b_{2i+1}]$; but in fact this zero lies in $[b_{2i-1}, t_{2i+1}] \subseteq J_i$ because $[a_{2i+1}, b_{2i+1}]$ satisfies the constant sign property (11). Now Lemma 9 implies f has some zero in $J_i \subseteq K_i$.

($\Rightarrow$) Suppose f has some zero in K_i. We must show that either (O) or (E) holds. From the definition of K_i, we know there are two distinct roots of $f^u f^d$ in K_i. If $f^u(t_{2i}) = 0$ iff $f^d(t_{2i+1}) = 0$, then clearly (O) holds. Otherwise, $f^d(t_{2i}) = 0$ iff $f^d(t_{2i+1}) = 0$. If $i = 0$, it is clear. If $i \geq 1$, without loss of generality, assume that t_{2i}, t_{2i+1} are zeros of f^d. We must show that $\frac{\partial f^u}{\partial x}$ has some zero z in $[b_{2i-1}, b_{2i+1}]$. By Lemma 9, $\frac{\partial f^u}{\partial x}$ has some zero z in $[t_{2i-1}, t_{2i+1}]$. So it is enough to show that z cannot lie in $[t_{2i-1}, b_{2i-1}]$. But this is a consequence of the constant sign property. □

We can use Sturm theorem to check whether a polynomial ($\frac{\partial f^u(x)}{\partial x}$ or $\frac{\partial f^d(x)}{\partial x}$) has real root in a given interval or isolate the real roots of them directly. In most cases, we need not to use this since Lemma 10 holds for almost all the cases in practice.

3. Bounds of Triangular System

Consider a triangular polynomial system F_n:

$$F_n = \{f_1(x_1), f_2(x_1, x_2), \ldots, f_n(x_1, \ldots, x_n)\} \quad (12)$$

where $f_i \in \mathbb{Z}[x_1, \ldots, x_i]$. Generalizing our univariate notation, if $B \subseteq \mathbb{R}^n$, let $\texttt{Zero}_B(F_n)$ denote the set of real zeros of F_n restricted to B.

Let $B = I_1 \times \cdots \times I_n$ be an n-dimensional box, $I_i = [a_i, b_i]$, and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{n-1}) \in \Box\xi = I_1 \times \cdots \times I_{n-1}$ a real zero of $F_{n-1} = \{f_1, \ldots, f_{n-1}\} = 0$. Consider

$$f(x) := f_n(\xi_1, \ldots, \xi_{n-1}, x). \tag{13}$$

We have a three-fold goal in this section: (1) Compute lower estimates on the evaluation bound $EB_{I_n}(f)$. (2) Construct a sleeve (I_n, f^u, f^d) for f that satisfies the monotonicity property. (3) Compute an upper estimate on the sleeve bound $SB_{I_n}(f^u, f^d)$.

3.1. Lower Estimate on Evaluation Bounds

We give two methods to compute lower estimates of $EB_{I_n}(f)$. The first method is based on a general result about multivariate zero bounds in (21); another is based on resultant computation.

Let $\Sigma = \{p_1, \ldots, p_n\} \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ be a zero dimensional equation system. Let $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ be one of these zeros. Suppose $d_i = \deg(p_i)$ and

$$K := \max\{\sqrt{n+1}, \|p_1\|_2, \ldots, \|p_n\|_2\},$$

where $\|p\|_2$ is the 2-norm of p. Then we have the following result (21, p. 341):

Theorem 12. Let $(\xi_1, \ldots, \xi_n)$ be a complex zero of Σ. For any i, if $|\xi_i| \neq 0$ then

$$|\xi_i| > MRB(\Sigma) := (2^{3/2} N K)^{-D}\, 2^{-(n+1) d_1 \cdots d_n}. \tag{14}$$

where $N := \binom{1+\sum_{i=1}^{n} d_i}{n}$, $D := (1 + \sum_{i=1}^{n} \frac{1}{d_i}) \prod_{i=1}^{n} d_i$.

Note that this theorem defines a numerical value $MRB(\Sigma)$ (the ***multivariate root bound***) for Σ. Given F_n as in (12), consider the polynomial set

$$\widehat{F}_n := \{f_1, \ldots, f_{n-1}, \frac{\partial f_n}{\partial x}, Y - f_n\} \tag{15}$$

in $\mathbb{Z}[x_1, \ldots, x_{n-1}, x, Y]$, where $f_n = f_n(x_1, \ldots, x_{n-1}, x)$.

Lemma 13. Use the notations in (13). Let $(\xi_1, \ldots, \xi_{n-1})$ be a zero of F_{n-1}. Then the evaluation bound $EB_{I_n}(f)$ of $f(x) \in \mathbb{R}[x]$ satisfies $EB_{I_n}(f) > MRB(\widehat{F}_n)$.

Proof. As F_n is zero-dimensional, so is $\widehat{F}_n$, which is easily seen. If $(\xi_1, \ldots, \xi_n, y)$ is a zero of $\widehat{F}_n$, then $f'(\xi_n) = 0$. Moreover, $y = f(\xi_n)$. By definition of $EB(f)$, we have $EB(f)$ is the minimum of all such non-zero $|y|$'s. By Theorem 12, $EB_{I_n}(f) > MRB(\widehat{F}_n)$. □

It is instructive to directly define the ***evaluation bound*** of a triangular system F_n: for $B \subseteq \mathbb{R}^n$, let $B' = B \times \mathbb{R}$. Then define $EB_B(F_n)$ to be

$$\min\{|y| : (x_1, \ldots, x_{n-1}, x, y) \in \mathtt{Zero}_{B'}(\widehat{F}_n), y \neq 0\}, \tag{16}$$

assuming $\min\{\emptyset\} = \infty$. Observe that (16) is a generalization of the corresponding univariate evaluation bound (1). For $i = 2, \ldots, n$, we similarly have evaluation bounds $EB_{B_i}(F_i)$ for F_i, where $F_i = \{f_1, \ldots, f_i\}$.

This multivariate evaluation bound is a lower bound on the univariate one: with f given by (13). In general, $MRB(\widehat{F}_n)$ is not a good estimation.We propose a computational way to compute such a lower estimate via resultants. Consider $\widehat{F}_n$ defined by (15). Let

$$e_i = \begin{cases} \mathtt{res}_X(Y - f_n, \frac{\partial f_n}{\partial X}) & i = n, \\ \mathtt{res}_{x_i}(e_{i+1}, f_i) & i = n-1, \ldots, 1 \end{cases} \tag{17}$$

where $\mathtt{res}_x(p,q)$ is the resultant of p and q relative to x. Thus $e_1 \in \mathbb{F}[Y]$. If $e_1 \not\equiv 0$, define

$$R(F_n) := \min\{|z| : e_1(z) = 0, z \neq 0\}.$$

If e_1 has no real roots, let $R(F_n) = \infty$. It is easy to show that

LEMMA 14. If $e_1 \not\equiv 0$, $EB(F_n) \geq R(F_n)$, and we can use $R(F_n)$ as the evaluation bound.

Therefore, we may isolate the real roots of $e_1(Y) = 0$ and take $\min\{l_1, -r_2\}$ as the evaluation bound for F_n, where (l_1, r_1) and (l_2, r_2) are the isolating intervals for the smallest positive root and the largest negative root of $e_1(Y) = 0$ respectively.

We can use the multiresultant (see (1)) to optimize the evaluation bound computation.

3.2. Sleeve and Sleeve Bound

We assume $I_i > 0$ for $i = 1, \ldots, n$ and will show how to treat other cases in Section 4.

Given $g \in \mathbb{R}[x_1, \ldots, x_n]$, we decompose it uniquely as $g = g^+ - g^-$, where $g^+, g^- \in \mathbb{R}[x_1, \ldots, x_n]$ each has only positive coefficients and with minimal number of monomials. Given f as in (13) and an isolating box $\Box\xi \in \Box\mathbb{F}^{n-1}$ for ξ, following (12; 16), we define

$$\begin{aligned} f^u(x) &= f_n^u(\Box\xi; x) = f_n^+(\mathbf{b}_{n-1}, x) - f_n^-(\mathbf{a}_{n-1}, x), \\ f^d(x) &= f_n^d(\Box\xi; x) = f_n^+(\mathbf{a}_{n-1}, x) - f_n^-(\mathbf{b}_{n-1}, x), \end{aligned} \tag{18}$$

where $\mathbf{a}_i = (a_1, \ldots, a_i)$, $\mathbf{b}_i = (b_1, \ldots, b_i)$, and $\Box\xi = [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}]$.

The bounding functions of the interval function of $f(x)$ (see (8; 10)) are similar to our sleeve polynomials. The functions in the paper (19) are not a sleeve. But in some special interval, they may have some properties of our sleeve polynomials.

From the construction, it is clear that $f^u \geq f \geq f^d$. Moreover, both inequalities are strict if $a_i = \xi_i = b_i$ does not hold for any $i = 1, \ldots, n-1$. Hence $(I_n, f^u(x), f^d(x))$ is a sleeve for $f(x)$ (12; 16). We further have:

LEMMA 15. Over any $I_n = [l, r] > 0$, we have:
(i) (Monotonicity) $\frac{\partial f^u}{\partial x} \geq \frac{\partial f}{\partial x} \geq \frac{\partial f^d}{\partial x}$.
(ii) $f^u(x) - f^d(x)$ is monotonously increasing over I_n.

Proof. Let $\mathbf{t}_{n-1} = (t_1, t_2, \ldots, t_{n-1})$ $(t = a, b, \xi)$, $f(x) = f_n^+(\boldsymbol{\xi}_{n-1}, x) - f_n^-(\boldsymbol{\xi}_{n-1}, x)$ and

$$\begin{aligned} T_1(x) = f^u(x) - f(x) &= (f_n^+(\boldsymbol{b}_{n-1}, x) - f_n^+(\boldsymbol{\xi}_{n-1}, x)) + (f_n^-(\boldsymbol{\xi}_{n-1}, x) - f_n^-(\boldsymbol{a}_{n-1}, x)), \\ T_2(x) = f(x) - f^d(x) &= (f_n^+(\boldsymbol{\xi}_{n-1}, x) - f_n^+(\boldsymbol{a}_{n-1}, x)) + (f_n^-(\boldsymbol{b}_{n-1}, x) - f_n^-(\boldsymbol{\xi}_{n-1}, x)), \\ T_3(x) = f^u(x) - f^d(x) &= (f_n^+(\boldsymbol{b}_{n-1}, x) - f_n^+(\boldsymbol{a}_{n-1}, x)) + (f_n^-(\boldsymbol{b}_{n-1}, x) - f_n^-(\boldsymbol{a}_{n-1}, x)). \end{aligned}$$

Since f_n^+, f_n^- are polynomials with positive coefficients and $0 < a_i \leq \xi_i \leq b_i$ for all i, $f_n^+(b_1, \ldots, b_{n-1},\ x) - f_n^+(\xi_1, \ldots, \xi_{n-1}, x), f_n^-(\xi_1, \ldots, \xi_{n-1}, x) - f_n^-(a_1, \ldots, a_{n-1}, x)$, and hence $T_1(x)$ are polynomials in X with positive coefficients. Similarly, $T_2(x)$ and $T_3(x)$ are polynomials with positive coefficients. For $x > 0$, we have $\frac{\partial T_1(x)}{\partial x} = \frac{\partial f^u(x)}{\partial x} - \frac{\partial f(x)}{\partial x} \geq 0$. Similarly, we can show that $\frac{\partial T_2(x)}{\partial x} = \frac{\partial f(x)}{\partial x} - \frac{\partial f^d(x)}{\partial x} \geq 0$, and $\frac{\partial T_3(x)}{\partial x} = \frac{\partial f^u(x)}{\partial x} - \frac{\partial f^d(x)}{\partial x} \geq 0$. Thus $\frac{\partial f^u}{\partial x} \geq \frac{\partial f}{\partial x} \geq \frac{\partial f^d}{\partial x}$. As consequence, $f^u(x) - f^d(x)$ is monotone increasing in I_n. $\Box$

As an immediate corollary, we have

Corollary 16. $SB_{I_n}(f^u, f^d) \le f^u(r) - f^d(r)$.

Our next goal is to give an upper bound on $f^u(r) - f^d(r)$ as a function of

$$b := \max\{b_1, \ldots, b_n\}, w := \max\{w_1, \ldots, w_n\},$$

where $w_i = b_i - a_i$. Also let $\mathbf{w} = (w_1, \ldots, w_n)$. For $f \in \mathbb{R}[x_1, \ldots, x_n]$, write $f = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}}(x_1, \ldots, x_n)$ where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and $p_{\boldsymbol{\alpha}}(x_1, \ldots, x_n)$ denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let $\|f\|_1 := \Sigma_{\boldsymbol{\alpha}} |c_{\boldsymbol{\alpha}}|$ denote its 1-norm. The inner product of two vectors, say $\mathbf{w}$ and $\boldsymbol{\alpha}$, is denoted $\langle \mathbf{w}, \boldsymbol{\alpha} \rangle$. Let $\mathbf{a}_i = (a_1, \ldots, a_i)$, $\mathbf{b}_i = (b_1, \ldots, b_i)$. We have:

Lemma 17. Let $m = \sum_{i=1}^n \alpha_i \ge 1$. Then $p_{\boldsymbol{\alpha}}(\mathbf{b}_n) - p_{\boldsymbol{\alpha}}(\mathbf{a}_n) \le b^{m-1} \langle \boldsymbol{\alpha}, \mathbf{w} \rangle \le wmb^{m-1}$.

Proof. We have

$$X^m - Y^m = (X - Y)(X^{m-1} + X^{m-2}Y + \cdots + Y^{m-1}) \le (X - Y)mX^{m-1}, \qquad (19)$$

provided $X \ge Y \ge 0$ and $m \ge 1$. Then, assuming each $\alpha_i \ge 1$ and by (19), $p_{\boldsymbol{\alpha}}(b_1, \ldots, b_n) - p_{\boldsymbol{\alpha}}(a_1, \ldots, a_n) = \sum_{i=1}^n \left\{ \left(\prod_{j=1}^{i-1} a_j^{\alpha_j} \right) (b_i^{\alpha_i} - a_i^{\alpha_i}) \left(\prod_{k=i+1}^n b_k^{\alpha_k} \right) \right\} \le \sum_{i=1}^n w_i \alpha_i \left\{ \left(\prod_{j=1}^{i-1} a_j^{\alpha_j} \right) (b_i^{\alpha_i - 1}) \left(\prod_{k=i+1}^n b_k^{\alpha_k} \right) \right\} \le \sum_{i=1}^n w_i \alpha_i \left\{ \left(\prod_{j=1}^{i-1} b^{\alpha_j} \right) (b^{\alpha_i - 1}) \left(\prod_{k=i+1}^n b^{\alpha_k} \right) \right\} = b^{m-1} \sum_{i=1}^n w_i \alpha_i$. In general, if any $\alpha_i = 0$, the corresponding term in the summation could be omitted in the above derivation, and the proof remains valid. □

The above lemma extends linearly to a polynomial f:

Corollary 18. Let $f = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}}(x_1, \ldots, x_n)$, $c_{\boldsymbol{\alpha}} > 0$, $c_{\boldsymbol{\alpha}} \in \mathbb{R}$, $m = \deg(f) \ge 1$. Then $f(\mathbf{b}_n) - f(\mathbf{a}_n) \le b^{m-1} \sum_{\boldsymbol{\alpha}} |c_{\boldsymbol{\alpha}}| \langle \mathbf{w}, \boldsymbol{\alpha} \rangle \le wmb^{m-1} \|f\|_1$.

Theorem 19. Let (I_n, f^u, f^d) be a sleeve as in (18), and $\square_{n-1}\xi = I_1 \times \cdots \times I_{n-1}$ an isolating box for $\xi \in \mathbb{R}^{n-1}$, where $I_i = [a_i, b_i] > 0$, $I_n = [l, r] > 0$, and $w = \max_{i=1}^{n-1}\{b_i - a_i\}$. Then $SB_I(f^u, f^d) \le wm\|f_n\|_1 b^{m-1}$, where $m = \deg(f_n)$, $b = \max\{b_1, \ldots, b_{n-1}, r\}$.

Proof. Let $f(x) = \sum_{i=0}^m C_i(\xi_1, \ldots, \xi_{n-1})X^i$ where $C_i \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$ has degree $\le m - i$, $C_i = C_i^+ - C_i^-$, $\mathbf{a} = (a_1, \ldots, a_{n-1})$, and $\mathbf{b} = (b_1, \ldots, b_{n-1})$. We have $f^u(x) = \sum_{i=0}^m (C_i^+(\mathbf{b}) - C_i^-(\mathbf{a}))X^i$, $f^d(x) = \sum_{i=0}^m (C_i^+(\mathbf{a}) - C_i^-(\mathbf{b}))X^i$. For $x \in I_n$, we have $f^u(x) - f^d(x) = \sum_{i=0}^m (C_i^+(\mathbf{b}) - C_i^+(\mathbf{a}) + C_i^-(\mathbf{b}) - C_i^-(\mathbf{a}))x^i \le \sum_{i=0}^m w(m-i)b^{m-i-1}(\|C_i^+\|_1 + \|C_i^-\|_1)b^i$ (By Corollary 18) $< wmb^{m-1} \sum_{i=0}^m \|C_i\|_1 = wmb^{m-1}\|f_n\|_1$. □

We give two corollaries to the above theorem.

Corollary 20. For a fixed F_n and I_n, when $w \to 0$, $SB_{I_n}(f^u, f^d) \to 0$.

So when $w \to 0$, $f^u \to f$ and $f^d \to f$, which implies that, with sufficient refinement, the sleeve-evaluation inequality (2) will eventually hold. The next corollary gives an explicit condition to guarantee this:

Corollary 21. The sleeve-evaluation inequality (2) holds if $w < \frac{EB_{I_n}(f)}{m\|f_n\|_1 b^{m-1}}$.

4. The Main Algorithm

In this section, we present our isolation algorithm: given F_n as in (12), to isolate the real zeros of F_n in a given n-dimensional box $B = I_1 \times \cdots \times I_n > 0$.

4.1. Refinement of Isolating Box

Refining an isolation box is a basic subroutine in our algorithm. Let $\Box_n\xi = \Box_{n-1}\xi \times [c,d] > 0$ be an isolating box for a zero $\xi = (\xi_1,\ldots,\xi_n)$ of F_n=0, $([c,d], f^d, f^u)$ a sleeve associated with $\Box_n\xi$ satisfying (2) and (7), $\Box'_{n-1}\xi$ an isolating box of F_{n-1} satisfying $\Box'_{n-1}\xi \subsetneq \Box_{n-1}\xi$, $f(x) = f_n(\xi_1,\ldots,\xi_{n-1},x)$, and $\bar{f}^u(x) = {f_n}^u(\Box'_{n-1}\xi, x)$, $\bar{f}^d(x) = {f_n}^d(\Box'_{n-1}\xi, x)$ (for definition, see (18)).

Refine(F_n, K, ϵ)
Input: F_n, K, ϵ.
Output: $\hat{K} = \hat{I}_1 \times \cdots \times \hat{I}_n$ with $w = \max_{j=1}^n\{|\hat{I}_j|\} \le \epsilon$.
1. If $n = 1$, subdivide I_n until $|I_n| < \epsilon$ and return I_n.
2. Let $K_{n-1} = I_1 \times \cdots \times I_{n-1}$, $w = \max_{j=1}^n\{|I_j|\}$.
    If $w \le \epsilon$, return K. Else, $\delta = \epsilon$.
3. while $w > \epsilon$, do
    3.1. $\delta = \delta/2$.
    3.2. If K_{n-1} is a point, $f(x) = f_n(\xi_1,\ldots,\xi_{n-1},x) \in \mathbb{F}[x]$.
        Isolate its roots under ϵ, return them.
    3.3. $K_{n-1} :=$ Refine$(F_{n-1}, K_{n-1}, \delta)$.
    3.4. Compute the sleeve: $f^u(x) := {f_n}^u(K_{n-1},x), f^d(x) := {f_n}^d(K_{n-1},x)$.
    3.5. Isolate the roots of $f^u f^d$ in I_n with precision δ.
    3.6. Denote the first two intervals as $[c_1,d_1],[c_2,d_2]$.
    3.7. $w := d_2 - c_1$.
4. Return $\hat{K} := K_{n-1} \times [c_1, d_2]$.

LEMMA 22. Let t_0, t_1 be the real roots of $f^u f^d = 0$ in $[c,d]$ and $t'_0 < t'_1$ the two smallest real roots of $\bar{f}^u \bar{f}^d = 0$ in $[c,d]$. If $\Box'_{n-1}\xi \neq [\xi_1,\xi_1]\times\cdots\times[\xi_{n-1},\xi_{n-1}]$, then $[t'_0,t'_1] \subset [t_0,t_1]$ and $\xi \in \Box'_{n-1}\xi \times [t'_0,t'_1]$.

Proof. From $\Box'_{n-1}\xi \subsetneq \Box_{n-1}\xi$, $\Box'_{n-1} \neq [\xi_1,\xi_1]\times\cdots\times[\xi_{n-1},\xi_{n-1}]$, and (18), we have

$$f^d(x) < \bar{f}^d(x) < f(x) < \bar{f}^u(x) < f^u(x), \forall x \in [c,d].$$

It is not difficult to check that sleeve-evaluation inequality (2) and the monotonicity property (7) hold for the sleeve $([c,d], \bar{f}^u, \bar{f}^d)$. Wlog, we assume $f^u(t_0) = 0, f^d(t_1) = 0$. The proofs for other cases are similar. We have $\bar{f}^u(t_0) < f^u(t_0) = 0$ and $\bar{f}^u(\xi_n) > f(\xi_n) = 0$. Then $\bar{f}^u$ has at least one root in (t_0,ξ_n). Since $(t_0,\xi_n) \subset A_{\xi_n}$, by Lemma 6, $\bar{f}^u(x)$ has a unique real root in (t_0,ξ_n). Let t'_0 be this root. Then, $t'_0 > t_0$. Since $\bar{f}^u(x) < f^u(x) < 0$, $\bar{f}^u$ has no real roots in $[c,t_0]$ and t'_0 is the smallest root of $\bar{f}^u\bar{f}^d = 0$ in $[c,d]$. Similarly, we could show that $\bar{f}^d(x) = 0$ has at least one root in (ξ_n, t_1). Let t'_1 be the smallest of these roots. Then t'_0 and t'_1 are the two smallest roots of $\bar{f}^u\bar{f}^d = 0$ in $[c,d]$ and $\xi_n \in (t'_0,t'_1) \subset [t_0,t_1]$. $\Box$

The lemma tells us how to refine an isolating box $K = I_1 \times \cdots \times I_n$ of a triangular system F_n without using Theorem 11. The algorithm Refine is to refine K of F_n to $\hat{K} = \hat{I}_1 \times \cdots \times \hat{I}_n$ under the precision ϵ.

4.2. Verifying Zeros

Let $\alpha = (\alpha_1,\ldots,\alpha_k)$ be a real root of the triangular system $\Sigma_k = \{h_1,\ldots,h_k\}$, $B = I_1 \times \cdots \times I_k$ an isolating box of α, and $g(x_1,\ldots,x_k) \in \mathbb{Z}[x_1,\ldots,x_k]$. We show how to check whether $g(\alpha_1, \ldots, \alpha_k) = 0$.

We call $\rho = \min\{|g(\alpha)| : g(\alpha) \neq 0, \forall \alpha \in \mathtt{Zero}_B(\Sigma_k)\}$ the ***zero bound*** of g on Σ_k. Let

$$\Sigma_B = \{h_1, \ldots, h_k, Y - g\}. \tag{20}$$

We have two methods to compute the zero bound. First, by Theorem 12, $MRB(\Sigma_B)$ can be taken as the zero bound. Second, we may compute the zero bound by resultant computation. Let $r_{k+1} = Y - g(x_1, \ldots, x_k)$ and $r_i = \mathtt{res}(h_i, r_{i+1}, x_i)$ for $i = k, \ldots, 1$. Then $r_1(Y)$ is a univariate polynomial in Y. If $r_1 \not\equiv 0$, chose a lower bound ρ for all the absolute values of the nonzero real roots of r_1. It is clear that ρ is smaller than the absolute value of any nonzero root of $r_1(Y) = 0$. We give the following algorithm.

ZeroTest$(\Sigma_n, B, g(x_1, \ldots, x_n))$
Input: $\Sigma_n, B = I_1 \times \cdots \times I_n > 0, g(x_1, \ldots, x_n)$.
Output: True if $g(\alpha) = 0$ or FALSE otherwise.
1. $\delta = \max_{j=1}^{n}\{|I_j|\}$.
2. $g^u = g^+(b_1, \ldots, b_n) - g^-(a_1, \ldots, a_n)$, $g^d = g^+(a_1, \ldots, a_n) - g^-(b_1, \ldots, b_n)$.
3. If $g^d = g^u$ (*Note that* $g = g^d = g^u$)
   If $g^d = 0$ return TRUE; else return FALSE. end
4. If $g^u g^d \geq 0$, then $g \neq 0$ and return FALSE. end
5. Compute the zero bound ρ if we didn't compute it before.
6. If $|g^u| < \rho$, and $|g^d| < \rho$, then $g < \rho$. Hence $g(\alpha) = 0$ and return TRUE. end
7. $\delta = \delta/2$, $B = \text{Refine}(\Sigma_n, B, \delta)$, and goto step 2.

4.3. Isolation Algorithm

We now give the real root isolation algorithm RootIsoTS for a triangular system. Note that Algorithm RootIsoTS can be improved in the following ways.

- The assumption $B_n > 0$ is reasonable. If we want to obtain the real roots of f in the interval $I = (a, b) < 0$, we may consider $g(x) = f(-x)$ in the interval $(-b, -a)$. If $0 \in (a, b)$, we can consider the two parts, $(a, 0)$ and $(0, b)$ respectively, since we can check whether 0 is a root of $f(x) = 0$.
- If $f(a)f(b) = 0$, we can ignore the first or last element in $SL_{f,I}$ to form effective candidate intervals. When $f(a) = 0$, the first effective candidate interval may or may not be an isolating interval, we need to check it by Theorem 11. And we need to use the first isolating interval in $SL_{f,I}$ to decide whether the first effective candidate interval is isolating if the first three elements in $SL_{f,I}$ are all isolating intervals of f^u (or f^d).
- If we want to find all real roots of f, we first isolate the real roots of f in $(0, 1)$, then isolate the real roots of $g(x) = X^n * f(1/x)$ in $(0, 1)$, and check whether 1 is a root of f. As a result, we can find all the roots of $f(x) = 0$ in $(0, +\infty)$. We can find the roots of $f(x) = 0$ in $(-\infty, 0)$ by isolating the roots of $f(-x) = 0$ in $(0, +\infty)$. Finally, check whether 0 is a root of $f(x) = 0$.
- In step 2.3.5, we have $f(x) \equiv 0$ when $\mathtt{Zero}_I(f^u f^d) = \emptyset$ and $f^u(x) f^d(x) < 0$, $\forall x \in I$. This can be used to check whether the given system is zero-dimensional or not.

RootIsoTS
Input: F_n, $B_n = \prod_{i=1}^{n} I_i (I_i = [l_i, r_i] > 0)$, $\epsilon > 0$.
Output: An isolating set $\Box\texttt{Zero}_{B_n}(F_n)$.
1. Compute $\Box\texttt{Zero}_{B_1}(F_1)$ for F_1 to precision ϵ.
   $Result := \Box\texttt{Zero}_{B_1}(F_1)$. $New := \emptyset$. If $Result = \emptyset$, return $Result$, end
2. For i from 2 to n, do
   2.1. Compute $EB_i := EB(F_i)$ for F_i.
   2.2. $\delta := \epsilon$.
   2.3. while $Result \neq \emptyset$, do
      2.3.1. Choose an element $\Box_{i-1}\xi$ from $Result$. $Result := Result \setminus \{\Box_{i-1}\xi\}$.
      2.3.2. Compute the sleeve: $f^u(x) = f_i{}^u(\Box_{i-1}\xi, x)$, $f^d(x) = f_i{}^d(\Box_{i-1}\xi, x)$.
      2.3.3. While $f^u(r_i) - f^d(r_i) \geq EB_i$,
         $\delta := \delta/2$. $\Box_{i-1}\xi :=$ Refine$(F_{i-1}, \Box_{i-1}\xi, \delta)$. Recompute $f^u(x)$ and $f^u(x)$.
      2.3.4. Isolate the real roots of $f^u f^d$ in I_i.
      2.3.5 If the set derived from 2.3.4 is empty and $f^u(a)f^d(a) < 0$, then $f(x) \equiv 0$.
         The input system is not zero dimensional. end
      2.3.6. Compute the parity of these roots.
      2.3.7. Construct the effective candidate intervals.
      2.3.8. for each effective candidate interval K,
         2.3.8.1. Check whether K is isolating.
            If K is odd, K is isolating;
            If K is even: If Lemma 10 holds, K is isolating;
               Else, ensure (11). K is isolating iff Theorem 11 (E) holds.
         2.3.8.2. If K is isolating, then $K :=$ Refine(F_i, K, ϵ). $New := New \bigcup \{\Box_{i-1}\xi \times K\}$.
   2.4. If $New = \emptyset$, return New, end
   2.5. $Result := New$. $New := \emptyset$.
3. return $Result$.

4.4. *Examples and Experimental Results*

We first gave two working examples.

Example 1: Consider the system $F_2 = \{f_1, f_2\}$ where $f_1 = x^4 - 3\,x^2 - x^3 + 2\,x + 2$, $f_2 = y^4 + xy^3 + 3\,y^2 - 6\,x^2y^2 + 4\,x\,y + 2\,xy^2 - 4\,x^2y + 4\,x + 2$. Set the precision to be 2^{-4}. Isolating the real roots of f_1 to precision 2^{-4}, we obtain the following isolating intervals: $[[\frac{-23}{16}, \frac{-11}{8}], [\frac{-5}{8}, \frac{-9}{16}], [\frac{11}{8}, \frac{23}{16}], [\frac{25}{16}, \frac{13}{8}]]$. Next consider $\Box_1\xi = [\frac{11}{8}, \frac{23}{16}]$, where $\xi \in \texttt{Zero}(f_1)$. We will isolate the real roots of $f_2(\xi, y) = 0$ in $[0, 2]$.

We derive $EB_2 = \frac{1}{2}$ by resultant computation. The sleeve computed using $\Box_1\xi$ is

$$f^u(y) = -\frac{175}{32}\,y^2 - \frac{29}{16}\,y + y^4 + \frac{23}{16}\,y^3 + \frac{31}{4}, f^d(y) = -\frac{851}{128}\,y^2 - \frac{177}{64}\,y + y^4 + \frac{11}{8}\,y^3 + \frac{15}{2}.$$

The sleeve bound of $([0, 2], f^u, f^d)$ is $SB = f^u(2) - f^d(2) = \frac{59}{8}$. Since (2) does not hold, we refine $\Box_1\xi$. Let $\Box_1\xi = \text{Refine}(f_1, \Box_1\xi, \frac{1}{2^8}) = [\frac{181}{128}, \frac{363}{256}]$. We have the new sleeve

$$f^u(y) = -\frac{50475}{8192}\,y^2 - \frac{9529}{4096}\,y + y^4 + \frac{363}{256}\,y^3 + \frac{491}{64}, f^d(y) = -\frac{204331}{32768}\,y^2 - \frac{39097}{16384}\,y + y^4 + \frac{181}{128}\,y^3 + \frac{245}{32}$$

with sleeve bound $SB = f^u(2) - f^d(2) = \frac{949}{2048} < \frac{1}{2} = EB_2$. It is easy to check that the sleeve $([0,2], f^u, f^d)$ is faithful. Isolating $f^u f^d$ in $[0,2]$ to precision 2^{-8}, we obtain $SL_{f_2,[0,2]}$: $[[\frac{165}{128}, \frac{331}{256}], [\frac{395}{256}, \frac{99}{64}]]$ both with parities 1. These intervals are both isolating intervals of f^d. It forms an isolating interval of $f_2(\xi, y)$ by Lemma 8. So there is an even root of $f_2(\xi, y)$ in $[0,2]$ by Theorem 11. It is in $[\frac{165}{128}, \frac{99}{64}]$. So $[\frac{11}{8}, \frac{23}{16}] \times [\frac{165}{128}, \frac{99}{64}]$ is an isolating box of triangular system F_2.

The isolating box does not satisfy our output precision requirement. Refine the isolating box with Refine, we obtain $[\frac{181}{128}, \frac{5793}{4096}] \times [\frac{1423}{1024}, \frac{2947}{2048}]$.

Eventually, we obtain all the isolating boxes for $F_2 = 0$ in 0.141s with RootIsoTS.

Using Theorem 12 to compute $MRB(F_2)$, we have $MRB(F_2) > \frac{1}{2^{289}}$ and the computing time is 9.282s. By Corollary 21, this precision is enough for us to isolate the roots of F_2.

Example 2: Consider the following system from (8).

$$f_1 = -12z^2 - 3yz + xz - 27z - 4y^2 - 11xy - 5y + 29x^2 + 11x - 27;$$
$$f_2 = -25z^2 - 23yz + 23xz + 4z + 2y^2 + 7xy + 21y + 4x^2 - 15x - 30;$$
$$f_3 = -14z^2 + 27yz - 29xz + 11z + 4y^2 - 31xy + 22y - 12x^2 - 28x - 9.$$

We first transform the system to a triangular system T with WSolve package (17) in 0.141s. The time for isolating the roots of T under the precision 2^{-20} is 0.406s. The C program in (8) uses 0.62s on a SUN4 with a 400 MHz CPU and 2GB of memory.

We implemented RootIsoTS in Maple 10 and tested our program with two sets of examples on a PC with a 3.2G CPU, 512M memory, and Windows OS. The coefficients of the polynomials are within -100 to 100. The precision is $\frac{1}{2^{10}}$. We use the method mentioned in the **Remarks** for RootIsoTS to compute *all the real solutions*. We estimate the evaluation bounds by resultant computation. The most time-consuming parts are the computation of the evaluation bounds and the refinement for the isolating boxes.

The first set of examples are random polynomials and the results are in Table 1. The *type* of $F_n = \{f_1, \ldots, f_n\}$ is a list $(d_1, \ldots, d_n)$ where $d_i = \deg_{x_i}(f_i)$. The column started with TYPE is the type of the tested triangular systems. TIME is the average running time for each triangular system in seconds. NS is the average number of real solutions. NT is the number of tested triangular systems. NE is the number of terms in each polynomial.

TYPE	TIME	NS	NT	NE
(3, 3)	0.05355	1.91	100	(3.99, 8.02)
(9, 8)	1.87486	4.26	100	(9.94, 43.98)
(11, 11)	8.782	4.5	80	(11.975, 72.5)
(16, 14)	50.22	6.0	100	(16.9, 127.13)
(21, 15)	164.23	6.22	100	(21.91, 176.8)
(3, 3, 3)	0.387	2.91	100	(3.99, 7.77, 13.01)
(5, 4, 4)	2.97	4.88	100	(5.99, 14.72, 24.24)
(5, 5, 5)	33.22	5.61	80	(5.9, 17.7, 42.1)
(8, 7, 6)	592.18	7.6	10	(8.9, 36.0, 79.8)
(3, 3, 3, 3)	119.94	6.96	50	(4.0, 8.1, 12.8, 20.9)
(5, 5, 5, 3)	551.44	3.4	10	(6.0, 32.1, 42.3, 21.5)

Table 1. Timings for dense triangular systems

The second set of examples are triangular systems with multiple roots and the results are given in Table 2. A triangular system of type $(d_1, \ldots, d_n)$ is generated as follows: f_1 is a random polynomial in x_1 and with degree d_1 in x_1 and $f_i = a_i^2 (b_i x_i + c_i)^{\lfloor \frac{d_i+1}{2} \rfloor - \lfloor \frac{d_i}{2} \rfloor}$ for $i = 2, \ldots, n$, where a_i is a random polynomial in $x_1, \ldots, x_i$ and with degree $\lfloor d_i/2 \rfloor$ in x_i, b_i, c_i are random polynomials in $x_1, \ldots, x_{i-1}$. In Table 2, NM is the average number of multiple roots for the tested systems.

TYPE	TIME	NS	NM	NT	NE
(5, 5)	0.712	3.71	1.57	100	(5.9, 34.4)
(9, 8)	0.604	3.1	3.1	100	(9.9, 18.9)
(13, 11)	32.44	6.55	3.92	100	(13.9, 107.6)
(23, 21)	466.0	6.15	3.75	20	(24.0, 183.4)
(3, 3, 3)	3.213	5.59	3.24	100	(3.9, 13.0, 31.7)
(9, 7, 5)	425.9	12.95	8.15	20	(9.9, 60.8, 100.3)
(3, 3, 3, 3)	130.6	11.15	6.1	20	(4.0, 12.2, 33.7, 62.9)

Table 2. Timings for dense triangular systems with multiple roots

From the above experimental results, we could conclude that our algorithm is capable of handling quite large triangular systems.

5. Triangular Systems Without Multiple Roots

As mentioned in the preceding section, one of the most time-consuming part of the algorithm is the computation of the evaluation bound. In this section, we will propose a root isolation algorithm for zero-dimensional triangular systems without multiple roots, which does not need to compute the evaluation bound.

5.1. Root Isolation of Univariate Equation without Multiple Roots

Consider a univariate polynomial $f(x) \in \mathbb{R}[x]$ without multiple roots, that is, f is square free. We will isolate its real roots in a given interval $I = [a, b] \in \square\mathbb{F}$.

We call (I, f^u, f^d) of f a ***normal sleeve*** if it satisfies condition (3) and the monotonicity property (7). As mentioned in the Remarks after Algorithm RootIsoTS, to assume that a sleeve is normal is reasonable. The following results show that Corollary 4 is valid in this case without the evaluation-sleeve inequality.

Lemma 23. Let (I, f^d, f^u) be a normal sleeve for f. Then we have $\#(\mathtt{ZERO}_I(f^u f^d)) =$ *even*. If $\mathtt{ZERO}(f^u f^d) = (t_0, \ldots, t_{2m-1})$ and $t_i \leq t_{i+1}$, then each real root of f in I is contained in a ***candidate interval*** of f: an interval like $(t_{2i}, t_{2i+1}), (0 \leq i \leq m-1)$.

Proof. Since (I, f^d, f^u) is a normal sleeve, we have (3). Wlog, we assume that $f(a) > 0, f^d(a) > 0, f^u(a) > 0$ and $f(b) < 0, f^d(b) < 0, f^u(b) < 0$. Let $\xi_0, \ldots, \xi_s$ be the roots of $f(x) = 0$ in I. Then, we have $f^u(x) > f(x) \geq 0$ for $x \in [a, \xi_0]$ and $f^d(\xi_0) < f(\xi_0) = 0$. Then, $f^u(x)$ has no roots in $[a, \xi_0]$ and $f^d(x) = 0$ must have roots in $J = [a, \xi_0]$. We will show that $\#\mathtt{ZERO}_J(f^d f^u) = \#\mathtt{ZERO}_J(f^d)$ is odd. Note that a univariate polynomial changes its sign after passing through an odd root. Since we considered multiplicities in $\mathtt{ZERO}_J$ and $f^d(a) > 0, f^d(\xi_0) < 0$, $\#\mathtt{ZERO}_J(f^d)$ must be odd. As a consequence, ξ_0 is in a candidate interval. Since $f(x) = 0$ has no multiple roots in I, there exists a number $c > \xi_0$

such that $f(x) < 0$ and $f^u(x) > 0$ on $(\xi_0, c]$. We can similarly show that $\#\mathtt{ZERO}_K(f^d f^u) = \#\mathtt{ZERO}_K(f^u)$ is even for $K = [c, \xi_1]$. Then, ξ_1 is also in a candidate interval. Similarly, ξ_s is in a candidate interval (t_{2u}, t_{2u+1}). We thus have $f^d(x) < f(x) \le 0$ on $[\xi_s, b]$. Also, $f^u(\xi_s) > f(\xi_s) = 0$ and $f^u(b) < 0$. Then for $L = [\xi_s, b]$, $\#\mathtt{ZERO}_L(f^d f^u) = \#\mathtt{ZERO}_L(f^u)$ is odd and with t_{2u+1} as the first root. We proved that in $[a, \xi_0]$ and $[\xi_s, b]$, the numbers of roots are odd; in $[\xi_i, \xi_{i+1}], i = 0, \ldots, s-1$, the numbers of roots are even. Then, $\#\mathtt{ZERO}_I(f^u f^d)$ is even and each root of $f(x) = 0$ is in a candidate interval. □

The above lemma shows that each root of $f = 0$ is in a candidate interval of $f^d f^u = 0$. But, one candidate interval may contain more than one roots of $f = 0$. The following lemma gives a sufficient criterion for a candidate interval to be an isolating one.

Lemma 24. Let (I, f^d, f^u) be a normal sleeve for f and $J = [t_{2i}, t_{2i+1}]$ a candidate interval. If J is an odd interval (definition see (6)) and

$$\left.\begin{array}{l} 1.\ \frac{\partial f^u}{\partial x} \text{ has no roots in } [t_{2i}, t_{2i+1}] \text{ if } t_{2i} \in \mathtt{Zero}(f^d), \\ 2.\ \frac{\partial f^d}{\partial x} \text{ has no roots in } [t_{2i}, t_{2i+1}] \text{ if } t_{2i} \in \mathtt{Zero}(f^u) \end{array}\right\} \tag{21}$$

then J is an isolating interval of a root of $f(x) = 0$.

Proof. Since J is an odd interval, it contains at least one root of $f(x) = 0$. If J contains more than one roots of $f(x) = 0$, the function $y = f(x)$ must has an extremal point x_0 and x_0 must be a root of $\frac{\partial f}{\partial x}(x) = 0$. On the other hand, we will show that for f satisfying the conditions in the lemma, $\frac{\partial f}{\partial x}$ cannot have a root in J, which means f has at most one root in J, and hence prove the lemma. Wlog, we assume that $f^d(t_{2i}) = f^u(t_{2i+1}) = 0$. Let ξ be a root of $f(x) = 0$ in J. Since $f^u(\xi) > 0$ and $f^u(t_{2i+1}) = 0$, there exists an $\eta \in (\xi, t_{2i+1})$ such that $\frac{\partial f^u}{\partial x}(\eta) < 0$. Since $\frac{\partial f^u}{\partial x}$ has no roots in J, we have $\frac{\partial f^u}{\partial x} < 0$ on J. From the monotonicity property (7), we have $\frac{\partial f^d}{\partial x} \le \frac{\partial f}{\partial x} \le \frac{\partial f^u}{\partial x}$, and hence $\frac{\partial f}{\partial x} < 0$ on J. Therefore, $\frac{\partial f}{\partial x} = 0$ have no roots in J. □

The following lemma shows that the conditions in Lemma 24 are also necessary in certain sense.

Lemma 25. Let $f \in R[x]$ be square free and (I, f^d, f^u) a normal sleeve for f constructed with formula (18). Then, when f^u and f^d sufficiently approximate f, each candidate interval is odd and satisfies condition (21).

Proof. Wlog, we assume that ξ is the first root of $f = 0$ in I, $C = (t_{2k}, t_{2k+1})$ is the candidate interval containing ξ, $f^d(a) > 0, f^u(a) > 0$, and $f^d(t_{2k}) = 0$. Since f has no multiple roots, we have $\frac{\partial f}{\partial x}(\xi) < 0$. Use the notations A_ξ and B_ξ introduced in (4). When f^u and f^d sufficiently approximate f, for example, when the sleeve-evaluation inequality (2) holds, $f = 0$ has no extremal points in $\bar{A}_\xi = [a_\xi, \xi]$ and $f^d = 0$ has no roots in $[a, a_\xi]$. Since $\frac{\partial f}{\partial x}(\xi) < 0$, for $x \in \bar{A}_\xi$, we have $\frac{\partial f}{\partial x}(x) < 0$ and $|\frac{\partial f}{\partial x}(x)| > \rho$ for a positive number ρ.

We first show that C must be an odd interval when f^d, f^u sufficiently approximate f. From the way to construct f^u and f^d, the coefficients of $\frac{\partial f^u}{\partial x}, \frac{\partial f^d}{\partial x}$ will sufficiently approximate that of $\frac{\partial f}{\partial x}$ when f^u, f^d sufficiently approximate f. Then, when f^d, f^u sufficiently approximate f, $\frac{\partial f^u}{\partial x}$ and $\frac{\partial f^d}{\partial x}$ will sufficiently approximate $\frac{\partial f}{\partial x}$. Since $\frac{\partial f}{\partial x}(x) < 0$ and $|\frac{\partial f}{\partial x}(x)| > \rho$ for $x \in \bar{A}_\xi$, when f^u, f^d sufficiently approximate f, we have $\frac{\partial f^u}{\partial x}(x) < 0$, $\frac{\partial f^d}{\partial x}(x) < 0$ for $x \in \bar{A}_\xi$. As a consequence, f^d has only one root in A_ξ and C is the first candidate interval. Since we assume that the sleeve-evaluation inequality (2) holds, C must be an odd interval. So we proved that C is the first candidate interval and is odd when f^d, f^u sufficiently approximate f. Other cases can be proved similarly and we proved that all candidate intervals must be odd when f^d, f^u sufficiently approximate f.

Since $\frac{\partial f^u}{\partial x}(x) < 0$ for $x \in \bar{A}_\xi$, condition (21) will be satisfied when C is contained in $\bar{A}_\xi$, which is possible when f^d, f^u sufficiently approximate f. □

Similar to the results in Section 2.3, we can obtain an effective version of the criteria given in Lemma 24. Let $\mathtt{ZERO}_I(f^u f^d)$ be the sorted list given in (5), and $[a_i, b_i]$ an isolating interval of t_i, where any two distinct intervals $[a_i, b_i]$ and $[a_j, b_j]$ are disjoint. We still call $[a_{2i}, b_{2i+1}]$ an ***effective candidate*** of f in I, which is called odd if $[t_{2i}, t_{2i+1}]$ is an odd interval. We have

THEOREM 26. Let $f \in \mathbb{R}[x]$ be square free, (I, f^u, f^d) a normal sleeve for f, and K the set of effective candidates of f in I. When f^d and f^u sufficiently approximate f and $[a_i, b_i]$ sufficiently approximates t_i, each $J \in K$ is an odd interval satisfying

$$\left.\begin{array}{l} 1.\ \frac{\partial f^u}{\partial x} \text{ has no roots in } [a_{2i}, b_{2i+1}] \text{ if } t_{2i} \in \mathtt{Zero}(f^d); \\ 2.\ \frac{\partial f^d}{\partial x} \text{ has no roots in } [a_{2i}, b_{2i+1}] \text{ if } t_{2i} \in \mathtt{Zero}(f^u). \end{array}\right\} \quad (22)$$

As a consequence, K is a set of isolating intervals for the roots of f in I.

Proof. By Lemma 25, when f^d and f^u sufficiently approximate f, each $J \in K$ is an odd interval satisfying (21). Wlog, consider the first case in (21). We have $|\frac{\partial f^u}{\partial x}(x)| > 0$ for $x \in [t_{2i}, t_{2i+1}]$. Since $[a_{2i}, b_{2i}]$ and $[a_{2i+1}, b_{2i+1}]$ are isolating intervals of t_{2i} and t_{2i+1} respectively, we can refine them so that $|\frac{\partial f^u}{\partial x}(x)| > 0$ for $x \in [a_{2i}, b_{2i+1}]$ and condition (22) is satisfied. By Lemma 24, these intervals are isolating intervals for some roots of f. By Lemma 23, these intervals are isolating intervals for all roots of f. □

RootIsoSQFree
Input: F_n: a zero dimesnional triangular system without multiple roots;
$B_n = \prod_{i=1}^n I_i (I_i = [l_i, r_i] > 0)$, $\epsilon > 0$.
Output: An isolating set $\square\mathtt{Zero}_{B_n}(F_n)$.
1. Compute $\square\mathtt{Zero}_{B_1}(F_1)$ for F_1 to precision ϵ.
   $Result := \square\mathtt{Zero}_{B_1}(F_1)$. $New := \emptyset$. If $Result = \emptyset$, return $Result$, end
2. For i from 2 to n, do
   2.1. $\delta := \epsilon$.
   2.2. while $Result \neq \emptyset$, do
      2.2.1. Choose an element $\square_{i-1}\xi$ from $Result$. $Result := Result \setminus \{\square_{i-1}\xi\}$.
      2.2.2. Compute the sleeve: $f^u(x) = f_i{}^u(\square_{i-1}\xi, x)$, $f^d(x) = f_i{}^d(\square_{i-1}\xi, x)$.
      2.3.3. Isolate the real roots of $f^u f^d$ in I_i with precision $\epsilon > 0$.
      2.2.4. Construct the set K of effective candidate intervals.
      2.2.5. While there is a $J \in K$ s.t. J is not odd or Condition (22) doesn't hold,
         $\delta := \delta/2$. $\square_{i-1}\xi := \text{Refine}(F_{i-1}, \square_{i-1}\xi, \delta)$.
         Reconstruct $f^u(x)$, $f^d(x)$ and recompute an effective candidate set K.
      2.2.6. $K := \text{Refine}(F_i, K, \epsilon)$. $New := New \bigcup \{\square_{i-1}\xi \times K\}$.
   2.3. If $New = \emptyset$, return New, end
   2.4. $Result := New$. $New := \emptyset$.
3. return $Result$.

5.2. Root Isolation Algorithm and Experiment Results

The idea is to construct and refine the sleeves until all the effective candidates are odd intervals and condition (22) is satisfied. After this, the effective candidate intervals are isolating intervals. Algorithm RootIsoSQFree is based on this idea.

We test a Maple version of Algorithm RootIsoSQFree on a PC with a 1.6G Core 2 Duo CPU, 512M memory, and Windows OS. Table 3 contains the results. The meaning of the parameters can be found in Section 4.4.

For triangular systems of types $(9, 8), (21, 15)$, and $(8, 7, 6)$, Algorithm RootIsoSQFree are 57, 1748, 4907 times faster than Algorithm RootIsoTS (Table 1) on a sample set of 100 problems for each type. Therefore, in terms of efficiency, the improvements of Algorithm RootIsoSQFree comparing to that of Algorithm RootIsoTS is significant. Also, we can see that Algorithm RootIsoSQFree is good enough to isolate the roots for large scale systems efficiently.

TYPE	TIME	NS	NT	NE
(9, 8)	0.03282	4.39	100	(9.9, 44.67)
(21, 15)	0.09391	5.75	100	(21.85, 135.37)
(119, 70)	4.33518	11.77	100	(119.38, 2543.39)
(219, 180)	54.33796	15.33	100	(218.7, 16387.83)
(8, 7, 6)	0.12077	8.08	100	(8.96, 35.86, 83.64)
(19, 17, 14)	1.22715	14.58	100	(19.94, 170.11, 676.38)
(39, 37, 31)	19.44103	24.63	100	(39.82, 737.1, 5954.7)
(139, 77, 41)	70.086375	36.25	40	(139.25, 3066.475, 13177.05)
(9, 7, 5, 3)	0.14828	8.05	100	(9.95, 35.85, 55.73, 34.8)
(19, 17, 15, 13)	9.49561	29.81	100	(19.96, 170.14, 811.96, 2368.19)
(59, 37, 25, 23)	71.62045	28.1	20	(59.45, 737.0, 3259.15, 17458.95)
(11, 9, 8, 7, 5, 3, 3)	3.41567	41.31	100	(11.9, 54.7, 163.31, 328.42, 250.94, 83.57, 119.38)

Table 3. Timings for dense triangular systems without multiple roots

Example 3 We also test examples from practical problems. We directly test the triangular sets of EX2, EX4, EX7 in the appendix in (19). Since they are square free, Algorithm RootIsoSQFree works for them. On a laptop with 1.73G Core 2 Duo CPU, 1G memory, and Windows OS, the computing times are $0.047, 0.125$, and 0.483 respectively.

6. Conclusion

This paper provides a complete numerical algorithm of isolating the real roots for arbitrary zero-dimensional triangular systems. The key idea is to use a sleeve satisfying the sleeve-evaluation inequality to isolate the roots for a univariate polynomial with algebraic number coefficients. Even with our current simple implementation, the algorithm is shown to be quite effective. We further propose a complete root isolation algorithm for zero-dimensional triangular systems without multiple roots, which does not need to compute the evaluation bound and is shown to be much faster than the general algorithm.

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