Complete Numerical Isolation of Real Zeros in Zero-dimensional Triangular Systems

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ABSTRACT

We present a complete numerical algorithm of isolating all the real zeros of a zero-dimensional triangular polynomial system $F_n \subseteq \mathbb{Z}[x_1, \ldots, x_n]$. Our system F_n is general, with no further assumptions. In particular, our algorithm successfully treat multiple zeros directly in such systems. A key idea is to introduce evaluation bounds and sleeve bounds. We implemented our algorithm and promising experimental results are shown.

Categories and Subject Descriptors

G.1.5 [**Mathematics of Computing**]: Roots of Nonlinear Equations - system of equations

General Terms

Algorithms, Theory

Keywords

Triangular system, real zero isolation, sleeve bound, evaluation bound

1. INTRODUCTION

Many problems in the computational sciences and engineering can be reduced to the solving of polynomial equations. There are two basic approaches to solving such polynomial systems – numerically or algebraically. Usually, the numerical methods have no global guarantees of correctness. Algebraic methods for solving polynomial systems include Gröbner bases, characteristic sets, CAD, and resultants (see [3, 4, 5, 7, 16, 18, 20, 25, 26]). One general idea in polynomial equation solving is to reduce the original system into a triangular system. Zero-dimensional polynomial systems are among the most important cases to solve. This paper considers zero-dimensional triangular systems only.

A zero-dimensional triangular system has the form $F_n = \{f_1, \ldots, f_n\}$, where each $f_i \in \mathbb{Z}[x_1, \ldots, x_i]$ $(i = 1, \ldots, n)$ and x_i is a variable of f_i. We are interested in real zeros of F_n. A real zero of F_n is $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ such that

$$f_1(\xi_1) = f_2(\xi_1, \xi_2) = \cdots = f_n(\xi_1, \ldots, \xi_n) = 0. \qquad (1)$$

The standard idea here is to first solve for $f_1(x_1) = 0$, and for each solution $x_1 = \xi_1$ of f_1, we find the solutions of $x_2 = \xi_2$ of $f_2(\xi_1, x_2) = 0$, etc. The problem is reduced to solving univariate polynomials of the form $f_i(\xi_1, \ldots, \xi_{i-1}, x_i) = 0$. Such polynomials have algebraic number coefficients. We could isolate roots of such polynomials by using standard root isolation algorithms (such as Sturm sequence method), but using algebraic number arithmetic. But even for $n = 2$ or 3, such algorithms are too slow. The numerical approach is to replace the ξ_i's by approximations, and thus reduce the problem to isolating roots of such numerical polynomials. The challenge is how to guarantee completeness of such numerical algorithms.

We will provide a numerical algorithm that solves such triangular systems completely in the following precise sense: given an n-dimensional box $R = J_1 \times \cdots \times J_n \subseteq \mathbb{R}^n$ where J_i are intervals, and any precision $\varepsilon > 0$, it will isolate the real zeros of F_n in R to precision ε.

Our solution places no restriction on F_n. The reason why we consider general zero-dimensional triangular systems is that the triangular systems derived in cylinder algebraic decomposition or topology determination [8] are generally with multiple roots and even non regular (for definition see [25]).

Our goal is to deal with such triangular systems directly without factorization or gcd computation over algebraic number fields. It is well known that these computations are expensive.

Many algorithms that seek to provide "exact numerical" solution assume computation over the rational numbers $\mathbb{Q}$. But this is much less efficient than using dyadic numbers: let $\mathbb{D} := \mathbb{Z}[\frac{1}{2}] = \{m2^n : m, n \in \mathbb{Z}\}$ denote the set of dyadic numbers (or bigfloats)[28]. Most current fast algorithms for bigfloats can be derived from Brent's work [6]. In the following, we use the symbol $\mathbb{F}$ to denote either $\mathbb{D}$ or with $\mathbb{Q}$. We use intervals to isolate real numbers: let $\square\mathbb{F}$ denote the set of intervals of the form $[a, b]$ where $a \leq b \in \mathbb{F}$.

Given a polynomial $f \in \mathbb{R}[X]$ and an interval $I = [a, b] \in \square\mathbb{F}$, we construct two polynomials $f^u, f^d \in \mathbb{F}[X]$ such that $f^u > f > f^d$ holds in I. We call (f^u, f^d) a **sleeve** of f over I and $SB_I(f^u, f^d) := \sup\{f^u(x) - f^d(x) : x \in I\}$ the **sleeve bound**. Note that the coefficients of $f^u f^d$ are in $\mathbb{F}$, but f have real coefficients which can be arbitrarily approximated. The key idea in this paper is the introduction of **evaluation bounds**. For a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and a

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subset $I \subseteq \mathbb{R}$, let its **evaluation bound** be

$$EB_I(f) := \inf\{|f(x)| : f'(x) = 0, f(x) \neq 0, x \in I\}. \quad (2)$$

If the following **sleeve-evaluation inequality**

$$SB_I(f^u, f^d) < EB_I(f) \quad (3)$$

holds, we show that the isolating intervals of $f^u f^d$ can be used to define isolating intervals of f. The use of evaluation bounds appear to be new. It is the ability to compute lower estimates on $EB_I(f)$ that allows us to detect zeros of even multiplicities.

As a consequence of the above analysis, the real roots isolation for f is reduced to real roots isolation for f^d and f^u. Univariate root isolation is a well-developed subject in its own right, with many efficient solutions known (see [1, 9, 11, 13, 15, 21, 23]). We can use any of these solutions in our algorithm.

The idea of using a sleeve to solve equations was used by [22] and [17]. Lu et al [17] proposed an algorithm to isolate the real roots of triangular systems. Their method could solve many problems in practice, but is not complete and cannot handle multiple zeros. Collins et al [10] considered the problem with interval arithmetic methods and Descartes' method using floating point computation. Again, they pointed out that if a real coefficient is implicitly zero, the method will fail. Xia and Yang [27] consider real root isolation of a semi-algebraic set. They ultimately considered the regular and square-free triangular systems. Their method can be viewed as a generalization of the Uspensky algorithm. They mentioned that the method is not complete and will fail in some cases. Eigenwillig et al considered root-isolation for real polynomials with bitstream coefficients [12]. Their algorithm requires f to be square-free. Our evaluation bound is similar to the curve separation bound in [29]. It seems difficult to define sleeves for non-triangular systems, because the variables appear simultaneously. Interesting work on general polynomial systems was done by Hong and Stahl [14].

In Section 2, we describe the basic technique of using sleeves and evaluation bounds of f. We next exploit a special property of sleeves called monotonicity. This leads to an effective criteria for isolating zeros of even multiplicity. Using these tools, we provide an algorithm to isolate the real roots of univariate polynomials with real coefficients. In Section 3, we give methods to compute evaluation bounds. We also show how to construct sleeves and derive a sleeve bound for a triangular system. In Section 4, we present the root isolation algorithm for triangular systems. Experimental results are also presented. We conclude in Section 5.

2. ROOT ISOLATION FOR REAL UNIVARIATE POLYNOMIALS

We give a framework for isolating the real roots of a univariate polynomial equation with real coefficients.

2.1 Evaluation and Sleeve Bounds

Let $\mathbb{Q}$ be the field of rational numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{D} := \mathbb{Z}[\frac{1}{2}] = \{m2^n : m, n \in \mathbb{Z}\}$ the set of dyadic numbers, and $\mathbb{F}$ denote either $\mathbb{D}$ or $\mathbb{Q}$. In this section, we fix f, f^u, f^d to be C^1 functions, and $I \in \square\mathbb{F}$.

We call (I, f^u, f^d) a **sleeve** for f if, for all $x \in I$, we have $f^u(x) > f(x) > f^d(x)$.

For any real function f, let $\mathtt{Zero}_I(f)$ denote the set of distinct real zeros of f in the interval I. If $I = \mathbb{R}$, then we simply write $\mathtt{Zero}(f)$. If $\#(\mathtt{Zero}_I(f)) = 1$, we call I an **isolating interval** of f. Sometimes, we need to count the zeros up to the parity (i.e., evenness or oddness) of their multiplicity. Call a zero $\xi \in \mathtt{Zero}(f)$ an **even zero** if its multiplicity is even, and **odd zero** if its multiplicity is odd. Define the multiset $\mathtt{ZERO}_I(f)$ whose underlying set is $\mathtt{Zero}_I(f)$ and where the multiplicity of $\xi \in \mathtt{ZERO}_I(f)$ is 1 (resp., 2) if ξ is an odd (resp., even) zero of f.

To avoid special treatment near the endpoints of an interval, we enforce the following conditions.

$$|f(a)| \geq EB_I(f), \quad f^u(b)f^d(b) > 0. \quad (4)$$

We say that the sleeve (I, f^u, f^d) is **faithful** for f if (4) and (3) are both satisfied. We can easily see that $|f(a)| \geq EB_I(f)$ implies $f^u(a)f^d(a) > 0$, using (3). An appendix will treat the case of non-faithful sleeves.

Intuitively, f is nicely behaved when if we restrict f to a neighborhood of a zero ξ where $|f| < EB(f)$. This is illustrated in Figure 1.

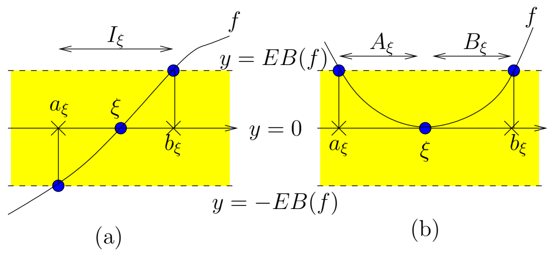

Figure 1: Neighborhood of ξ: $I_\xi = A_\xi \cup \{\xi\} \cup B_\xi$.

Given f and I, define the polynomials

$$\widehat{f}(X) := f(X) - EB_I(f), \qquad \overline{f}(X) := f(X) + EB_I(f).$$

If $\xi \in \mathtt{Zero}_I(f)$, we define the points a_ξ, b_ξ as follows:

$$a_\xi \;:=\; \max\{\{a\} \cup \mathtt{Zero}(\widehat{f} \cdot \overline{f}) \cap (-\infty, \xi)\}, \quad (5)$$

$$b_\xi \;:=\; \min\{\{b\} \cup \mathtt{Zero}(\widehat{f} \cdot \overline{f}) \cap (\xi, +\infty)\}. \quad (6)$$

Then define the open intervals (see Figure 1):

$$A_\xi := (a_\xi, \xi),\ B_\xi := (\xi, b_\xi) \text{ and } I_\xi := (a_\xi, b_\xi). \quad (7)$$

Basic properties of these intervals are captured below. The proofs are omitted.

LEMMA 1. *Let (I, f^u, f^d) be a faithful sleeve for f. For all $\xi, \zeta \in \mathtt{Zero}_I(f)$, we have:*
(i) If $\xi \neq \zeta$ then I_ξ and I_ζ are disjoint.
(ii) $\mathtt{Zero}_I(f^u f^d) \subseteq \bigcup_\xi I_\xi$.
(iii-a) $A_\xi \cap \mathtt{Zero}(f^u)$ is empty iff $A_\xi \cap \mathtt{Zero}(f^d)$ is non-empty.
(iii-b) $B_\xi \cap \mathtt{Zero}(f^u)$ is empty iff $B_\xi \cap \mathtt{Zero}(f^d)$ is non-empty.
(iv) The derivative f' has a constant sign in A_ξ or B_ξ for any $\xi \in \mathtt{Zero}_I(f)$.

If $s, t \in \mathtt{Zero}_I(f^u f^d)$ such that $s < t$ and $(s, t) \cap \mathtt{Zero}_I(f^u f^d)$ is empty, then we call (s, t) a **sleeve interval** of (I, f^u, f^d). The following is immediate from the preceding lemma (iii):

COROLLARY 2. *Each zero of $\mathtt{Zero}_I(f)$ is isolated by some sleeve interval of (I, f^u, f^d).*

Lemma 3. *Let (I, f^u, f^d) be a faithful sleeve. For all $\xi \in \mathtt{Zero}_I(f)$, the multiset $ZERO_{B_\xi}(f^u \cdot f^d)$ has odd size. Similarly, the multiset $ZERO_{A_\xi}(f^u \cdot f^d)$ has odd size.*

It follows from the preceding lemma that for each zero ξ of f, the multiset $\mathtt{ZERO}_{I_\xi}(f^u f^d)$ has even size. Hence the multiset $\mathtt{ZERO}_I(f^u f^d)$ has even size, say $2m$. So we may denote the sorted list of zeros of $\mathtt{ZERO}_I(f^u f^d)$ by

$$(t_0, t_1, \ldots, t_{2m-1}). \tag{8}$$

where $t_0 \le t_1 \le \cdots \le t_{2m-1}$. Note that $t_i = t_{i+1}$ iff t_i is an even zero of $f^u f^d$. Intervals of the form $J_i := [t_{2i}, t_{2i+1}]$ where $t_{2i} < t_{2i+1}$ are called **candidate intervals** of the sleeve. We immediately obtain:

Corollary 4. *Each $\xi \in \mathtt{Zero}_I(f)$ is contained in some candidate interval of a faithful sleeve (I, f^u, f^d).*

Which of these candidate intervals actually contain zeros of f? To do this, we classify a candidate interval $[t_{2j}, t_{2j+1}]$ in (8) into two types:

$$\left.\begin{array}{ll}\text{(Odd):} & t_{2j} \in \mathtt{Zero}(f^d) \text{ iff } t_{2j+1} \in \mathtt{Zero}(f^u) \\ \text{(Even):} & t_{2j} \in \mathtt{Zero}(f^d) \text{ iff } t_{2j+1} \in \mathtt{Zero}(f^d)\end{array}\right\} \tag{9}$$

We call a candidate interval J an **odd** or **even candidate interval** if it satisfies (9)(Odd) or (9)(Even). We now treat the easy case of deciding which candidate intervals are isolating intervals of f:

Lemma 5 (Odd Zero). *Let J be a candidate interval. The following are equivalent:*
(i) J is an odd candidate interval.
(ii) J contains a unique zero ξ of f. Moreover ξ is an odd zero of f.

Proof. Let $J = [t, t']$.
(i) implies (ii): Without loss of generality, let $f^u(t) = 0$ and $f^d(t') = 0$. Thus, $f(t) < 0$ and $f(t') > 0$. Thus f has an odd zero in J. By Corollary 2, we know that candidate intervals contain at most one distinct zero.
(ii) implies (i): Since ξ is an odd zero, we see that f must be monotone over J. Without loss of generality, assume f is increasing. This implies $f^d(t) < 0$ and hence $f^u(t) = 0$. Similarly, $f^u(t') > 0$ and hence $f^d(t') = 0$. Hence J is an odd candidate. **Q.E.D.**

Isolating zeros of even multiplicity is more subtle and will be dealt with in the next section. To do this we need to look at the sign of $\frac{\partial f^u}{\partial X}$ and $\frac{\partial f^d}{\partial X}$. We make a first observation along this line:

Lemma 6. *Let $t_i \in ZERO(f^u f^d)$.*
(a) If t_i is a zero of f^u, then i is even implies $\frac{\partial f^u}{\partial X}(t_i) \ge 0$, and i is odd implies $\frac{\partial f^u}{\partial X}(t_i) \le 0$.
(b) If t_i is a zero of f^d, then i is even implies $\frac{\partial f^d}{\partial X}(t_i) \le 0$. and i is odd implies $\frac{\partial f^d}{\partial X}(t_i) \ge 0$.

2.2 Monotonicity Property

We will exploit a special property of (I, f^u, f^d) for f:

$$\frac{\partial f^u}{\partial X} \ge \frac{\partial f}{\partial X} \ge \frac{\partial f^d}{\partial X} \quad \text{holds in } I \tag{10}$$

We call this the **monotonicity property**. In this subsection, we assume (10) and the faithfulness of the sleeve. We now strengthen one half of Lemma 3 above.

Lemma 7. *For any $\xi \in \mathtt{Zero}_I(f)$, there is a unique zero of odd multiplicity of $f^u \cdot f^d$ in $A_\xi = (a_\xi, \xi)$.*

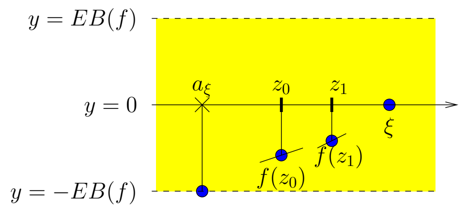

Figure 2: A_ξ has a unique zero of $f^u \cdot f^d$: CASE of $f^u(z_0) = f^u(z_1) = 0$.

Corollary 8. *If t_{2j} is an even zero of $f^u f^d$, then $J_j = [t_{2j}, t_{2j+1}]$ contains no zero of f.*

If t_{2j} is an even zero we have either $t_{2j} = t_{2j+1}$ or $t_{2j} = t_{2j-1}$. But for the former case, (t_{2j}, t_{2j+1}) clearly has no zeros of f. The next result is a consequence of monotonicity and faithfulness:

Lemma 9. *The interval $J_0 = [t_0, t_1]$ is a candidate interval and it isolates a zero of f.*

In Lemma 5, we showed that (9)(Odd) holds iff J_j isolates an odd zero of f. The next result shows what condition must be added to (9)(Even) in order to to characterize the isolation of even zeros.

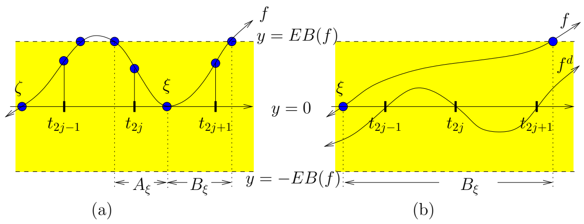

Figure 3: Detection of even zero when $t_{2j}, t_{2j+1} \in \mathtt{Zero}_I(f^d)$: (a) even zero, (b) no zero

Lemma 10 (Even Zero). *Let $J_j = [t_{2j}, t_{2j+1}]$ $(j > 0)$ be an even candidate interval. Then J_j isolates an even zero ξ of f iff (i) $f^d(t_{2j}) = 0$ and $\frac{\partial f^u}{\partial X}$ has real zero in (t_{2j-1}, t_{2j+1}), or (ii) $f^u(t_{2j}) = 0$ and $\frac{\partial f^d}{\partial X}$ has real zero in (t_{2j-1}, t_{2j+1}).*

Note: Since $j > 0$, then t_{2j-1} is a zero of f^d iff t_{2j} is a zero of f^d.
Proof. Let t_{2j} be a zero of f^d. So $f^d(t_{2j+1}) = 0$ and $t_{2j+1} \in B_\xi$ for a zero ξ of f. This means $\frac{\partial f}{\partial X}$ is positive in (ξ, t_{2j+1}). There are two cases: (a) $t_{2j} < \xi < t_{2j+1}$ or (b) $\xi < t_{2j} < t_{2j+1}$. If (a), then $t_{2j-1} \in B_\zeta$ for some zero ζ of f and $\zeta \ne \xi$(see Figure 3(a)). By (3), we have $0 < f^u(t_{2j-1}) < EB(f)$, $0 < f^u(t_{2j}) < EB(f)$. Since $t_{2j-1} \in B_\zeta, t_{2j} \in A_\xi$ and $\zeta \ne \xi$, there exists a point $\eta \in (t_{2j-1}, t_{2j})$ such that $f(\eta) \ge EB(f)$. So $f^u(\eta) > EB(f)$. That means there is an extremum point of f^u in (t_{2j-1}, t_{2j}). That is, there exists

a zero of $\frac{\partial f^u}{\partial X}$ in $(t_{2j-1}, t_{2j}) \subset (t_{2j-1}, t_{2j+1})$. If (b), then $\frac{\partial f^u}{\partial X}(x) > 0$ for all $x \in (t_{2j-1}, t_{2j+1})$ since $\frac{\partial f}{\partial X}$ has constant sign in B_ξ (see Figure 3(b)). We finish the proof. **Q.E.D.**

2.3 Effective Root Isolation of f

So far, we have been treating the roots t_j of $f^u f^d$ exactly. But in our algorithms, we only have numbers in $\mathbb{F}$. We now want to replace t_j by their isolating intervals $[a_j, b_j]$. As usual, we assume that (I, f^u, f^d) is faithful and satisfies the monotonicity property (10). Let $\texttt{ZERO}_I(f^u f^d)$ be the sorted list given in (8), and $[a_i, b_i]$ an isolating interval of t_i, where any two distinct intervals $[a_i, b_i]$ and $[a_j, b_j]$ are disjoint. Let

$$SL_{f,I} = ([a_0, b_0], [a_1, b_1], \ldots, [a_{2m-1}, b_{2m-1}]) \tag{11}$$

be the isolating intervals for roots of $f^u f^d$ in $\texttt{ZERO}_I(f^u f^d)$. Assume that $[a_i, b_i] = [a_j, b_j]$ iff $t_i = t_j$. Note that $t_i = t_j$ implies $|i - j| \leq 1$. Let $K_i := [a_{2i}, b_{2i+1}]$.

By Corollary 8, J_i is not an isolating interval if t_{2i} is an even zero. Hence, we call K_i an **effective candidate** iff $t_{2i} < t_{2i+1}$ and t_{2i} is an odd zero. Thus, K_i contains the candidate interval $J_i = [t_{2i}, t_{2i+1}]$. Furthermore, K_i is called an **effective even candidate** (resp., **effective odd candidate**) if J_i is an even (resp., odd) candidate interval (cf. (9)).

Our next theorem characterizes when K_i is an isolating interval of f. This is the "effective version" of Lemma 5 and Lemma 10. But before this theorem, we provide a useful partial criterion:

LEMMA 11. *Let $K_i = [a_{2i}, b_{2i+1}]$ be an effective even candidate. Then K_i isolates an even zero provided one of the following conditions hold:*
(E')^d: $t_{2i} \in \texttt{Zero}(f^d)$ and $\frac{\partial f^u}{\partial X}$ is negative at a_{2i} or b_{2i},
(E')^u: $t_{2i} \in \texttt{Zero}(f^u)$ and $\frac{\partial f^d}{\partial X}$ is positive at a_{2i} or b_{2i}.

For the even effective candidates, we shall need a **constant sign property**:

$$\left.\begin{array}{l} \text{Let } t_{2j}, t_{2j+1} (j \geq 1) \text{ all be real zeros of } f^u \text{ or } f^d. \\ \quad \text{If } t_{2j}, t_{2j+1} \in \texttt{Zero}(f^d) \text{ then } \frac{\partial f^u}{\partial X} \text{ is positive} \\ \text{in } [a_{2j-1}, b_{2j-1}] \text{ and } [a_{2j+1}, b_{2j+1}]. \\ \quad \text{If } t_{2j}, t_{2j+1} \in \texttt{Zero}(f^u) \text{ then } \frac{\partial f^d}{\partial X} \text{ is negative} \\ \text{in } [a_{2j-1}, b_{2j-1}] \text{ and } [a_{2j+1}, b_{2j+1}]. \end{array}\right\} \tag{12}$$

Note that $t_{2j-1} \in B_\zeta, t_{2j+1} \in B_\xi$ for some $\zeta, \xi \in \texttt{Zero}_I(f)$. And we know $\frac{\partial f^u}{\partial X}(x) > 0$ ($\frac{\partial f^d}{\partial X}(x) < 0$) for all $x \in B_\eta(\eta = \xi, \zeta)$ when $t \in \texttt{Zero}_I(f^d)(t = t_{2j-1}, t_{2j+1})$ ($t \in \texttt{Zero}_I(f^u)$). So the constant sign can be reached. We strengthen this to a necessary and sufficient criterion:

THEOREM 12 (EFFECTIVE ISOLATION CRITERIA). *Let $K_i = [a_{2i}, b_{2i+1}]$ be an effective candidate. If K_i is an even effective candidate, further assume that constant sign property holds. Then K_i is an isolating interval of f iff one of the following conditions hold:*
(O) K_i is an effective odd candidate.
(E): K_i is an effective even candidate and, $i = 0$ or $i > 0$ and $\frac{\partial f^u}{\partial X}$ (resp., $\frac{\partial f^d}{\partial X}$) has some zero in $[b_{2i-1}, b_{2i+1}]$ if f^d (resp., f^u) has two distinct zeros in K_i.

Proof. As a preliminary remark, we note that K_i contains at most one zero of f.

($\Leftarrow$) We first show that (O) or (E) implies that K_i is an isolating interval. Suppose (O) holds. We may assume that f^u has a zero in $[a_{2i}, b_{2i}]$ and f^d has a zero in $[a_{2i+1}, b_{2i+1}]$. Thus $[a_{2i}, b_{2i+1}]$ contains a candidate interval $J_i = [t_{2i}, t_{2i+1}]$ satisfying the conditions of Lemma 5, and J_i has an odd zero of f. Suppose (E) holds. Without loss of generality, assume f^u has two distinct zeros in K_i. If $i = 0$, then clearly, K_i has a zero of f. Otherwise, these zeros must be t_{2i} and t_{2i+1}. By assumption, $\frac{\partial f^d}{\partial X}$ has some zero in $[b_{2i-1}, b_{2i+1}]$; but in fact this zero lies in $[b_{2i-1}, t_{2i+1}] \subseteq J_i$ because $[a_{2i+1}, b_{2i+1}]$ satisfies the constant sign property (12). Now Lemma 10 implies f has some zero in $J_i \subseteq K_i$.

($\Rightarrow$) Suppose f has some zero in K_i. We must show that either (O) or (E) holds. From the definition of K_i, we know there are two distinct roots of $f^u f^d$ in K_i. If $f^u(t_{2i}) = 0$ iff $f^d(t_{2i+1}) = 0$, then clearly (O) holds. Otherwise, $f^d(t_{2i}) = 0$ iff $f^d(t_{2i+1}) = 0$. If $i = 0$, it is clear. If $i \geq 1$, without loss of generality, assume that t_{2i}, t_{2i+1} are zeros of f^d. We must show that $\frac{\partial f^u}{\partial X}$ has some zero z in $[b_{2i-1}, b_{2i+1}]$. By Lemma 10, $\frac{\partial f^u}{\partial X}$ has some zero z in $[t_{2i-1}, t_{2i+1}]$. So it is enough to show that z cannot lie in $[t_{2i-1}, b_{2i-1}]$. But this is a consequence of the constant sign property.

Q.E.D.

We can use Sturm theorem to check whether a polynomial ($\frac{\partial f^u(X)}{\partial X}$ or $\frac{\partial f^d(X)}{\partial X}$) has real root in a given interval. In most cases, we need not to use this since Lemma 11 holds for almost all the cases in practice.

3. BOUNDS OF TRIANGULAR SYSTEM

Consider a triangular polynomial system F_n:

$$F_n = \{f_1(x_1), f_2(x_1, x_2), \ldots, f_n(x_1, \ldots, x_n)\} \tag{13}$$

where $f_i \in \mathbb{Z}[x_1, \ldots, x_i]$. Generalizing our univariate notation, if $B \subseteq \mathbb{R}^n$, let $\texttt{Zero}_B(F_n)$ denote the set of real zeros of F_n restricted to B.

Let $B = I_1 \times \cdots \times I_n$ be an n-dimensional box, $I_i = [a_i, b_i]$, and $\xi = (\xi_1, \ldots, \xi_{n-1}) \in \Box\xi = I_1 \times \cdots \times I_{n-1}$ be a real zero of $F_{n-1} = \{f_1, \ldots, f_{n-1}\} = 0$. Consider

$$f(X) := f_n(\xi_1, \ldots, \xi_{n-1}, X). \tag{14}$$

We have a three-fold goal in this section:
1. Compute lower estimates on $EB_{I_n}(f)$.
2. Compute a sleeve (I_n, f^u, f^d) for f.
3. Compute an upper estimate on $SB_{I_n}(f^u, f^d)$.

3.1 Lower Estimate on Evaluation Bounds

We give two methods to compute lower estimates of $EB_{I_n}(f)$. The first method is based on a general result about multivariate zero bounds in [28]; another is based on resultant computation.

Let $\Sigma = \{p_1, \ldots, p_n\} \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ be a zero dimensional equation system. Let $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ be one of these zeros. Suppose $d_i = \deg(p_i)$ and $K := \max\{\sqrt{n+1}, \|p_1\|_2, \ldots, \|p_n\|_2\}$ where $\|p\|_2$ is the 2-norm of p. Then we have the following result [28, p. 341]:

PROPOSITION 13. *Let $(\xi_1, \ldots, \xi_n)$ be a complex zero of Σ. For any $i = 1, \ldots, n$, if $|\xi_i| \neq 0$ then*

$$|\xi_i| > \mathrm{MRB}(\Sigma) := (2^{3/2} N K)^{-D}\, 2^{-(n+1)d_1 \cdots d_n}. \tag{15}$$

where $N := \binom{1+\sum_{i=1}^n d_i}{n}$, $\quad D := (1 + \sum_{i=1}^n \frac{1}{d_i}) \prod_{i=1}^n d_i$.

Note that this proposition defines a numerical value $MRB(\Sigma)$ (the **multivariate root bound**) for Σ. Given F_n as in (13), consider the polynomial set

$$\widehat{F}_n := \{f_1, \ldots, f_{n-1}, \frac{\partial f_n}{\partial X}, Y - f_n\} \quad (16)$$

in $\mathbb{Z}[x_1, \ldots, x_{n-1}, X, Y]$, where $f_n = f_n(x_1, \ldots, x_{n-1}, X)$.

LEMMA 14. *Use the notations in (14). Let $(\xi_1, \ldots, \xi_{n-1})$ be a zero of F_{n-1}. Then the evaluation bound $\mathrm{EB}_{I_n}(f)$ of $f(X) \in \mathbb{R}[X]$ satisfies $\mathrm{EB}_{I_n}(f) > \mathrm{MRB}(\widehat{F}_n)$.*

It is instructive to directly define the **evaluation bound** of a triangular system F_n: for $B \subseteq \mathbb{R}^n$, let $B' = B \times \mathbb{R}$. Then define $EB_B(F_n)$ to be

$$\min\{|y| : (x_1, \ldots, x_{n-1}, x, y) \in \mathtt{Zero}_{B'}(\widehat{F}_n), y \neq 0\}, \quad (17)$$

assuming $\min\{\} = \infty$. Observe that (17) is a generalization of the corresponding univariate evaluation bound (2). For $i = 2, \ldots, n$, we similarly have evaluation bounds $EB_{B_i}(F_i)$ for F_i, where $F_i = \{f_1, \ldots, f_i\}$.

This multivariate evaluation bound is a lower bound on the univariate one: with f given by (14). Since $MRB(\widehat{F}_n)$ is easily computed, our algorithm can use $MRB(\widehat{F}_n)$ as the lower bound on $EB(F_n)$.

In general, $MRB(\widehat{F}_n)$ is not a good estimation (see Examples in Section 5). We propose a computational way to compute such a lower estimate via resultants. Consider $\widehat{F}_n$ defined by (16). Let

$$e_i = \begin{cases} \mathtt{res}_X(Y - f_n, \frac{\partial f_n}{\partial X}) & i = n, \\ \mathtt{res}_{x_i}(e_{i+1}, f_i) & i = n-1, \ldots, 1 \end{cases} \quad (18)$$

where $\mathtt{res}_x(p, q)$ is the resultant of p and q relative to x. Thus $e_1 \in \mathbb{F}[Y]$. If $e_1 \not\equiv 0$, define

$$R(F_n) := \min\{|z| : e_1(z) = 0, z \neq 0\}.$$

If e_1 has no real roots, let $R(F_n) = \infty$.

LEMMA 15. *If $e_1 \not\equiv 0$, $\mathrm{EB}(F_n) \geq R(F_n)$, and we can use $R(F_n)$ as the evaluation bound.*

Therefore, we may isolate the real roots of $e_1(Y) = 0$ and take $\min\{l_1, -r_2\}$ as the evaluation bound for F_n, where (l_1, r_1) and (l_2, r_2) are the isolating intervals for the smallest positive root and the largest negative root of $e_1(Y) = 0$ respectively.

In fact, the multiresultant can be used to optimize our computation of evaluation bounds (see [3]).

3.2 Sleeve and Sleeve Bound

We assume a positive sign in I_i, that is, $I_i > 0$ for $i = 1, \ldots, n$ and will show how to treat other cases in Section 4.

Given a polynomial $g \in \mathbb{R}[x_1, \ldots, x_n]$, we may decompose it uniquely as $g = g^+ - g^-$, where $g^+, g^- \in \mathbb{R}[x_1, \ldots, x_n]$ each has only positive coefficients, and the support of g^+ and g^- are both minimum. Here, the support of a polynomial g is the set of power products with non-zero coefficients in g.

Given f as in (14) and an isolating box $\square\xi \in \square\mathbb{F}^{n-1}$ for ξ, following [17, 22], we define

$$\begin{aligned} f^u(X) &= f_n^u(\square\xi; X) = f_n^+(\mathbf{b}_{n-1}, X) - f_n^-(\mathbf{a}_{n-1}, X), \\ f^d(X) &= f_n^d(\square\xi; X) = f_n^+(\mathbf{a}_{n-1}, X) - f_n^-(\mathbf{b}_{n-1}, X), \end{aligned} \quad (19)$$

where $f_n = f_n^+ - f_n^-$, $\mathbf{a}_i = (a_1, \ldots, a_i)$, $\mathbf{b}_i = (b_1, \ldots, b_i)$, and $\square\xi = [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}]$.

The bounding functions of the interval function of $f(X)$ are similar to our sleeve polynomials (see [10, 14]). The functions in [27] are not a sleeve but in some special interval, they may have the some properties of our sleeve polynomials.

From the construction, it is clear that $f^u \geq f \geq f^d$. Moreover, both inequalities are strict if $a_i = \xi_i = b_i$ does not hold for any $i = 1, \ldots, n-1$. Hence $(I_n, f^u(X), f^d(X))$ is a sleeve for $f(X)$ [17, 22]. We further have:

LEMMA 16. *Over any $I_n = [l, r] > 0$, we have:*
(i) (Monotonicity) $\frac{\partial f^u}{\partial X} \geq \frac{\partial f}{\partial X} \geq \frac{\partial f^d}{\partial X}$.
(ii) $f^u(X) - f^d(X)$ is monotonously increasing over I_n.

As an immediate corollary, we have

COROLLARY 17. $\mathrm{SB}_{I_n}(f^u, f^d) \leq f^u(r) - f^d(r)$.

Our next goal is to give an upper bound on $f^u(r) - f^d(r)$ as a function of $b := \max\{b_1, \ldots, b_n\}, w := \max\{w_1, \ldots, w_n\}$ where $w_i = b_i - a_i$. Also let $\mathbf{w} = (w_1, \ldots, w_n)$. For $f \in \mathbb{R}[x_1, \ldots, x_n]$, write $f = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}}(x_1, \ldots, x_n)$ where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and $p_{\boldsymbol{\alpha}}(x_1, \ldots, x_n)$ denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let $\|f\|_1 := \Sigma_{\boldsymbol{\alpha}} |c_{\boldsymbol{\alpha}}|$ denote its 1-norm. The inner product of two vectors, say $\mathbf{w}$ and $\boldsymbol{\alpha}$, is denoted $\langle \mathbf{w}, \boldsymbol{\alpha} \rangle = \sum_{i=1}^n w_i \alpha_i$. Let $\mathbf{a}_i = (a_1, \ldots, a_i)$, $\mathbf{b}_i = (b_1, \ldots, b_i)$. We have the following result.

THEOREM 18. *Let (I_n, f^u, f^d) be a sleeve as in (19), and $\square_{n-1}\xi = I_1 \times \cdots \times I_{n-1}$ an isolating box for $\xi \in \mathbb{R}^{n-1}$, where $I_i = [a_i, b_i] > 0$, $I_n = [l, r] > 0$, and $w = \max_{i=1}^{n-1}\{b_i - a_i\}$. Then*

$$\mathrm{SB}_I(f^u, f^d) \leq wm\|f_n\|_1 b^{m-1},$$

where $m = \deg(f_n)$, $b = \max\{b_1, \ldots, b_{n-1}, r\}$.

Proof. Let $f(X) = \sum_{i=0}^m C_i(\xi_1, \ldots, \xi_{n-1})X^i$ where $C_i \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$ has degree $\leq m - i$, $C_i = C_i^+ - C_i^-$, $\mathbf{a} = (a_1, \ldots, a_{n-1})$, and $\mathbf{b} = (b_1, \ldots, b_{n-1})$. We have $f^u(X) = \sum_{i=0}^m (C_i^+(\mathbf{b}) - C_i^-(\mathbf{a}))X^i$, $f^d(X) = \sum_{i=0}^m (C_i^+(\mathbf{a}) - C_i^-(\mathbf{b}))X^i$. For $x \in I_n$, we have

$$\begin{aligned} & f^u(x) - f^d(x) \\ = & \sum_{i=0}^m (C_i^+(\mathbf{b}) - C_i^+(\mathbf{a}) + C_i^-(\mathbf{b}) - C_i^-(\mathbf{a}))x^i \\ \leq & \sum_{i=0}^m w(m-i)b^{m-i-1}(\|C_i^+\|_1 + \|C_i^-\|_1)b^i \\ < & wmb^{m-1}\sum_{i=0}^m \|C_i\|_1 = wmb^{m-1}\|f_n\|_1. \end{aligned}$$

Q.E.D.

We give two corollaries to the above theorem.

COROLLARY 19. *For a fixed F_n and I_n, when $w \to 0$, $\mathrm{SB}_{I_n}(f^u, f^d) \to 0$.*

So when $w \to 0$, $f^u \to f$ and $f^d \to f$, which implies that, with sufficient refinement, the sleeve-evaluation inequality (3) will eventually hold. The next corollary gives an explicit condition to guarantee this:

COROLLARY 20. *The sleeve-evaluation inequality (3) holds if*

$$w < \frac{\mathrm{EB}_{I_n}(f)}{m\|f_n\|_1 b^{m-1}}.$$

4. THE MAIN ALGORITHM

In this section, we present our isolation algorithm: given F_n as in (13), to isolate the real zeros of F_n in a given n-dimensional box $B = I_1 \times \cdots \times I_n$.

4.1 Refinement of Isolating Box

Refining an isolation box is a basic subroutine in our algorithm. Let $\Box_n\xi = \Box_{n-1}\xi \times [c,d] > 0$ be an isolating box for a zero $\xi = (\xi_1, \ldots, \xi_n)$ of F_n=0, $([c,d], f^d, f^u)$ a sleeve associated with $\Box_n\xi$ satisfying (3) and (10), $\Box'_{n-1}\xi$ an isolating box of F_{n-1} satisfying $\Box'_{n-1}\xi \subsetneq \Box_{n-1}\xi$, $f(X) = f_n(\xi_1, \ldots, \xi_{n-1}, X)$, and $\bar{f}^u(X) = f_n^u(\Box'_{n-1}\xi, X)$, $\bar{f}^d(X) = f_n^d(\Box'_{n-1}\xi, X)$ (for definition, see (19)).

LEMMA 21. *Let t_0, t_1 be the real roots of $f^u f^d = 0$ in $[c,d]$ and $t'_0 < t'_1$ the two smallest real roots of $\bar{f}^u \bar{f}^d = 0$ in $[c,d]$. If $\Box'_{n-1} \neq [\xi_1, \xi_1] \times \cdots \times [\xi_{n-1}, \xi_{n-1}]$, then $[t'_0, t'_1] \subset [t_0, t_1]$ and $\xi \in \Box'_{n-1}\xi \times [t'_0, t'_1]$.*

The lemma tells us how to refine an isolating box $K = I_1 \times \cdots \times I_n$ of a triangular system F_n without using Theorem 12. The following algorithm is to refine K of F_n to $\hat{K} = \hat{I}_1 \times \cdots \times \hat{I}_n$ under the precision ϵ.

Refine(F_n, K, ϵ)
Input: F_n, K, ϵ.
Output: $\hat{K} = \hat{I}_1 \times \cdots \times \hat{I}_n$ with $w = \max_{j=1}^n\{|\hat{I}_j|\} \le \epsilon$.
1. If $n = 1$, subdivide I_n until $|I_n| < \epsilon$ and return I_n.
2. Let $K_{n-1} = I_1 \times \cdots \times I_{n-1}$, $w = \max_{j=1}^n\{|I_j|\}$.
   If $w \le \epsilon$, return K. Else, $\delta = \epsilon$.
3. while $w > \epsilon$, do
   3.1. $\delta = \delta/2$.
   3.2. $K_{n-1} :=$ Refine(F_{n-1}, K_{n-1}, δ).
   3.3. If K_{n-1} is a point, $f(X) = f_n(\xi_1, \ldots, \xi_{n-1}, X)$ $\in \mathbb{F}[X]$. Isolate its roots under ϵ, return them.
   3.4. Compute the sleeve:
   $f^u(X) := f_n^u(K_{n-1}, X), f^d(X) := f_n^d(K_{n-1}, X)$.
   3.5. Isolate the roots of $f^u f^d$ in I_n with precision δ.
   3.6. Denote the first two intervals as $[c_1, d_1], [c_2, d_2]$.
   3.7. $w := d_2 - c_1$.
4. Return $\hat{K} := K_{n-1} \times [c_1, d_2]$.

4.2 Verifying Zeros

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a real root of the triangular system $\Sigma_k = \{h_1, \ldots, h_k\}$, $B = I_1 \times \cdots \times I_k$ an isolating box of α, and $g(x_1, \ldots, x_k) \in \mathbb{Z}[x_1, \ldots, x_k]$. We show how to check whether $g(\alpha_1, \ldots, \alpha_k) = 0$.

We call $\rho = \min\{|g(\alpha)| : g(\alpha) \neq 0, \forall \alpha \in \mathtt{Zero}_B(\Sigma_k)\}$ the *zero bound* of g on Σ_k. Let

$$\Sigma_B = \{h_1, \ldots, h_k, Y - g\}. \tag{20}$$

We have two methods to compute the zero bound. First, by Proposition 13, $MRB(\Sigma_B)$ can be taken as the zero bound. Second, we may compute the zero bound by resultant computation. Let $r_{k+1} = Y - g(x_1, \ldots, x_k)$ and $r_i = \mathtt{res}(h_i, r_{i+1}, x_i)$ for $i = k, \ldots, 1$. Then $r_1(Y)$ is a univariate polynomial in Y. If $r_1 \not\equiv 0$, chose a lower bound ρ for all the absolute values of the nonzero real roots of r_1. It is clear that ρ is smaller than the absolute value of any nonzero root of $r_1(Y) = 0$.

We give the following algorithm.

ZeroTest($\Sigma_n, B, g(x_1, \ldots, x_n)$)
Input: $\Sigma_n, B = I_1 \times \cdots \times I_n, g(x_1, \ldots, x_n)$.
Output: True if $g(\alpha) = 0$ or FALSE otherwise.
1. $\delta = \max_{j=1}^n\{|I_j|\}$.
2. Compute bound ρ similar to a sleeve of g:
   $g^u = g^+(b_1, \ldots, b_n) - g^-(a_1, \ldots, a_n)$,
   $g^d = g^+(a_1, \ldots, a_n) - g^-(b_1, \ldots, b_n)$.
3. If $g^d = g^u$, then $g = g^d = g^u$.
   If $g^d = 0$ return TRUE; else return FALSE. end
4. If $g^u g^d \ge 0$, then $g \neq 0$ and return FALSE. end
5. Compute the zero bound ρ if it does not exist.
6. If $|g^u| < \rho$, and $|g^d| < \rho$, then $g < \rho$ and hence $g = 0$ and return TRUE. end
7. $\delta = \delta/2$, $B =$ Refine(Σ_n, B, δ), and goto step 2.

4.3 Isolation Algorithm

We now give the real root isolation algorithm RootIsol for a triangular system.

RootIsol
Input: F_n,$B_n = \prod_{i=1}^n I_i (I_i = [l_i, r_i] > 0)$, $\epsilon > 0$.
Output: An isolating set $\Box\mathtt{Zero}_{B_n}(F_n)$.

1. Compute $\Box\mathtt{Zero}_{B_1}(F_1)$ for F_1 to precision ϵ.
   $Result := \Box\mathtt{Zero}_{B_1}(F_1)$. $New := \emptyset$.
   If $Result = \emptyset$, return $Result$, end
2. For i from 2 to n, do
   2.1. Compute $EB_i := EB(F_i)$ for F_i.
   2.2. $\delta := \epsilon$.
   2.3. while $Result \neq \emptyset$, do
      2.3.1. Choose an element $\Box_{i-1}\xi$ from $Result$.
      $Result := Result \setminus \{\Box_{i-1}\xi\}$.
      2.3.2. $f(X) = f_i(x_1, \ldots, x_{i-1}, X) = \sum_k c_k X^k$.
      If ZeroTest($F_{i-1}, \Box_{i-1}\xi, c_k$) =TRUE for all c_k then F_n is nonzero dimensional. end
      2.3.3. Compute the sleeve:
      $f^u(X) = f_i^u(\Box_{i-1}\xi, X)$,
      $f^d(X) = f_i^d(\Box_{i-1}\xi, X)$.
      2.3.4. While $f^u(r_i) - f^d(r_i) \ge EB_i$,
         $\delta := \delta/2$.
         $\Box_{i-1}\xi :=$ Refine($F_{i-1}, \Box_{i-1}\xi, \delta$).
         Recompute $f^u(X)$ and $f^u(X)$.
      2.3.5. Isolate the real roots of $f^u f^d$ in I_i.
      2.3.6. Compute the parity of these roots.
      2.3.7. Construct the effective candidate intervals.
      2.3.8. for each effective candidate interval K,
         2.3.8.1. Check whether K is isolating.
            If K is odd, K is isolating;
            If K is even:
               If Lemma 11 holds, K is isolating;
               Else, ensure (12). K is isolating iff Theorem 12 (E) holds.
         2.3.8.2. If K is isolating, then
            $K :=$ Refine(F_i, K, ϵ).
            $New := New \bigcup \{\Box_{i-1}\xi \times K\}$.
   2.4. If $New = \emptyset$, return New, end
   2.5. $Result := New$. $New := \emptyset$.
3. return $Result$.

Remarks: Algorithm RootIsol can be improved or made more complete in the following ways.

- The assumption $B_n > 0$ is reasonable. If we want to obtain the real roots of f in the interval $I = (a,b) < 0$, we may consider $g(X) = f(-X)$ in the interval $(-b, -a)$. If $0 \in (a,b)$, we can consider the two parts, $(a,0)$ and $(0,b)$ respectively, since we can check if 0 is a root of $f(X) = 0$.
- If we want to find all real roots of f, we first isolate the real roots of f in $(0,1)$, then isolate the real roots

of $g(X) = X^n * f(1/X)$ in $(0, 1)$, and check whether 1 is a root of f. As a result, we can find all the roots of $f(X) = 0$ in $(0, +\infty)$. We can find the roots of $f(X) = 0$ in $(-\infty, 0)$ by isolating the roots of $f(-X) = 0$ in $(0, +\infty)$. Finally, check whether 0 is a root of $f(X) = 0$.

- Theorem 12 assumes that the sleeves are faithful (see (4)). In fact, if we replace $EB_I(f)$ with

$$ET_I(f) := \min\{|f(z)| : z \in \mathtt{Zero}_I(f') \cup \{a, b\} \backslash \mathtt{Zero}_I(f)\}, \quad (21)$$

then almost all the sleeve (I, f^u, f^d) is faithful except for the cases $f(a) = 0$ or $f(b) = 0$. If $f(a) = 0$ or $f(b) = 0$, we can ignore the first or last element in $SL_{f,I}$ to form effective candidate intervals of f. When $f(a) = 0$, the first effective candidate interval may or may not be the isolating interval of f, we need to check it by Theorem 12. And we need to use the first isolating interval in $SL_{f,I}$ to decide whether the first effective candidate interval is isolating if the first three elements in $SL_{f,I}$ are all isolating intervals of f^u (or f^d).

  Although we can simply solve the non-faithful problem as mentioned above, when $f(a)$ or $f(b)$ is very close but not equal to 0, $ET_I(f)$ is very small. It is expensive to construct (I, f^u, f^d) in order to satisfy the sleeve-evaluation inequality (3). In order to avoid this case, we just use $EB_I(f)$ directly and deal with the non-faithful sleeve case as in the appendix.

4.4 Examples and Experimental Results

We first gave two working examples. The timings are collected on a PC with a 3.2G CPU, 512M memory, and Windows OS.

Example 1: Consider the system $F_2 = \{f_1, f_2\}$ where

$$f_1 = x^4 - 3x^2 - x^3 + 2x + 2,$$
$$f_2 = y^4 + xy^3 + 3y^2 - 6x^2y^2 + 4xy + 2xy^2 - 4x^2y + 4x + 2.$$

Set the precision to be 2^{-4}. Isolating the real roots of f_1 to precision 2^{-4}, we obtain the following isolating intervals: $[[\frac{-23}{16}, \frac{-11}{8}], [\frac{-5}{8}, \frac{-9}{16}], [\frac{11}{8}, \frac{23}{16}], [\frac{25}{16}, \frac{13}{8}]]$. Next consider $\Box_1\xi = [\frac{11}{8}, \frac{23}{16}]$, where $\xi \in \mathtt{Zero}(f_1)$. We will isolate the real roots of $f_2(\xi, y) = 0$ in $[0, 2]$.

We derive $EB_2 = \frac{1}{2}$ by resultant computation. The sleeve computed using the interval $\Box_1\xi$ is

$$f^u(y) = -\frac{175}{32}y^2 - \frac{29}{16}y + y^4 + \frac{23}{16}y^3 + \frac{31}{4},$$
$$f^d(y) = -\frac{851}{128}y^2 - \frac{177}{64}y + y^4 + \frac{11}{8}y^3 + \frac{15}{2}.$$

The sleeve bound of $([0, 2], f^u, f^d)$ is $SB = f^u(2) - f^d(2) = \frac{59}{8}$. Since the sleeve-evaluation inequality (3) does not hold, we refine $\Box_1\xi$. Let $\Box_1\xi = \text{Refine}(f_1, \Box_1\xi, \frac{1}{2^8}) = [\frac{181}{128}, \frac{363}{256}]$. We have the new sleeve

$$f^u(y) = -\frac{50475}{8192}y^2 - \frac{9529}{4096}y + y^4 + \frac{363}{256}y^3 + \frac{491}{64},$$
$$f^d(y) = -\frac{204331}{32768}y^2 - \frac{39097}{16384}y + y^4 + \frac{181}{128}y^3 + \frac{245}{32}$$

with sleeve bound $SB = f^u(2) - f^d(2) = \frac{949}{2048} < \frac{1}{2} = EB_2$. The sleeve $([0, 2], f^u, f^d)$ is faithful (4) since $f^u(0) = \frac{491}{64} > \frac{1}{2}$, $f^d(0) = \frac{245}{32} > \frac{1}{2}$, $f^u(2) = \frac{2927}{512} > \frac{1}{2}$, $f^d(2) = \frac{10759}{2048} > \frac{1}{2}$. Isolating $f^u f^d$ in $[0, 2]$ to precision 2^{-8}, we obtain $SL_{f_2,[0,2]}$: $[[\frac{165}{128}, \frac{331}{256}], [\frac{395}{256}, \frac{99}{64}]]$ both with parities 1. These intervals are both isolating intervals of f^d. It forms an isolating interval of $f_2(\xi, y)$ by Lemma 9. So there is an even root of $f_2(\xi, y)$ in $[0, 2]$ by Theorem 12. It is in $[\frac{165}{128}, \frac{99}{64}]$. So $[\frac{11}{8}, \frac{23}{16}] \times [\frac{165}{128}, \frac{99}{64}]$ is an isolating box of triangular system F_2.

The isolating box does not satisfy our output precision requirement. Refine the isolating box with Refine, we obtain $[\frac{181}{128}, \frac{5793}{4096}] \times [\frac{1423}{1024}, \frac{2947}{2048}]$.

Eventually, we obtain all the isolating boxes for $F_2 = 0$ in 0.141s with RootIsol. If using Proposition 13 to compute $MRB(F_2)$, we have $MRB(F_2) > \frac{1}{2^{289}}$ and the computing time is 9.282s.

Example 2: Consider the following system from [10].

$$f_1 = -12z^2 - 3yz + xz - 27z - 4y^2 - 11xy - 5y + 29x^2 + 11x - 27;$$
$$f_2 = -25z^2 - 23yz + 23xz + 4z + 2y^2 + 7xy + 21y + 4x^2 - 15x - 30;$$
$$f_3 = -14z^2 + 27yz - 29xz + 11z + 4y^2 - 31xy + 22y - 12x^2 - 28x - 9.$$

We first transform the system to a triangular system with WSolve package ([24]) in 0.141s. The isolating time for the roots of the triangular system under the precision 2^{-20} is 0.406s. The C program in [10] uses 0.62s on a SUN4 with a 400 MHz CPU and 2GB of memory.

We implemented RootIsol in Maple 10 and tested our program with three sets of examples. The coefficients of the polynomials are within −100 to 100. The precision is $\frac{1}{2^{10}}$. We use the method mentioned in the **Remarks** for RootIsol to compute *all the real solutions*. We estimate the evaluation bounds by resultant computation. The most time-consuming parts are the computation of the evaluation bounds and the refinement for the isolating boxes.

The first set of examples are sparse polynomials and the results are given in Table 1. The *type* of a triangular system $F_n = \{f_1, \ldots, f_n\}$ is a list $(d_1, \ldots, d_n)$ where $d_i = \deg_{x_i}(f_i)$. The column started with TYPE is the type of the tested triangular systems. TIME is the average running time for each triangular system in seconds. NS is the average number of real solutions for each triangular system. NT is the number of tested triangular systems. NE is the number of terms in each polynomial.

TYPE	TIME	NS	NT	NE
(3, 3)	0.04862	2.04	100	(4, 10)
(9, 7)	0.52717	3.99	100	(10, 10)
(21, 21)	108.9115	5.45	20	(10, 10)
(3, 3, 3)	0.15783	3.48	100	(4, 10, 10)
(9, 7, 5)	16.20573	8.36	100	(10, 10, 10)
(3, 3, 3, 3)	1.69115	5.64	100	(4, 10, 10, 10)
(3, 3, 3, 3, 3)	159.1199	8.0	10	(4, 10, 10, 10, 10)

Table 1: Timings for sparse triangular systems

The second set of examples are dense polynomials and the results are in Table 2. A triangular system $F_n = \{f_1, \ldots, f_n\}$ of type $(d_1, \ldots, d_n)$ is called *dense* if $f_i = \sum_{k=0}^{d_i} c_k x_i^k$ and $\deg_{x_j}(c_k) = d_j - 1$ for all k and $i > j$.

TYPE	TIME	NS	NT	NE
(3, 3)	0.05355	1.91	100	(3.99, 8.02)
(9, 8)	1.87486	4.26	100	(9.94, 43.98)
(11, 11)	8.782	4.5	80	(11.975, 72.5)
(16, 14)	50.22	6.0	100	(16.9, 127.13)
(21, 15)	164.23	6.22	100	(21.91, 176.8)
(3, 3, 3)	0.387	2.91	100	(3.99, 7.77, 13.01)
(5, 4, 4)	2.97	4.88	100	(5.99, 14.72, 24.24)
(5, 5, 5)	33.22	5.61	80	(5.9, 17.7, 42.1)
(8, 7, 6)	592.18	7.6	10	(8.9, 36.0, 79.8)
(3, 3, 3, 3)	119.94	6.96	50	(4.0, 8.1, 12.8, 20.9)
(5, 5, 5, 3)	551.44	3.4	10	(6.0, 32.1, 42.3, 21.5)

Table 2: Timings for dense triangular systems

The third set of examples are triangular systems with multiple roots and the results are given in Table 3. A triangular system of type $(d_1, \ldots, d_n)$ is generated as follows: f_1 is a random polynomial in x_1 and with degree d_1 in x_1 and $f_i = a_i^2(b_i x_i + c_i)^{\lfloor \frac{d_i+1}{2} \rfloor - \lfloor \frac{d_i}{2} \rfloor}$ for $i = 2, \ldots, n$, where a_i is a random polynomial in $x_1, \ldots, x_i$ and with degree $\lfloor d_i/2 \rfloor$ in x_i, b_i, c_i are random polynomials in $x_1, \ldots, x_{i-1}$, and $\lfloor d \rfloor$ is the maximal integer which is less than d. In Table 3, NM is the average number of multiple roots for the tested systems.

TYPE	TIME	NS	NM	NT	NE
(5, 5)	0.712	3.71	1.57	100	(5.9, 34.4)
(9, 8)	0.604	3.1	3.1	100	(9.9, 18.9)
(13, 11)	32.44	6.55	3.92	100	(13.9, 107.6)
(23, 21)	466.0	6.15	3.75	20	(24.0, 183.4)
(3, 3, 3)	3.213	5.59	3.24	100	(3.9, 13.0, 31.7)
(9, 7, 5)	425.9	12.95	8.15	20	(9.9, 60.8, 100.3)
(3, 3, 3, 3)	130.6	11.15	6.1	20	(4.0, 12.2, 33.7, 62.9)

Table 3: Timings for dense triangular systems

From the above experimental results, we could conclude that our algorithm is capable of handling quite large triangular systems.

5. CONCLUSION

This paper provides a complete numerical algorithm of isolating the real roots for arbitrary zero-dimensional triangular systems. The key idea is to use a sleeve satisfying the the sleeve-evaluation inequality to isolate the roots for a univariate polynomial with algebraic number coefficients. Even with our current simple implementation, the algorithm is shown to be quite effective. To solve larger problems, the bottle neck of the algorithm is the computation of the evaluation bound. It is worth exploring sharper evaluation bounds or new methods that use alternative bounds.

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* QECAD means Quantifier Elimination and Cylindrical Algebraic Decomposition.

The appendix is omitted in this abstract.