Assignments

1. Let R be a Euclidean domain, $f_1, \ldots, f_n \in R$, and $h = \gcd(f_1, \ldots, f_n)$.
   - Prove that there exist $s_1, \ldots, s_n \in R$ such that $s_1 f_1 + \cdots + s_n f_n = h$.
   - Prove
     - The linear Diophantine equation
       $$s_1 f_1 + \cdots + s_n f_n = a \tag{1}$$
       is solvable for $s = (s_1, \ldots, s_n) \in R^n$ if and only if $\gcd(f_1, \ldots, f_n)$ divides a.
     - If $s \in R^n$ is a solution of $s \cdot f + t \cdot g = a$, then the set of all integral solutions is $s + U$.
     - $U = Ru_2 + \cdots + Ru_n$, where
       $$u_i = \left(\frac{f_i}{h_i} s_{i1}, \ldots, \frac{f_i}{h_i} s_{i,i-1}, -\frac{h_{i-1}}{h_i}, 0, \ldots, 0 \right),$$
       $$h_{i-1} = \gcd(f_1, \ldots, f_{i-1}) = s_{i1} f_1 + \ldots + s_{i,i-1} f_{i-1}$$
       for $2 \leq i \leq n$ and $h_n = \gcd(f_1, \ldots, f_n)$, with suitably chosen $s_{ij} \in R$.
   - Let $l = \operatorname{lcm}(f_1, \ldots, f_n)$. Show that if $R = F[x]$ for a field F, $h \neq 0$, and $\deg a < \deg l$, then there exist $s_1, \ldots, s_n \in R$ solving (1) such that $\deg s_i < \deg l - \deg f_i$.