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# Determination of approximate symmetries of differential equations

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Abstract. There has been considerable progress in the theory and computer implementation of symbolic computation algorithms to automatically determine and exploit exact symmetries of exact differential equations. Such programs usually apply a finite number of exact differentiations and eliminations to the overdetermined linearized equations for the unknown symmetries (the symmetry defining equations), to complete them to certain involutive or standard forms. The symmetry properties can be determined from these involutive forms.

In many applications, however, the differential equations describing a model are only known approximately. For example they may contain parameters that are only known approximately. Symbolic methods are unstable if applied to the symmetry defining equations directly, and indirect techniques (e.g. replacing approximate parameters by symbolic ones) might not be practical in cases with many parameters. Discussion and examples are given of such difficulties.

A new generation of symbolic-numeric methods is described and applied to the problem of determining symmetries of differential equations.

We introduce a class of differential-elimination methods which uses Numerical Linear Algebra, and in particular the Singular Value Decomposition, to perform the elimination process on the symmetry defining equations. Our approach uses symbolic differentiations but not symbolic eliminations. Substitution of an appropriate random point in the independent variables, followed by numerical projection is used to test for the conditions of completion to a projective involutive form. We prove that this form is equivalent to the involutive form of the Cartan-Kuranishi theory of partial differential equations.

Our method is applied to determining symmetry properties of 50 ODE from the collection in Kamke's book.

## 1. Introduction

The continuous (exact) point symmetries of a differential equation with independent variables $x$ and dependent variables $y$ are the group of transformations

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$g$:

$$g : \hat{x} = X(x, y),\ \hat{y} = Y(x, y) \tag{1}$$

leaving invariant its solution set $S$:

$$g(S) = S \tag{2}$$

Applications of symmetries to differential equations include determination of invariant solutions, mappings from intractable to tractable equations (e.g. non-linear to linear), and the determination of conservation laws. They have become one of the major tools for determining analytic features and solutions of differential equations [**1, 11**].

The (exact) symmetries of a differential equation are usually not known a priori and have to be determined. By associating the differential equation with a submanifold $\mathcal{M}_S$ of a *Jet space*, the condition is replaced with

$$pr(g)\,(\mathcal{M}_S) = \mathcal{M}_S \tag{3}$$

where the group has been prolonged to act on derivatives up to the order of the differential equation.

The Infinitesimal Lie symmetries are the linearized form of such symmetries about the identity transformation:

$$\hat{x} = X(x, y; \epsilon) = x + \epsilon\xi(x, y) + \mathcal{O}(\epsilon^2), \hat{y} = Y(x, y; \epsilon) = y + \epsilon\eta(x, y) + \mathcal{O}(\epsilon^2) \tag{4}$$

In particular, the $\mathcal{O}\epsilon$) terms of condition (3) leads to a linear homogeneous system of over-determined partial differential equations on the function $\xi$ and $\eta$ [**1, 11**], which we call the symmetry defining system.

There are many computer algebra programs for automatically generating such symmetry defining systems (see the review article by Hereman [**8**]). They can be large and highly over-determined. Indeed such systems can be simplified to canonical forms by a finite number of differentiations and eliminations. This process is a differential generalization of Gröbner Bases [**2, 10, 15, 18, 20**].

The canonical forms can then be used for explicit integration of the system and a number of symbolic computation approaches have been based on this method. Other approaches [**27**] take advantage of a mixture of the two approaches.

Such symbolic differential-elimination algorithms tend to explode in memory on some non-trivial examples, reflecting their underlying exponential complexity. Moreover they are not guaranteed to deal with models with floating point numbers in them. This motivated us to develop a symbolic-numeric differential elimination algorithms by the fact that often on complicated examples,

An underlying principle of our approach, is our strong emphasis on geometry. In particular we emphasize jet space geometry, the geometry of differential systems [**13, 15, 20, 25, 7**]. In comparison to the symbolic differentiation-elimination approaches, where symbolic or algebraic manipulations of the equations are emphasized, our approach focuses on the solutions of the system regarded as algebraic equations. This is the jet space picture of a differential equation, in which one regards all the appearing variables (derivatives etc.) as formal unknowns on an equal footing.

Our approach is based on combining techniques from numerical analysis and symbolic computation. This work falls under the new area of hybrid symbolic-numeric computation. Symbolic elimination algorithms (e.g. the Gaussian elimination and Buchberger algorithms) can be executed with systems with floating point arithmetic. However, the output is not well defined (for example the Buchberger algorithm relies on using a field and floating point numbers do not form a field). Moreover, due to round-off errors, the output is almost always unstable with respect to small changes in the coefficients of the input systems.

Our approach uses only symbolic differentiation, and avoids symbolic elimination, in an attempt to address the significant expression swell seen in many symbolic elimination algorithms. In contrast, symbolic differentiation is relatively cheap. Techniques from Numerical Linear Algebra [**6**], and in particular the Singular Value Decomposition (SVD) are used. It was first outlined briefly in [**26**].

Our approach is motivated by the geometric differential elimination methods that arose in the approaches of Cartan and Kuranishi. This work has also been influenced by [**19**] in which an approach is proposed for ordinary differential systems that generate prime differential ideals. That approach uses symbolic differentiations, and avoids nonlinear eliminations, instead using linear eliminations based on substitution of a random point.

In this paper we concentrate on symmetries of second order ordinary differential equations of the form

$$y_{xx} = f(x, y, y_x) \tag{5}$$

Extensive studies have been made of this class of equations about of its exact symmetry starting with Lie and equivalence properties dating back to Tresse, Cartan and others [**12**]. Kamke's book [**9**] lists ODE of equations which are significant for their analytic properties, including exact solution etc.

First we introduce (exact) Jet geometry, and the symbolic (exact) method underlying our completion method. Theorems are given on the relation between our method and the classical Cartan-Kuranishi method. The approximate implementation of this method which uses the Singular Value Decomposition is discussed.

We use 50 ODE taken from Kamke [**9**] as a test bed for our symbolic-numeric methods.

Finally we mention that this work is part of a series in which we are developing the theory and applications of symbolic-numeric methods for general systems of differential equations. An early example of this series [**16**] has been that of constant coefficient linear homogeneous PDE in particular as used in the problem of determining a camera's orientation from reference data (the *camera pose problem* an important problem in computer vision). In that paper, we showed experimentally that the methods presented here, could stably predict orientations, even in singular configurations which had been difficult for other methods in the literature.

In another work [**14**], we build on the new methods of Numerical Algebraic Geometry due to Sommese, Verschelde and Wampler [**23**], and applied these methods to the problem of completion of systems of ordinary differential equations with constraints.

In the current work, we focus on linear homogeneous PDE with variable coefficients. Ultimately we believe that efficient algorithms for symbolic-numeric treatments of nonlinear differential systems, will contain features of the linear ones presented here (since nonlinear systems become linear in their highest derivatives when differentiated), and nonlinear methods for the lower order equations based in spirit on the methods of Sommese, Verschelde and Wampler.

## 2. Symmetries and Introductory Examples

We illustrate some of the difficulties and associated with symbolic methods when applied to cases containing floating point numbers. In particular consider:

$$y_{xx} + 6.708203932\, yy_x + 5\, y^3 = 0 \tag{6}$$

which will be used to motivate the need to have truly symbolic-numeric methods for determining approximate features of systems.

**2.1. Defining system for symmetries of $\mathbf{y_{xx} + 6.708203932\, yy_x + 5\, y^3 = 0}$.** Lie point symmetries of (6) are transformations of form $(x, y) \mapsto (\hat{x}, \hat{u}) = (x + \xi(x, y)\epsilon + \mathcal{O}(\epsilon^2), y + \eta(x, y)\epsilon + \mathcal{O}(\epsilon^2))$. They generate a Lie algebra of vector fields $\xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$ which follows the flow of the group. The symmetry defining system of (6) is:

$$\begin{aligned}
&\xi_{yy} = 0,\\
&\eta_{yy} - 2.0\,\xi_{xy} + 13.41640787\, y\xi_y = 0,\\
&2.0\,\eta_{xy} - 1.0\,\xi_{xx} + 15.0\, y^3\xi_y + 6.708203928\, y\xi_x + 6.708203932\,\eta = 0,\\
&\eta_{xx} - 5.0\, y^3\eta_y + 10.0\, y^3\xi_x + 6.708203932\, y\eta_x + 15.0\, y^2\eta = 0.
\end{aligned} \tag{7}$$

where subscript notation has been used for partial derivatives. This system has been produced by the Maple command `DEtools[odepde]`.

There are two differential elimination packages in Maple, `RifSimp` and `diffalg`. The theory that both are based on assume that the coefficients in the input system to be from a field. Since floating point numbers do not form a field (e.g. the associative property fails), computations with floats with these packages are not guaranteed to produce a theoretically well-founded result.

**2.2. Direct use of floats in exact differential elimination methods.** The package `diffalg` will process floats as floats in its internal computations, and when applied to system (7) yields:

$$\xi_x = 0.0,\ \xi_y = 0.0,\ \eta = 0.0 \tag{8}$$

which represents a 1 parameter translation symmetry in $x$ for (6). The Maple package `RifSimp` will by default, convert floats into their exact rational approximations, in the input system, and this strategy is discussed next.

**2.3. Rational Substitution Strategy.** A common strategy, for symbolic programs is to replace input floats with rational numbers. Executing this strategy with system (7) (so that 6.708203928 is replaced with $6.708203928 = 6708203928/10^9$ etc.) both the packages `RifSimp` and `diffalg` output the same simplified system:

$$\xi_x = 0,\ \xi_y = 0,\ \eta = 0, \tag{9}$$

which clearly has a one parameter solution space. In the formation of the defining system (7) small round-off errors have been introduced (e.g. 6.708203928 appears in one coefficient and 6.708203932 in another). So an alternative strategy is to make the rational replacement $6.708203932 = 6708203932/10^9$ in the original ODE. The result of this strategy is that both Maple packages produce simplified form of the defining system as:

$$\xi_{xx} = 0,\ \xi_y = 0,\ \eta = -y\xi_x\ . \tag{10}$$

Integration easily shows that $\xi = ax+b$, $\eta = -ay$. These correspond to the obvious symmetries of translation in $x$ and a mutual scaling in $(x, y) \mapsto (x/c, cy)$ possessed by the input ODE. As the above example illustrates the symbolic simplification methods are not continuous with respect to small changes in the coefficients. The method can, for this example, be regarded as more successful since it found a larger group of symmetries of (6) than the previous method.

**2.4. Symbolic Substitution Strategy.** A common strategy by symmetry and computer algebra researchers to address difficulties faced when using floats is to replace such quantities with symbolic parameters (or in general with unspecified functions).

In particular when this is done with the above example, it is embedded in the one parameter class of ODE:

$$y_{xx} + \alpha y y_x + 5y^3 = 0 \tag{11}$$

Application of `RifSimp` and `diffalg` which can be called using the convenient interface `DEtools[casesplit](sys, option)` with `option = rif` or `option = diffalg` yields:

Case 1 ($\alpha^2 \neq 45$):

$$\xi_{xx} = 0,\ \xi_y = 0,\ \eta = -y\xi_x \tag{12}$$

and Case 2 ($\alpha^2 = 45$):

$$\begin{aligned}
&\eta = \frac{2}{15}\,\xi_{xxxy} + 2\,y^2\xi_{xy} - y\xi_x - \frac{2}{15} y\,\alpha\xi_{xxy} + \frac{1}{15}\,\alpha\xi_{xx} - \frac{1}{3}\,\alpha\xi_y y^3,\\
&\xi_{xxxx} = \frac{30\,y^2}{\alpha}\xi_{xxxy} - \frac{30\,y}{\alpha}\xi_{xxx},\\
&\xi_{yy} = 0.
\end{aligned} \tag{13}$$

Further algorithms can determine that the dimension of the solution space for Case 2 is eight dimensional. Thus ODE (11) has an 8 dimensional Lie group of symmetries when $\alpha^2 = 45$ and a 2 dimensional group when $\alpha^2 \neq 45$. ODE with larger symmetry groups are amenable to a wider class of analytical techniques. For example second order ODE are linearizable [**12**] if and only if they have an eight dimensional group. Hence the ODE (6) is very close to a linearizable ODE.

**2.5. Discussion of difficulties with symbolic approaches.** As the above examples illustrate, care has to be taken with symbolic approaches. They are not designed for using floating point numbers and can produce unstable results when used with numeric coefficients. Strategies such as rational replacement need to be used with care, since round-off errors can mean that the rational approximation does not inherit desired properties of the original system. Further the more general approach of symbolic replacement, although powerful in certain situations, can be impractical due to the greater complexity of the calculations involved.

In summary it is useful to explore approaches that more thoroughly integrate symbolic and numeric methods, and such methods need to consider *close-by* systems. For example it would be desirable for a symbolic-numeric approach to be able to determine that the case $\alpha = \sqrt{45} \approx 6.708203932$ was close by with a desirably large (8) dimensional symmetry group characterizing a linearizable ODE.

We could compare our goal to the problem of finding rank deficient matrices $\tilde{A}$ near a matrix $A$. This linear algebra problem is well known to be solved using the Singular Value Decomposition (SVD). For more details on SVD, see [**24**] or [**6**]. Given a tolerance $\epsilon$, the SVD of $A$ can determine the lowest rank (equivalently largest dimension for Null($A$)) that the family of matrices $\tilde{A}$ satisfying $\|\tilde{A}-A\|_2 \leq \epsilon$ can have. Thus the SVD gives information on how close rank deficient matrices are to $A$.

The objective of the work initiated in this article is to stably seek close-by systems admitting large symmetry groups in the presence of round off errors from the original system.

## 3. Jet Space and Involution

In this section we describe the symbolic algorithm underlying our symbolic-numeric method. It depends heavily on the methods of Jet Space Geometry.

The most basic step in the theory of differential elimination algorithms is to replace differential equations by algebraic ones, by regarding derivatives as formal variables.

A polynomially nonlinear $q$-th order system with $n$ independent variables $x = (x_1, x_2, ..., x_n)$ and $m$ dependent variables $(\Psi^1, \Psi^2, ..., \Psi^m)$ is represented as a polynomial in $x, u, \underset{1}{u}, \underset{2}{u}, \ldots$ Equivalently and more formally it is represented as an element of the differential ring $\mathbb{F}[x, u, \underset{1}{u}, \underset{2}{u}, ...]$ where the field $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$ and

$$\begin{aligned} u = \underset{0}{u} \quad &\leftrightarrow \quad (\Psi^1, \Psi^2, ..., \Psi^m), \\ \underset{1}{u} \quad &\leftrightarrow \quad \frac{\partial \Psi^l}{\partial x_j}, \\ \underset{2}{u} \quad &\leftrightarrow \quad \frac{\partial^2 \Psi^l}{\partial x_j \partial x_k}, \\ &\ldots \end{aligned}$$

The formal total derivative is:

$$D_{x_j} = \frac{\partial}{\partial x_j} + \sum_l u^l_{x_j} \frac{\partial}{\partial u^l} + ... \tag{14}$$

For sake of notational brevity, it has become customary to use the same variable names as the names of the variables in the original physical formulation of the differential system. We follow this convention in our paper.

Thus we consider systems of total derivative order $q$, of form $R^1 = 0, ..., R^s = 0$, or more concisely $R = 0$, where

$$R^k : J^q \to \mathbb{C}, \quad J^q = \mathbb{C}^{N_q}, \tag{15}$$

and $N_q = n + m \begin{pmatrix} q+n \\ q \end{pmatrix}$, is the number of jet variables of order less than or equal to $q$.

The spirit of the geometric approach to differential equations is that it is concerned with the (jet) variety of the system:

$$\mathcal{R} = V(R) := \{(x, u, \underset{1}{u}, ..., \underset{q}{u}) \in J^q : R^k(x, u, \underset{1}{u}, ..., \underset{q}{u}) = 0\}, \tag{16}$$

where $\underset{r}{u}$ represents the $r$th order derivatives. If all of the equations in the system have derivative of order exactly $q$ then a single symbolic prolongation of the system is taken to be:

$$D(R) := \{(x, u, \underset{1}{u}, ..., \underset{q+1}{u}) \in J^{q+1} : R^k = 0,\ D_{x_i} R^k = 0\}. \tag{17}$$

If there are equations with derivative order less than $q$ then to form $DR$ derivatives of these equations are appended to the system, and this process continued until no undifferentiated equations of lower order remain.

The prolongations of a system can be obtained more efficiently by omitting the obviously equivalent equations resulting from the commutativity of partial differentiation. For example, given $\frac{\partial R^1}{\partial x_1}$, $\frac{\partial R^1}{\partial x_2}$, in the first prolongation, $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}R^1$ only needs to be listed once in the second prolongation. These and many more refined efficiency improvements are discussed in [**20**].

The corresponding geometric operation to elimination is that of geometric projection. A single geometric projection is defined as:

$$E(R) := \{(x, u, \underset{1}{u}, ..., \underset{q-1}{u}) \in J^{q-1} : R^k(x, u, \underset{1}{u}, ..., \underset{q-1}{u}, \underset{q}{u}) = 0\}. \tag{18}$$

We call the systems: $ER$, $E^2R$, ... , $E^qR$ projected systems. They are of respective orders: $q-1, q-2, \ldots, 0$.

The symbol of a $q$th order system is the linearized highest order part of the system, which is given by the matrix:

$$\text{Symbol } R := \frac{\partial R}{\partial \underset{q}{u}}. \tag{19}$$

The symbol of a $q$th order system $R = 0$ is involutive if

$$\text{rank Symbol}(DR) = \Sigma_{j=1}^{n} j \beta_j^{(q)} \tag{20}$$

and the $\beta_j^{(q)}$ are the dimensions of certain subspaces of the Null Space of the Symbol of $R$. Details of this test are given in [**20, 13**].

Definition 1 (Involutive System). *A differential system of order $q$ is said to be involutive* [**13, 20**]*, in a $\delta$-regular system of coordinates if it passes the elimination test: $(E \circ D)R = R$, has involutive symbol and satisfies the constant rank conditions given below. Almost all coordinate systems are $\delta$-regular, and can be achieved by a random (linear) change of coordinates if necessary.*

Definition 2 (Constant Rank Conditions). *Let $R(x, \underset{q}{v}) = 0$ be a $q$th order system with independent variables $x$ and jet variables $\underset{q}{v} = (u, \underset{1}{u}, ..., \underset{q}{u})$ corresponding to dependent variables and their derivatives. Then this system satisfies the constant rank conditions at $(x^0, \underset{q}{v}^0) = (x^0, u^0, \underset{1}{u}^0, ..., \underset{q}{u}^0) \in V(R) \subseteq J^q$ if there exist nonzero constants $\alpha, \beta$ such that*

$$rank \frac{\partial R(x, \underset{q}{v})}{\partial (x, \underset{q}{v})} = \alpha = rank \frac{\partial R}{\partial \underset{q}{v}},\ \ rank \frac{\partial R}{\partial \underset{q}{u}} = \beta \tag{21}$$

*in a neighbourhood of $(x^0, \underset{q}{v}^0)$ in the usual Euclidean norm. We call $(x^0, \underset{q}{v}^0)$ a Euclidean point.*

Involution enables a local existence and uniqueness theorem (the Cartan-Kähler Theorem) to be stated for the system [**3**]. It has been used to determine consistent initial conditions for the numerical solution of systems.

In this article we confine ourselves, unless otherwise stated, to linear homogeneous systems of PDE. The constant rank conditions are then satisfied automatically except at a lower ($< n$) dimensional set of non-Euclidean points in the space of independent variables. The more complicated situation encountered in non-linear systems, of decomposition into components of different dimensions does not occur. As a special case of the Cartan-Kuranishi prolongation theorem [**3**] we have:

THEOREM 3.1 (Cartan-Kuranishi Prolongation Theorem). *At regular points a linear homogeneous system becomes involutive after a finite number of projections and prolongations. That is given an input system* $R = 0$ *there exist* $C_r, ...C_2, C_1$ *with* $C_j = E$ *or* $C_j = D$ *for each* $j$*:*

$$C_r ... C_2 C_1(R) \tag{22}$$

*is an involutive system of order greater than or equal to* $q$ *except at a lower* ($< n$) *dimensional set of non-Euclidean points in the space of independent variables.*

Typical symbolic implementations to obtain involutive systems try to keep the order of the systems as low as possible through use of symbolic elimination (as is the case with the *RifSimp* algorithm). Symbolic implementations of involutive form algorithms use Gröbner Bases, symbolic Gauss Elimination or Ritt reduction to perform the symbolic eliminations [**20**].

The approach which will be described here does not use symbolic elimination, and instead constructs prolonged systems $D^k R$ and finds a $l$ which for some $k$, yields $E^l D^k R$ as an involutive system (in other words we always apply prolongations first). To prove that such a $k, l$ exist we need the following easily derived results concerning dimensions and permuting $D$ and $E$.

THEOREM 3.2 (Monotonicity Properties of D and E). *Given a system* $R$ *the following properties hold:*

$$R \subseteq S \Rightarrow DR \subseteq DS \text{ and } ER \subseteq ES; \tag{23}$$

$$EDR \subseteq V(R),\ EDR \subseteq DER; \tag{24}$$

$$\dim E^{l+1} D^k R \leq \dim E^l D^k R,\ \ \dim E^{l+1} D^{k+1} R \leq \dim E^l D^k R. \tag{25}$$

PROOF. The property $R \subseteq S \Rightarrow DR \subseteq DS$ and $ER \subseteq ES$ follows directly from the definitions of $D$ and $E$.

Note that $EDR = \{(x, u, \underset{1}{u}, ..., \underset{q}{u}) \in J^q : R(x, u, \underset{1}{u}, ..., \underset{q}{u}) = 0, D_{x^j} R = 0\} \subseteq V(R)$, so we have the first property in (24). So given $(x, u, \underset{1}{u}, ..., \underset{q}{u}) \in EDR$ then $(x, u, \underset{1}{u}, ..., \underset{q}{u}) \in V(R)$, so $(x, u, \underset{1}{u}, ..., \underset{q-1}{u}) \in ER$. Therefore for some $\underset{q+1}{w}$ we have $(x, u, \underset{1}{u}, ..., \underset{q}{u}, \underset{q+1}{w}) \in DER$. Thus $EDR \subseteq DER$ and the second property of (24) is proved.

We have $\dim E^{l+1} D^k R \leq \dim E^l D^k R$ since $\dim ES < \dim S$ for any system $S$. Let $S = D^k R$, then since $EDS \subseteq V(S)$ from (24) we have $\dim EDS \leq \dim S$. By repeated application of this property we obtain $\dim E^{l+1} DS \leq \dim E^l S$. Consequently we have the monotonicity property $\dim E^{l+1} D^{k+1} R \leq \dim E^l D^k R$. □

THEOREM 3.3 (Projected Involutive Systems). *For a given linear homogeneous system of* PDE *there exist non-negative integer $l$, and an integer $0 \leq k \leq l$ such that $E^k D^l R$ is involutive except at a lower ($< n$) dimensional set of non-Euclidean points in the space of independent variables.*

PROOF. Applying theorem 3.1, there exist $C_1, \ldots, C_r$ where each $C_j$ is either $D$ or $E$ such that $C_r \cdots C_2 C_1(R)$ is an involutive system. Let $k = \#j$ such that $C_j = D$ and $l = \#j$ such that $C_j = E$ then $0 \leq l \leq k$ since the total order of the system $q + k - l \geq q$.

From the repeated application of the permutation rules (24) we have

$$E^l D^k R \subseteq C_r ... C_2 C_1 R. \tag{26}$$

We will prove that $E^l D^k R = C_r ... C_2 C_1 R$. Suppose by contradiction that $E^l D^k R \neq C_r ... C_2 C_1 R$ and let $w \in C_r ... C_2 C_1 R \backslash E^l D^k R$. Since $w \in C_r ... C_2 C_1 R$ by the Cartan-Kähler Theorem [**3**] there exists a local analytic solution passing through $w$. However such a solution must also satisfy $E^l D^k R$. But $w \notin E^l D^k R$ contradicting the existence of a local solution. Hence:

$$E^l D^k R = C_r ... C_2 C_1 R \tag{27}$$

is an involutive system. □

The symbolic algorithm on which our symbolic-numeric method is based is to determine the $k, l$ whose existence is given above, such that $E^l D^k R$ is involutive. However the classical tests for involution, for example the classical elimination test:

$$\dim EDE^l D^k R = \dim E^l D^k R \tag{28}$$

together with the classical involutive symbol test, depend on systems which are not in the form $E^s D^r R$. Our need to develop tests for involution based solely on systems of the form $E^s D^r R$ motivates us to define a projectively involutive system in the following manner.

DEFINITION 3 (Projectively Involutive System). *A differential system $E^l D^k R$ is said to be projectively involutive if it passes the projective elimination test:*

$$\dim E^{l+1} D^{k+1} R = \dim E^l D^k R \tag{29}$$

*and passes the projective involutive symbol test*

$$rank\, E^l D^{k+1} R = \sum_j j \beta_j \tag{30}$$

*where the $\beta_j \equiv \beta_j^{(q+k-l)}$ are the characters for the $(q+k-l)$-th order system $E^l D^k R$ in a $\delta$-regular system of coordinates.*

THEOREM 3.4. *If a system is projectively involutive if and only if it is involutive.*

PROOF. Suppose that $E^l D^k R$ is projectively involutive.

Then the properties (24) and (26) imply:

$$\dim E^{l+1} D^{k+1} R \leq \dim EDE^l D^k R \leq \dim E^l D^k R \tag{31}$$

and consequently from (29) that

$$\dim EDE^l D^k R = \dim E^l D^k R. \tag{32}$$

So the elimination test for classical involution is satisfied.

Next we want to show that the projective involutive symbol test (30) implies the classical involutive symbol test (20). Rewriting the ranks in terms of co-dimensions we wish to show that

$$N_{q+k+1-l} - \dim E^l D^{k+1} R = \sum_j j\beta_j \Rightarrow N_{q+k+1-l} - \dim DE^l D^k R = \sum_j j\beta_j \tag{33}$$

In $\delta$-regular coordinates, $N_{q+k+1-l} - \dim DE^l D^k R \geq \sum_j j\beta_j$, so we only need to show that $N_{q+k+1-l} - \dim DE^l D^k R \leq \sum_j j\beta_j$, or equivalently that $N_{q+k+1-l} \leq \dim DE^l D^k R + \sum_j j\beta_j$. Now

$$N_{q+k+1-l} = \dim E^l D^{k+1} R + \sum_j j\beta_j \leq \dim DE^l D^k R + \sum_j j\beta_j \tag{34}$$

by using the permutation properties of dimensions and consequently the symbol is involutive.

Suppose that $E^l D^k R$ is involutive. Then $\dim EDE^l D^k R = \dim E^l D^k R$. Also $\dim E^{l+1} D^{k+1} R \leq \dim EDE^l D^k R = \dim E^l D^k R$.

Then repeated application of the properties (24):

$$E^l D^{k+1} R \subseteq DE^l D^k R \tag{35}$$

Then $DE^l D^k R$ is also involutive since any prolongation of an involutive system is also involutive [**13**].

We will prove that $E^l D^{k+1} R = DE^l D^k R$. Suppose by contradiction that $E^l D^{k+1} R \neq DE^l D^k R$ and let $w \in DE^l D^k R \backslash E^l D^{k+1} R$. Since $w \in DE^l D^k R$ by the Cartan-Kähler Theorem [**3**] there exists a local analytic solution passing through $w$. However such a solution must also satisfy $E^l D^{k+1} R$. But $w \notin E^l D^{k+1} R$ contradicting the existence of a local solution. Hence:

$$E^l D^{k+1} R = DE^l D^k R. \tag{36}$$

Consequently applying $E$ to (36) gives

$$E^{l+1} D^{k+1} R = EDE^l D^k R. \tag{37}$$

Since $EDE^l D^k R = E^l D^k R$ we have from (37) that $\dim E^{l+1} D^{k+1} R = \dim E^l D^k R$ and so the projective elimination test is satisfied.

From (37) we have $\operatorname{rank} E^l D^{k+1} R = \operatorname{rank} DE^l D^k R$, so $E^l D^k R$ is projectively involutive.

□

In summary the symbolic algorithm on which our symbolic-numeric method is based is as follows. A $q$-th order system is prolonged (differentiated) until a projection of the prolonged system (of order $\geq q$) satisfies the projected elimination test. If any of these projections also satisfy the projected involutive symbol test, then we have found an involutive system (without loss we can choose the minimum order ($\geq q$) such system).

## 4. Symbol-Numeric Differential Elimination Method

The method is outlined for linear homogeneous systems. A linear homogeneous differential system can be written in the matrix form:

$$B^{(q)}(x) \underset{q}{v} = \mathbf{0}, \quad \underset{q}{v} = \begin{pmatrix} \underset{q}{u} \\ \cdot \\ \cdot \\ \underset{1}{u} \\ u \end{pmatrix}. \tag{38}$$

For example, consider again the determining system from our simple introductory example (7). When we write this system in matrix form, rounding the floats for compactness of presentation, it becomes:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 13.416y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -5.0y^3 & 10.0y^3 & 6.708y & 0 & 15.0y^2 \\ 0 & 0 & 0 & 2 & -1 & 1 & 15.0y^3 & 0 & 6.708y & 0 & 0 & 6.708 \end{bmatrix} \underset{2}{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\underset{2}{v}$ is the transpose of the vector $[\xi_{yy}, \eta_{yy}, \xi_{xy}, \eta_{xy}, \xi_{xx}, \eta_{xx}, \xi_y, \eta_y, \xi_x, \eta_x, \xi, \eta]$.

Successive prolongations (differentiations) of the system yield $R, DR, D^2R, ...$, and a sequence of corresponding linear homogeneous matrix systems:

$$B^{(q)}(x) \underset{q}{v} = \mathbf{0}, \;\; B^{(q+1)}(x) \underset{q+1}{v} = \mathbf{0}, \;\; B^{(q+2)}(x) \underset{q+2}{v} = \mathbf{0}, ... \tag{39}$$

where the zero vector in the right hand side of each system has the appropriate dimension for that system.

For linear homogeneous systems our hybrid symbolic-numeric approach is to choose a random point $x = x^0$, and substitute it into the sequence of systems above. This yields a sequence of constant matrix systems:

$$B^{(q)}(x^0) \underset{q}{v} = \mathbf{0}, \;\; B^{(q+1)}(x^0) \underset{q+1}{v} = \mathbf{0}, \;\; B^{(q+2)}(x^0) \underset{q+2}{v} = \mathbf{0}, ... \tag{40}$$

For each prolongation, $D^kR$, the projected systems $E^lD^kR$, $l = 0, 1, ..., k$, are numerically constructed, by replacing the symbolic projection operator $E$ with a numeric projection operator $\hat{E}$. This results in the family of systems $\hat{E}^l(D^k(R))$.

The numerical implementation of the projected involution test is briefly discussed. In the numerical implementation, the symbolic elimination operator $E$ is replaced with a numerical projection $\hat{E}$. We first find the singular value decomposition (SVD) of $B^{(q+k)}(x^0)$ at a random point $x^0$:

$$B^{(q+k)}(x^0) = U\Sigma V^t \tag{41}$$

using one of the available numerical packages (in our case the NAG library). Here $U$ and $V$ are unitary matrices.

$\Sigma$ is a diagonal matrix whose diagonal entries, called *singular values*, are real decreasing non-negative numbers $\sigma_1 \geq \sigma_2 \geq \ldots$. The submatrix of $V^t$ obtained by deleting the first rank$(B^{(q+k)}(x^0)$ rows of $V^t$ is a basis for the Null Space of $B^{(q+k)}(x^0)$.

When dealing with numeric coefficients, the matrices $B^{(q+k)}(x^0)$ contain floating point numbers. Instead of computing an exact rank, we compute an approximate rank by computing the SVD $B^{(q+k)}(x^0) = U\Sigma V^t$. The approximate rank $r$ is the number of singular values bigger thant a fixed tolerance.

Deleting the first $r$ rows of $V^t$ yields an approximate basis for the Null Space of $B^{(q+k)}(x^0)$. This yields an estimate for dim $(D^k R)$. To estimate $\dim(\hat{E}(D^k R))$ the components of the approximate basis for $D^k R$ corresponding to the highest order ($(q+k)$th order) derivatives are deleted. This projected basis yields an approximate spanning set for $\hat{E}(D^k R)$. Proceeding in the same way, deleting components corresponding to highest order jet variables from the approximate spanning set just obtained, yields an approximate spanning set for $\hat{E}^2(D^k R)$, and then for $(\hat{E}^3(D^k R))$, etc. Application of the SVD to each of these approximate spanning sets yields the approximate dimensions of $E(D^k R), E^2(D^k R), E^3(D^k R), ....$ In this way the dimensions necessary for the application of the approximate projected elimination test (29) are determined.

To execute the projected involutive symbol test (20) bases for the spaces $E(D^k R)$ are first constructed as above. Then the subspaces of their symbols are extracted by using the SVD based Subspace Intersection Algorithm described in [**6**]. The integer from the left hand side of this test is easily approximated using the methods discussed above. The approximation of the $\beta_k^{(q)}$ is more complicated. In this case, the symbol, or rather approximate bases for each of the subspaces corresponding to the $\beta_k^{(q)}$ are determined by further use of the SVD based Subspace Intersection Algorithm [**6**].

## 5. Examples

**5.1. $\mathbf{y_{xx} + 6.708203932\, y y_x + 5\, y^3 = 0}$.** When the symbolic-numeric method is applied to (7) we get the table of dimensions:

**Table 1: $\mathbf{\dim \hat{E}^l D^k R}$ for (6)**

| | $k=0$ | $k=1$ | $k=2$ |
|---|---|---|---|
| $l=0$ | 8 | 8 | 8 |
| $l=1$ | 6 | $\underline{8}$ | 8 |
| $l=2$ | | 6 | 8 |
| $l=3$ | | | 6 |

We have calculated slightly more than needed here, by calculating an additional projection. This enables us to calculate the symbol dimensions via:

$$\dim(\text{Symbol } E^l(D^k R)) = \dim E^l(D^k R) - \dim E^{l+1}(D^k R) \tag{42}$$

When $\dim(\text{Symbol } E^l(D^k R)) = 0$, or equivalently when that the symbol has full rank, it is easily shown that the symbol of $E^l(D^k R)$ is involutive [**13**], so that the projected involutive symbol test amounts to testing:

$$\dim E^l(D^k R) = \dim E^{l+1}(D^k R) \tag{43}$$

in this case (which occurs in this paper).

We seek the smallest $k$ such that there exists an $l = 0, ..., k$ with $\hat{E}^l D^k R$ approximately involutive (choosing the largest such $l$ if there are several such values for the given $k$). Passing the projected elimination test (29), amounts to finding the first column with an equal entry in the next column entry with an neighboring equal entry diagonally downwards to its right with both entries being on or above the main diagonal $k = l$. This first occurs for $k = 0$ and $l = 0$. However in this case the version of the involutive symbol test in the form (43) is not passed.

At the next prolongation $k = 1$, the approximate projected elimination test is passed when $l = 0$, $l = 1$. Examining the dimensions of the symbol shows that the involutive symbol test is passed when $l = 1$ (the underlined entry in Table 1). Consequently $EDR$ is approximately involutive, and we expect approximately an 8 dimensional solution space and symmetry group. This result coincide with the dimension expected for the case $\alpha = \sqrt{45}$ whose approximate value is 6.708203932.

This example shows that the algorithm was able to determine the dimension of the symmetry group of the the close-by equation $y_{xx} + \sqrt{45} y y_x + 5y^3 = 0$. However, to be able to speak properly of close-by systems, we would need to introduce a notion of distance between systems. This challenging topic is the focus of a forth-coming paper.

**5.2. $\mathbf{y_{xx} + 7.1\, y y_x + 5\, y^3 = 0}$.** Another case is if we choose a different value of the parameter, say $\alpha = 7.1$ then for

$$y_{xx} + 7.1\, y y_x + 5\, y^3 = 0 \tag{44}$$

we obtain using the symbolic-numeric method:

**Table 2: $\dim \hat{\mathbf{E}}^{\mathbf{l}} \mathbf{D}^{\mathbf{k}} \mathbf{R}$ for (44)**

| | $k=0$ | $k=1$ | $k=2$ | $k=3$ | $k=4$ | $k=5$ |
|---|---|---|---|---|---|---|
| $l=0$ | 8 | 8 | 6 | 3 | 2 | 2 |
| $l=1$ | 6 | 8 | 6 | 3 | 2 | 2 |
| $l=2$ | | 6 | 6 | 3 | 2 | 2 |
| $l=3$ | | | 4 | 3 | 2 | 2 |
| $l=4$ | | | | 3 | $\underline{\mathbf{2}}$ | 2 |
| $l=5$ | | | | | 2 | 2 |
| $l=6$ | | | | | | 2 |

Applying the same process as in Section 5.1, we conclude that $\hat{E}^l D^k R$ is a approximately involutive for $k = 4$ and $l = 4$ (the underlined entry in Table 2). Consequently $E^4 D^4 R$ is approximately involutive, and we expect approximately a 2 dimensional solution space and symmetry group. This result is consistent with the one we found by the symbolic substitution strategy, namely that the case $\alpha \neq \sqrt{45}$, there should be a 2 dimensional symmetry group.

**5.3. Test Examples from the Kamke Collection.** Kamke [**9**] provided a large collection of ODE with interesting features (usually amenable to some analytical solution technique). This set has become popular as a suite for testing the effectiveness of ODE symbolic solving software. Many of the ODE have symmetries, and systematic methods using invariants of the symmetries, can be used to assist in integrating them. Thus they are also used as a suite to bug-check and test software to reduce the overdetermined systems for the symmetries of the ODE.

We took, fairly randomly, 50 ODE from Kamke's collection, and generated the overdetermined systems of PDE for their symmetries using the Maple's `DEtools[odepde]` command. ODE with parameters in them, had integer values substituted for the parameters, since our method does not currently allow for parameters.

The 50 overdetermined systems were symbolically reduced (there were no floating point quantities in the systems), without difficulty by the Maple differential elimination package *RifSimp*. Using the *RifSimp* function *initialdata*, the number of parameters in the solution spaces of the systems (i.e. the dimensions of their symmetry groups) were determined. Additionally the dimensions of the projected spaces both for the system and its symbol were constructed at involution, symbolically, to provide exact data to test the output of our symbolic-numeric method.

We briefly mention that one of the methods we used to detect numerical instability, was lack of monotonicity in the tables of dimensions for the projected systems and their symbol spaces. In particular the dimensions of the projected systems must decrease (non-strictly) from on downwards sloping diagonals and also from top to bottom along the columns of the projected dimension table.

Using a tolerance of $10^{-9}$, and performing calculations with Maple's Digits := 15, we applied our symbolic-numeric method to the 50 examples.

In the initial versions of the program, 45 of the 50 cases ran successfully. The remaining 5 cases exhibited instability for the symmetries of ODE $24, 26, 27, 28, 33$ had high degree monomial terms. ODE 26 $y'' + y' + 2\,x^3 y^8 = 0$, had the highest degree monomial (an 11th degree monomial) in Table 3. We conjectured that the instability was due to either very large (or very small) entries in the matrices after substitution of the random points with values greater than 1 (less than 1). This would disguise linear dependencies and independencies in the system.

This motivated us to substitute random points of modulus 1 in the complex exponential form $\exp(\mathrm{i}\theta_j)$, and in some cases random real numbers in a sufficiently small real interval about 1. These changes removed the instability observed in the remaining cases and gave results consistent with those determined by the (exact) symbolic differential elimination algorithms.

**Table 3:** Statistics for application of Symbolic-Numeric Method to 50 Kamke ODE where $k$, $l$ are integers such that $E^l D^k R$ is approximately involutive. $\dim(\mathcal{G})$ is the dimension of the symmetry group of the ODE calculated using the symbolic numeric results.

| | ODE | $k$ | $l$ | $\dim(\mathcal{G})$ |
|---|---|---|---|---|
| 1 | $y'' - y^2 = 0$ | 5 | 5 | 2 |
| 2 | $y'' - 6\,y^2 - x^4 = 0$ | 7 | 7 | 0 |
| 3 | $y'' - 6\,y^2 - x = 0$ | 7 | 7 | 0 |
| 4 | $y'' - 6\,y^2 + 4\,y = 0$ | 6 | 6 | 1 |
| 5 | $y'' + y^2 + 2\,x + 3 = 0$ | 7 | 7 | 0 |
| 6 | $y'' - 2\,y^3 - yx + 1 = 0$ | 5 | 5 | 0 |
| 7 | $y'' - y^3 = 0$ | 4 | 4 | 2 |
| 8 | $y'' - 2\,y^3 + 4\,yx - 2 = 0$ | 5 | 5 | 0 |
| 9 | $y'' + 4 + 2\,yx + 3\,y + y^3 = 0$ | 5 | 5 | 0 |
| 10 | $y'' + 4 + 2\,y^2 + 3\,y + y^3 = 0$ | 4 | 4 | 1 |
| 11 | $y'' + x^4 y^9 = 0$ | 4 | 4 | 1 |
| 12 | $y''\,y - 1 = 0$ | 4 | 4 | 2 |
| 13 | $y''\,y = 0$ | 1 | 0 | 8 |
| 14 | $y''\,y - x^2 = 0$ | 4 | 4 | 1 |
| 15 | $2(1 + x^2)y'' - xy'^2(x + 4y') + 2(x + y')y' - 2y = 0$ | 4 | 4 | 0 |
| 16 | $y''' - \left(1 - x^3 y'\right)^3 = 0$ | 2 | 2 | 1 |
| 17 | $y'' - \frac{y'^2}{y+4} - \frac{y'}{x+2} = 0$ | 1 | 0 | 8 |
| 18 | $y' - y = 0$ | 0 | 0 | $\infty$ |
| 19 | $y'''' + 9\,y = 0$ | 2 | 2 | 6 |
| 20 | $y'' - 3\,y' - y^2 - 2\,y = 0$ | 6 | 6 | 1 |
| 21 | $y'' - 7\,y' - y^3 + 12\,y = 0$ | 4 | 4 | 1 |
| 22 | $y'' + 5\,y' - 6\,y^2 + 6\,y = 0$ | 5 | 5 | 2 |
| 23 | $y'' - 3\,y' - 2\,y^3 + 2\,y = 0$ | 4 | 4 | 2 |
| 24 | $y'' - \frac{7}{2}\,y' - \frac{45}{16}\,y\left(y^9 - 1\right) = 0$ | 4 | 4 | 1 |
| 25 | $y'' + y' + 2\,y^8 = 0$ | 4 | 4 | 1 |
| 26 | $y'' + y' + 2\,x^3 y^8 = 0$ | 5 | 5 | 0 |
| 27 | $x^4 y'' + y^8 = 0$ | 4 | 4 | 1 |
| 28 | $x^4 y'' - x\left(x^2 + 2\,y\right)y' + 4\,y^2 = 0$ | 4 | 4 | 2 |
| 29 | $x^4 y'' - x^2\left(x + y'\right)y' + 4\,y^2 = 0$ | 4 | 4 | 1 |
| 30 | $x^4 y'' + \left(xy' - y\right)^3 = 0$ | 1 | 0 | 8 |
| 31 | $y'' + yy' - y^3 + y = 0$ | 4 | 4 | 1 |
| 32 | $y'' + (y + 3)\,y' - y^3 + y^2 + 2\,y = 0$ | 4 | 4 | 2 |
| 33 | $x^4 y'' + \left(x^4 y + 3\,x^3\right)y' - x^4 y^3 + x^3 y^2 + 2\,x^2 y = 0$ | 4 | 4 | 1 |
| 34 | $y'' + 2\,yy' + xy' + y = 0$ | 5 | 5 | 0 |
| 35 | $y'' + 2\,yy' + x\left(y' + y^2\right) - 1 = 0$ | 5 | 5 | 0 |
| 36 | $y'' + 3\,yy' + y^3 + yx - 1 = 0$ | 1 | 0 | 8 |
| 37 | $y'' + (3\,y + x)\,y' + y^3 + xy^2 = 0$ | 1 | 0 | 8 |
| 38 | $y'' - 3\,yy' - 3\,y^2 - 4\,y - 2 = 0$ | 4 | 4 | 1 |
| 39 | $y'' - (3\,y + x)\,y' + y^3 + xy^2 + x^2 y + x^4 = 0$ | 1 | 0 | 8 |
| 40 | $y'' - 2\,yy' = 0$ | 4 | 4 | 2 |
| 41 | $y'' + yy' + 2\,y^3 = 0$ | 4 | 4 | 2 |
| 42 | $y'' + x^2 y' + x^3 = 0$ | 1 | 0 | 8 |
| 43 | $y'' + y'^2 + 2\,y = 0$ | 4 | 4 | 1 |
| 44 | $y'' + y'^2 + 2\,y' + 3\,y = 0$ | 4 | 4 | 1 |
| 45 | $y'' + y'^2 + 2\,y' + 3\,y^3 = 0$ | 4 | 4 | 1 |
| 46 | $y'' + y'^2 + 2\,x^3 y = 0$ | 5 | 5 | 0 |
| 47 | $y'' + y'^2 + 2 = 0$ | 1 | 0 | 8 |
| 48 | $y'' + yy'^2 + 2\,y = 0$ | 4 | 4 | 1 |
| 49 | $y'' + x^2 y'^2 + y' = 0$ | 4 | 4 | 1 |
| 50 | $4\,x^2 y'' - x^4 y'^2 + 4\,y = 0$ | 4 | 4 | 1 |

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