QUANTUM STRASSEN'S THEOREM

SHMUEL FRIEDLAND, JINGTONG GE, AND LIHONG ZHI

Abstract. Strassen's theorem circa 1965 gives necessary and sufficient conditions on the existence of a probability measure on two product spaces with given support and two marginals. In the case where each product space is finite Strassen's theorem is reduced to a linear programming problem which can be solved using flow theory. A density matrix of bipartite quantum system is a quantum analog of a probability matrix on two finite product spaces. Partial traces of the density matrix are analogs of marginals. The support of the density matrix is its range. The analog of Strassen's theorem in this case can be stated and solved using semidefinite programming. The aim of this paper is to give analogs of Strassen's theorem to density trace class operators on a product of two separable Hilbert spaces, where at least one of the Hilbert spaces is infinite dimensional.

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1. Introduction

1.1. **Finite dimensional marginals problems.** Let μ be a probability measure on the discrete space $\Omega = [m] \times [n]$, where m, n are positive integers and $[m] = \{1, \ldots, m\}$. A probability measure on Ω is a nonnegative $m \times n$ matrix $A = [a_{ij}] \in \mathbb{R}_+^{m\times n}$, such that the sum of its entires is 1. Let $\mathbf{1}_m = (1, \ldots, 1)^\top \in \mathbb{R}^m$. Then the marginals: $\mu_1 = A\mathbf{1}_n$ and $\mu_2 = A^\top \mathbf{1}_m$ are the probability measures on $[m]$ and $[n]$ respectively. The support of μ, denoted as $\operatorname{supp} \mu$, is the following bipartite graph $G = (V, E)$, where $V = [m] \cup [n]$ and $E = \{(i, j), i \in [m], j \in [n], a_{ij} > 0\}$. The following inverse problem is natural:

Problem 1.1. Given probability measures μ_1 and μ_2 on $[m]$ and $[n]$ respectively, find necessary and sufficient conditions for existence of a probability measure μ on $[m] \times [n]$, whose support is contained in a given bipartite graph $G = ([m] \cup [n], E)$ and whose marginals are μ_1 and μ_2.

This problem is a classical problem in combinatorial optimization [8], and can be solved using the standard flow theory [11]. See [16]. Strassen [25] gave a solution of Problem 1.1 to a measure on the Borel σ-algebra of the product of two compact metric spaces. (Strassen did not bother to state the finite space case. Actually, Strassen considered a more general $\varepsilon \geq 0$ version of Problem 1.1.)

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We now consider the analog of Problem 1.1 in the quantum setting: Let $\mathcal{H}$ be a finite dimensional inner product space of dimension n over the complex numbers $\mathbb{C}$. We identify $\mathcal{H}$ with $\mathbb{C}^n$ with the standard inner product $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{y}^*\mathbf{x}$. Then $\mathrm{B}(\mathcal{H})$, the set of linear operators $A : \mathcal{H} \to \mathcal{H}$, is the algebra of $n \times n$ matrices $\mathbb{C}^{n\times n}$. The set of selfadjoint operators $\mathrm{S}(\mathcal{H})$ is the real space of $n \times n$ Hermitian matrices. Then $\mathrm{S}_+(\mathcal{H}) \supset \mathrm{S}_{+,1}(\mathcal{H})$ is the cone of positive semidefinite matrices in $\mathrm{S}(\mathcal{H})$ and the convex set of positive semidefinite matrices of trace 1. On $\mathrm{S}(\mathcal{H})$ we have a partial order $A \succeq B$ if $A-B \in \mathrm{S}_+(\mathcal{H})$. The set $\mathrm{S}_{+,1}(\mathcal{H})$, which is called the space of density matrices, is the analog of the set of probability measure in quantum physics. For $\rho \in \mathrm{S}(\mathcal{H})$ the support of ρ, denoted as $\operatorname{supp} \rho$, is $\rho(\mathcal{H})$, i.e. the subspace spanned by the eigenvectors of ρ corresponding to nonzero eigenvalues, which is the range of ρ. A density matrix $\rho \in \mathrm{S}_{+,1}(\mathcal{H})$ is called pure state, it is rank one matrix, i.e. it has one positive eigenvalue equal to 1. Equivalently, $\dim \sup \rho = 1$.

Let $\mathcal{H}_1 \equiv \mathbb{C}^m, \mathcal{H}_2 \equiv \mathbb{C}^n$. Then $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \equiv \mathbb{C}^m \otimes \mathbb{C}^n \equiv \mathbb{C}^{m\times n}$ is called the bipartite space. The space $\mathrm{B}(\mathcal{H})$ can be viewed as $(mn) \times (mn)$ matrices $T = [t_{(i,p)(j,q)}] \in \mathbb{C}^{(mn)\times(mn)}$, where $i, j \in [m], p, q \in [n]$. There are two natural maps, which are called partial traces:

$$\operatorname{Tr}_2 : \mathrm{B}(\mathcal{H}) \to \mathrm{B}(\mathcal{H}_1); \operatorname{Tr}_2 T = \left[\sum_{p=1}^{n} t_{(i,p)(j,p)}\right], i, j \in [m],$$

$$\operatorname{Tr}_1 : \mathrm{B}(\mathcal{H}) \to \mathrm{B}(\mathcal{H}_2); \operatorname{Tr}_1 T = \left[\sum_{i=1}^{m} t_{(i,p)(i,q)}\right], p, q \in [n].$$

A density matrix $\rho \in \mathrm{S}_{+,1}(\mathcal{H})$ is an analog of a probability measure μ on $[m]\times[n]$. Clearly $\rho_1 = \operatorname{Tr}_2 \rho \in \mathrm{S}_{+,1}(\mathcal{H}_1)$ and $\rho_2 = \operatorname{Tr}_1 \rho \in \mathrm{S}_{+,1}(\mathcal{H}_2)$ are the analogs of marginals μ_1 and μ_2. Hence the analog of Problem 1.1 is the quantum marginals and coupling problems [3, 4, 5, 27]:

Problem 1.2. Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional inner product spaces. Let $\mathcal{X} \subseteq \mathcal{H}$ be a closed subspace. Given $\rho_i \in \mathrm{S}_{+,1}(\mathcal{H}_i)$, $i = 1, 2$, what are necessary and sufficient conditions for the existence of $\rho \in \mathrm{S}_{+,1}(\mathcal{H})$, $\operatorname{supp} \rho \subseteq \mathcal{X}$, such that ρ_1, ρ_2 are its partial traces?

This problem is a variation of the classical 2-representability problem: Given $\rho_i \in \mathrm{S}_{+,1}(\mathcal{H}_i)$ for $i \in [2]$, does there exists a pure state $\rho \in \mathrm{S}_{+,1}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that ρ_1, ρ_2 are its partial traces? This problem was solved by Klyachko in [18] for a more general case: Namely the spectrum of the mixed bipartite state is prescribed. The N-representation problem [9, 6, 19] was considered to be in mid 90's one of ten most prominent research challenges in quantum chemistry [24]. Recently, some aspects of the three partite quantum marginals problem were discussed in [4].

Problem 1.2 can be stated in terms of semidefinite problem (SDP): Let $P_{\mathcal{X}}$ be the projection on the $\mathcal{X}$. Consider the maximum problem

$$\max\{\operatorname{Tr} X P_{\mathcal{X}}, X \in \mathrm{S}_+(\mathcal{H})\}, \operatorname{Tr}_j X = \rho_i \in \mathrm{S}_{+,1}(\mathcal{H}), \{i, j\} = \{1, 2\}, i = 1, 2\}.$$

Then Problem 1.2 is solvable if an only if the above maximum is 1. It is possible to convert this problem to an equivalent SDP problem where the admissible set is bounded and has an interior in $S_+(\mathcal{H})$, see Section 4.1. Thus one can use interior methods to find the maximum within a given precision $\varepsilon > 0$ in polynomial time in the given data and ε.

Zhou et al. [29] gave necessary and sufficient conditions for the solution of Problem 1.2. These conditions are analogous to the conditions for the solution of Problem 1.1 [16]. They pointed out that quantum coupling can be used to extend quantum Hoare logic [26] for proving relational properties between quantum programs and further for verifying quantum cryptographic protocols and differential privacy in quantum computation [28]. The second named author generalized some of the results of [29] in [15].

1.2. **Quantum marginals problem in the infinite dimensional case.** The aim of this paper is to answer Problem 1.2 in the case when at least one of the Hilbert spaces is an infinite dimensional separable Hilbert space. The most challenging and interesting parts of this paper are tackling the weak operator convergence in the trace class operators on the tensor product of two Hilbert spaces, (bipartite space), under the partial trace mapping. As shown in Example 2.4 the weak operator convergence is not preserved under the partial trace. This paper offers some tools and approaches for the quantum marginals problem. We hope that our results will be useful to other problems on trace class operators with partial traces.

Our main idea to solve Problem 1.2 is by stating a countable number of necessary conditions on finite dimensional Hilbert spaces. Then to show that these conditions are sufficient using compactness arguments. This was a successful approach in finding infinite dimensional generalizations of Choi's theorem for characterization of quantum channels [13].

It turns out that the most difficult case is when $\mathcal{H}_1, \mathcal{H}_2, \mathcal{X}$ are infinite dimensional separable spaces. We now outline briefly our main result in this case.

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Denote by $\mathrm{B}(\mathcal{H}) \supset \mathrm{K}(\mathcal{H})$ the space of bounded linear operators, with the operator norm $\|\cdot\|$, and the ideal of compact operators respectively. Let $\mathrm{S}(\mathcal{H}) \supset \mathrm{S}_+(\mathcal{H})$ be the subspace of selfadjoint operators and the cone of positive semidefinite operators in $\mathrm{B}(\mathcal{H})$. Assume that $A \in \mathrm{K}(\mathcal{H})$. Recall that A has the Schmidt decomposition, which is the singular value decomposition for the finite dimensional $\mathcal{H}$, with a nonnegative nonincreasing sequence of singular values $\|A\| = \sigma_1(A) \ge \cdots \ge \sigma_i(A) \ge \cdots \ge 0$, which converges to 0. For $A \in \mathrm{S}_+(\mathcal{H}) \cap \mathrm{K}(\mathcal{H})$ the Schmidt decomposition is the spectral decomposition. For $p \in [1, \infty)$, denote by $\mathrm{T}_p(\mathcal{H}) \subset \mathrm{K}(\mathcal{H})$ the Banach space of all compact operators with the p-Schatten norm $\|A\|_p = (\sum_{i=1}^{\infty} \sigma_i(A)^p)^{1/p}$. The Banach space $\mathrm{T}_1(\mathcal{H})$ is the space of trace class operators, which will be abbreviated to $\mathrm{T}(\mathcal{H})$. For $A \in \mathrm{T}(\mathcal{H})$ the trace $\operatorname{Tr} A$ is a bounded linear functional $A \mapsto \operatorname{Tr} A$ satisfying $|\operatorname{Tr} A| \le \|A\|_1$. For $A \in \mathrm{T}(\mathcal{H}) \cap \mathrm{S}(\mathcal{H})$, $\operatorname{Tr} A$ is the sum of the eigenvalues of A. The cone of positive semidefinite operators in trace class is denoted as $\mathrm{T}_+(\mathcal{H}) = \mathrm{T}(\mathcal{H}) \cap \mathrm{S}_+(\mathcal{H})$. Note that $\|A\|_1 = \operatorname{Tr} A$ if and only if $A \in \mathrm{T}_+(\mathcal{H})$. (See [14, Appendix A].) Denote by $\mathrm{S}_{+,1}(\mathcal{H}) \subset \mathrm{T}_+(\mathcal{H})$ the convex set of positive semidefinite trace class operators with trace 1, i.e., the density operators.

Assume that $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable Hilbert spaces. Suppose that $\rho \in \mathrm{T}(\mathcal{H})$. Then there are two partial trace maps: $\operatorname{Tr}_1 : \mathrm{T}(\mathcal{H}) \to \mathrm{T}(\mathcal{H}_2)$ and $\operatorname{Tr}_2 : \mathrm{T}(\mathcal{H}) \to \mathrm{T}(\mathcal{H}_1)$ which are both contractions: $\|\operatorname{Tr}_i(A)\|_1 \le \|A\|_1$ for $i = 1, 2$, see Section 2. Denote

$$\Phi : \mathrm{T}(\mathcal{H}) :\to \mathrm{T}(\mathcal{H}_1) \oplus \mathrm{T}(\mathcal{H}_2), \quad \Phi(\rho) = (\operatorname{Tr}_2 \rho, \operatorname{Tr}_1 \rho). \tag{1.1}$$

Then $\|\Phi\| \le 2$. Let $\Sigma = \Phi(\mathrm{T}_+(\mathcal{H}))$. Then

$$\Sigma = \{(\rho_1, \rho_2) \mid \rho_1 \in \mathrm{T}_+(\mathcal{H}_1), \rho_2 \in \mathrm{T}_+(\mathcal{H}_2), \operatorname{Tr} \rho_1 = \operatorname{Tr} \rho_2\}. \tag{1.2}$$

For $(\rho_1,\rho_2)\in\Sigma$, let

$$\mathcal{M}(\rho_1,\rho_2)=\Phi^{-1}(\rho_1,\rho_2)\cap \mathrm{T}_+(\mathcal{H}). \tag{1.3}$$

Thus if ρ_1,ρ_2 are density operators then $\mathcal{M}(\rho_1,\rho_2)$ is the convex set of bipartite density operates with marginals ρ_1,ρ_2. Observe that $\mathrm{T}_+(\mathcal{H})$ fibers over Σ: $\mathrm{T}_+(\mathcal{H})=\cup_{(\rho_1,\rho_2)}\mathcal{M}(\rho_1,\rho_2)$.

1.3. **Summary of the main results.**

Theorem 1.3. *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are infinite dimensional separable Hilbert spaces. Assume that $\rho_i\in\mathrm{S}_{+,1}(\mathcal{H}_i)$ for $i=1,2$. Let $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$. Define on $\mathrm{T}(\mathcal{H})$ the following Lipschitz convex function with respect to $\|\cdot\|_1$:*

$$f(X)=\|\operatorname{Tr}_2 X-\rho_1\|_1+\|\operatorname{Tr}_1 X-\rho_2\|_1. \tag{1.4}$$

Suppose that $\mathcal{X}\subset\mathcal{H}$ is infinite dimensional closed subspace, with an orthonormal basis $\mathbf{x}_i, i\in\mathbb{N}$. Let $\mathcal{X}_n$ be the subspace spanned by $\mathbf{x}_1,\dots,\mathbf{x}_n$ for $n\in\mathbb{N}$. Consider the minimization problem

$$\mu_n(\rho_1,\rho_2)=\inf\{f(X),\ X\in\mathrm{S}_+(\mathcal{X}_n)\}, \tag{1.5}$$

for $n\in\mathbb{N}$. This infimum is attained for some $X_n\in\mathcal{X}_n$ which satisfies $\|X_n\|\le 2$. Then there exists $\rho\in\mathrm{S}_{+,1}(\mathcal{H}),\operatorname{supp}\rho\subseteq\mathcal{X}$ such that $\operatorname{Tr}_2\rho=\rho_1,\operatorname{Tr}_1\rho=\rho_2$ if and only if

$$\lim_{n\to\infty}\mu_n(\rho_1,\rho_2)=0. \tag{1.6}$$

We now comment on the above theorem. The Lipschitz and convexity properties of f on $\mathrm{T}(\mathcal{H})$ follows straightforward from the triangle inequality for norms and the fact that the partial traces are contractions. Since $\mathcal{X}_n$ has dimension n the minimum $\mu_n(\rho_1,\rho_2)$ can be computed efficiently. Furthermore, the sequence $\mu_n(\rho_1,\rho_2)$ is decreasing. It is also straightforward to show that that if there exists $\rho\in\mathrm{T}_1(\mathcal{H}),\operatorname{supp}\rho\subseteq\mathcal{X}$ such that $\operatorname{Tr}_1\rho=\rho_2$ and $\operatorname{Tr}_2\rho=\rho_1$ then (1.6) holds. The nontrivial part of the above theorem is that the condition (1.6) yields the existence of ρ. This part follows from the following nontrivial interesting result:

Theorem 1.4. *Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are infinite dimensional separable Hilbert spaces. Suppose that $\rho_i\in\mathrm{T}_+(\mathcal{H}_i)$ for $i=1,2$. Assume that the sequence $\rho^{(n)}\in\mathrm{T}_+(\mathcal{H}),n\in\mathbb{N}$ converges in the weak operator topology to $\rho\in\mathrm{T}_+(\mathcal{H})$. Suppose furthermore that*

$$\lim_{n\to\infty}\|\operatorname{Tr}_1\rho^{(n)}-\rho_2\|_1+\|\operatorname{Tr}_2\rho^{(n)}-\rho_1\|_1=0. \tag{1.7}$$

Then

$$\lim_{n\to\infty}\|\rho^{(n)}-\rho\|_1=0. \tag{1.8}$$

In particular $\operatorname{Tr}\rho=\operatorname{Tr}\rho_1=\operatorname{Tr}\rho_2$. Hence ρ is a density operator if and only if ρ_1 and ρ_2 are density operators.

Our proof is long and computational.

The above theorem implies the following results. First, $\mathcal{M}(\rho_1,\rho_2)$ is a compact metric space with the distance induced by the norm $\|\cdot\|_1$ on $\mathrm{T}_+(\mathcal{H}_1\otimes\mathcal{H}_2)$. Second, Theorem 5.2 shows that this Hausdorff distance $\mathrm{hd}(\mathcal{M}(\rho_1,\rho_2),\mathcal{M}(\sigma_1,\sigma_2))$ is a complete metric on the fibers $\mathcal{M}(\rho_1,\rho_2)$.

1.4. **Survey of the content of the paper.** In this paper we use many standard and known results for compact operators, trace class operators and Hilbert-Schmidt operators ($\mathrm{T}_2(\mathcal{H})$), on a separable Hilbert space. In our arXiv preprint [14] we elaborated explicitly with full details the proofs of the results on operators we use. In this paper we stated explicitly the Lemmas and other results that we use. For known results we give the appropriate references, and sketch the ideas of proofs of results, which are less known. The full details are given in [14].

We now survey briefly the content of this paper. In Section 2 we discuss some basic results on operators on separable Hilbert spaces. We recall the Schmidt decomposition of compact operators and its properties. We discuss in detail the trace class operators $\mathrm{T}(\mathcal{H})$, the Hilbert-Schmidt operators $\mathrm{T}_2(\mathcal{H})$ and relations between these Banach spaces. Next we consider these classes of operators for bipartite Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. We discuss in detail the partial trace operators and their properties under the weak operator convergence.

In Section 3 we give proofs to Theorems 1.4 and 1.3. Most of this Section is devoted to the proof of Theorem 1.4, which is long and computational. The proof of Theorem 1.3 follows quite simply from Theorem 1.4.

Section 4 discusses a simpler case of quantum marginals problem, where the support of ρ is contained in a finite dimensional subspace $\mathcal{X}$ of the bipartite space $\mathcal{H}$. In this case we can replace the minimum problem (1.5), which boils down to the minimum of Lipschitz convex function on a finite dimensional compact convex set, to a maximum problem in semidefinite programming(SDP), on a bounded compact set of positive semidefinite matrices, which has an interior. In Subsection 4.1 we discuss a more general SDP problem than the one considered in [29], and its dual problem. Most of Subsection 4.2 is devoted to the case where $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable infinite dimensional. The main result of this subsection is Theorem 4.5, which is an analog of Theorem 1.3, where $\mu_n(\rho_1, \rho_2)$ is replaced by $\mu_n(\rho_1, \rho_2, \mathcal{X})$, which is the maximum of an appropriate SDP problem.

In Section 5, we prove that the Hausdorff metric on the space of fibers $\mathcal{M}(\rho_1, \rho_2)$ over Σ is a compact metric space with respect to the Hausdorff metric.

2. Preliminary results on operators in Hilbert spaces

We now recall some results needed in this paper on operators in a separable Hilbert space $\mathcal{H}$. Our main references are [2, 7, 17, 21, 22, 23]. We will follow closely the notions in [13]. The elements of $\mathcal{H}$ are denoted by lower bold letters as $\mathbf{x}$. We denote the inner product in $\mathcal{H}$ by $\langle \mathbf{x}, \mathbf{y} \rangle$, which is linear in $\mathbf{x}$ and antilinear in $\mathbf{y}$. The norm $\|\mathbf{x}\|$ is equal to $\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. We denote by $\mathcal{H}^\vee$ the dual space of the linear functional on $\mathcal{H}$. Recall that a linear functional $\mathbf{f} \in \mathcal{H}^\vee$ represented by $\mathbf{y} \in \mathcal{H}$: $\mathbf{f}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x} \in \mathcal{H}$. We denote this $\mathbf{f}$ by $\mathbf{y}^\vee$. Note

$$(a_1 \mathbf{y}_1 + a_2 \mathbf{y}_2)^\vee = \bar{a}_1 \mathbf{y}_1^\vee + \bar{a}_2 \mathbf{y}_2^\vee.$$

Denote by $\mathbb{N}$ the set of positive integers. For $n \in \mathbb{N}$ we denote $[n] = \{1, \ldots, n\}$, and let $[\infty] = \mathbb{N}$. Recall that $\mathcal{H}$ is separable if it has an orthormal basis $\mathbf{e}_i$ for $i \in [N]$, where $N \in \mathbb{N} \cup \{\infty\}$. In this paper we assume that $\mathcal{H}$ is separable. Then $\dim \mathcal{H} = N$. Thus $\mathcal{H}$ is finite dimensional if $N \in \mathbb{N}$.

We denote by $\mathrm{B}(\mathcal{H})$ the space of bounded linear operators $L : \mathcal{H} \to \mathcal{H}$. The bounded linear operators are denoted by the capital letters. The operator norm of L is given by $\|L\| = \sup\{\|L\mathbf{x}\|, \|\mathbf{x}\| \le 1\}$. The adjoint operator of L is denoted by $L^\vee$ and is given by the equality $\langle L\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, L^\vee \mathbf{y}\rangle$. L is called a selfadjoint operator if $L^\vee = L$. Denote by $\mathrm{S}(\mathcal{H}) \subset \mathrm{B}(\mathcal{H})$ the real space of selfadjoint operators. $L \in \mathrm{S}(\mathcal{H})$ is called nonnegative (positive) if $\langle L\mathbf{x}, \mathbf{x}\rangle \ge 0$ ($\langle L\mathbf{x}, \mathbf{x}\rangle > 0$) for all $\mathbf{x} \ne \mathbf{0}$. Denote by $\mathrm{S}_{++}(\mathcal{H}) \subset \mathrm{S}_+(\mathcal{H})$ the open set of positive and nonnegative (selfadjoint bounded) operators. So $\mathrm{S}_+(\mathcal{H})$ is a closed cone and $\mathrm{S}_{++}(\mathcal{H})$ its interior. Recall that $L \in \mathrm{S}_+(\mathcal{H})$ has a unique root $L^{1/2} \in \mathrm{S}_+(\mathcal{H})$. If L is positive then $L^{1/2}$ is positive. For $L \in \mathrm{B}(\mathcal{H})$ we have that $L^\vee L, LL^\vee \in \mathrm{S}_+(\mathcal{H})$, and $|L| = (L^\vee L)^{1/2} \in \mathrm{S}_+(\mathcal{H})$. For $A, B \in \mathrm{S}(\mathcal{H})$ we denote $A \succeq B (A \succ B)$ if $A - B \in \mathrm{S}_+(\mathcal{H})(A - B \in \mathrm{S}_{++}(\mathcal{H}))$.

L is called rank one operator if $L = \mathbf{x}\mathbf{y}^\vee$, where $\mathbf{x}, \mathbf{y} \ne \mathbf{0}$. Thus $L(\mathbf{z}) = \langle \mathbf{z}, \mathbf{y}\rangle \mathbf{x}$. L is selfadjoint if and only if $\mathbf{y} = a\mathbf{x}$ for some $a \in \mathbb{R}$. $L \in \mathrm{S}_+(\mathcal{H})$ if and only if $a \ge 0$.

Assume that $\dim \mathcal{H} = \infty$. Denote by $\mathrm{K}(\mathcal{H})$ the closed ideal (left and right) of compact operators. The operator L is in $\mathrm{K}(\mathcal{H})$ if and only if L has singular value decomposition (SVD), (or Schmidt decomposition):

$$L = \sum_{i=1}^{\infty} \sigma_i(L) \mathbf{g}_i \mathbf{f}_i^\vee, \tag{2.1}$$
$$\|L\| = \sigma_1(L) \ge \cdots \ge \sigma_n(L) \ge \cdots \ge 0, \lim_{i\to\infty} \sigma_i(L) = 0.$$

Here $\{\mathbf{g}_1, \ldots, \mathbf{g}_n, \ldots\}, \{\mathbf{f}_1, \ldots, \mathbf{f}_n, \ldots\}$ are two orthonormal sets of vectors of $\mathcal{H}$. The nth singular value of L denoted by $\sigma_n(L)$, and $\mathbf{g}_n, \mathbf{f}_n$ are called left and right nth singular vectors of L. L is selfadjoint if and only if $\mathbf{f}_i = \varepsilon_i \mathbf{g}_i, \varepsilon_i \in \{-1, 1\}$ for all $i \in \mathbb{N}$. Then (2.1) is the spectral decomposition of L where $\varepsilon_i \sigma_i(L)$ is the eigenvalue of L with the corresponding eigenvector $\mathbf{g}_i$. Furthermore $L \in \mathrm{S}_+(\mathcal{H}) \cap \mathrm{K}(\mathcal{H})$ if and only if $\mathbf{f}_i = \mathbf{g}_i$ for all $i \in \mathbb{N}$. Hence all positive $\sigma_i(L)^2$ are the positive eigenvalues of compact operators $LL^\vee, L^\vee L \in \mathrm{S}_+(\mathrm{H}) \cap \mathrm{K}(\mathcal{H})$. Note that

$$\|L - \sum_{i=1}^{n} \sigma_i(L) \mathbf{g}_i \mathbf{f}_i^\vee\| = \sigma_{n+1}(L), \quad n \in \mathbb{N}.$$

Recall that if $A \in \mathrm{B}(\mathcal{H})$ and $L \in \mathrm{K}(\mathcal{H})$ then $AL, LA \in \mathrm{K}(\mathcal{H})$. Furthermore, one has the inequalities

$$\sigma_i(AL), \sigma_i(LA) \le \sigma_i(L)\|A\|, \ i \in \mathbb{N}. \tag{2.2}$$

([14, Appendix A].) The above inequalities on singular values yield that if $L \in \mathrm{T}(\mathcal{H})$ then $AL, LA \in \mathrm{T}(\mathcal{H})$. Furthermore, $\|AL\|_1, \|LA\|_1 \le \|L\|_1 \|A\|$ [7, 1.11 Theorem].

If $L \in \mathrm{T}(\mathcal{H})$, then for each orthonormal basis $\mathbf{e}_i, i \in \mathbb{N}$, we have the inequality $\sum_{i=1}^{\infty} |\langle L\mathbf{e}_i, \mathbf{e}_i\rangle| \le \|L\|_1$. (See [14, Lemma A.3], or [7, Proof of 1.9 Proposition].) Furthermore the value of the sum $\sum_{i=1}^{\infty} \langle L\mathbf{e}_i, \mathbf{e}_i\rangle$ is independent of a choice of the basis, is denoted as the trace of L. Thus the SVD decomposition (2.1) of $L \in \mathrm{T}(\mathcal{H})$ yields that

$$\operatorname{Tr} L = \sum_{i=1}^{\infty} \sigma_i(L) \langle \mathbf{g}_i, \mathbf{f}_i\rangle. \tag{2.3}$$

Thus $|\operatorname{Tr} L| \le \|L\|_1$ and equality holds if and only of $zL \in \mathrm{T}_+(\mathcal{H})$ for some $z \in \mathbb{C}, |z| = 1$. Note that if $L \in \mathrm{S}(\mathcal{H}) \cap \mathrm{T}(\mathcal{H})$ then the trace of L is the sum of the eigenvalues of L. (See [14, Appendix A], or [7, 1.11 Theorem].)

Next we recall the following known result that we need later:

$$\operatorname{Tr} LA = \operatorname{Tr} AL = \operatorname{Tr} A^{1/2} L A^{1/2} \ge 0 \text{ if } L \in \mathrm{T}_+(\mathcal{H}) \text{ and } A \in \mathrm{S}_+(\mathcal{H}).$$

Recall that $\mathrm{T}_p(\mathcal{H}) \subset \mathrm{T}_q(\mathcal{H})$ for $1 \le p < q < \infty$. (Usually $\mathrm{T}_\infty(\mathcal{H})$ is identified with $\mathrm{B}(\mathcal{H})$.) In particular, $\mathrm{T}(\mathcal{H}) \subset \mathrm{T}_2(\mathcal{H})$. The space $\mathrm{T}_2(\mathcal{H})$ is the Hilbert-Schmidt space of compact operators. Fix an orthonormal basis $\{\mathbf{e}_i\}, i \in \mathbb{N}$. Then $A_1, A_2 \in \mathrm{T}_2(\mathcal{H})$ have representations

$$A_l = \sum_{i=j=1}^{\infty} a_{ij,l}\mathbf{e}_i\mathbf{e}_j^\vee, \quad \|A_l\|_2 = (\sum_{i=j=1}^{\infty} |a_{ij,l}|^2)^{1/2},\ l \in [2].$$

Thus $\mathrm{T}_2(\mathcal{H})$ is a Hilbert space with the inner product

$$\langle A_1, A_2\rangle = \sum_{i=j=1}^{\infty} a_{ij,1}\bar{a}_{ij,2}.$$

It is well known that if $A_1, A_2 \in \mathrm{T}_2(\mathcal{H})$ then $A_1A_2 \in \mathrm{T}(\mathcal{H})$:

$$A_1A_2 = \sum_{i=j=1}^{\infty}(\sum_{k=1}^{\infty} a_{ik,1}a_{kj,2})\mathbf{e}_i\mathbf{e}_j^\vee.$$

Furthermore

$$\langle A_1, A_2\rangle = \operatorname{Tr} A_1A_2^\vee, \quad \|A_1A_2\|_1 \le \|A_1\|_2\|\|A_2\|_2,$$
$$A_1A_1^\vee \in \mathrm{T}_+(\mathcal{H}), \quad \|A_1A_1^\vee\|_1 = \|A_1\|_2^2 = \operatorname{Tr} A_1A_1^\vee.$$

See [14, Lemma A.4], or [7, 1.8 Proposition].

We next discuss the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of two separable Hilbert spaces. It is called in quantum physics bipartite states. Assume that the inner product in $\mathcal{H}_i$ is $\langle\cdot,\cdot\rangle_i$. Then $\mathcal{H}_1 \otimes \mathcal{H}_2$ has the induced inner product satisfying the property $\langle \mathbf{x}\otimes\mathbf{y}, \mathbf{u}\otimes\mathbf{v}\rangle = \langle\mathbf{x},\mathbf{u}\rangle_1\langle\mathbf{y},\mathbf{v}\rangle_2$. We assume that $\mathcal{H}_l$ has an orthonormal basis $\mathbf{e}_{i,l}, i \in [N_l]$, where $N_l \in \mathbb{N}\cup\{\infty\}$ for $l \in [2]$. These two orthonormal bases induce the orthonormal basis $\mathbf{e}_{i,1}\otimes\mathbf{e}_{j,2}$ for $i \in [N_1], j \in [N_2]$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$. A vector $\mathbf{a} \in \mathcal{H}_1 \otimes \mathcal{H}_2$ has the expansion

$$\mathbf{a} = \sum_{i=j=1}^{N_1,N_2} a_{ij}\mathbf{e}_{i,1}\otimes\mathbf{e}_{j,2},\ \|\mathbf{a}\| = \sqrt{\sum_{i=j=1}^{N_1,N_2}|a_{ij}|^2} < \infty. \tag{2.4}$$

Note that $\mathbf{a}$ induces two bounded linear operators $A(\mathbf{a}) : \mathcal{H}_2 \to \mathcal{H}_1$ and $A(\mathbf{a})^\vee : \mathcal{H}_1 \to \mathcal{H}_2$ given by

$$A(\mathbf{a}) = \sum_{i=j=1}^{N_1,N_2} a_{ij}\mathbf{e}_{i,1}\mathbf{e}_{j,2}^\vee, \quad A(\mathbf{a})^\vee = \sum_{i=j=1}^{N_1,N_2}\bar{a}_{ij}\mathbf{e}_{j,2}\mathbf{e}_{i,1}^\vee. \tag{2.5}$$

We can view $A(\mathbf{a})$ as a matrix $\hat{A} = [a_{ij}]_{i=j=1}^{N_1,N_2}$. We denote by $\hat{A}^\dagger = [a_{pq}^\dagger]_{p=q=1}^{N_2,N_1}$, where $a_{pq}^\dagger = \bar{a}_{qp}$ for all $p \in [N_2], q \in [N_1]$. ($\hat{A}^\dagger$ is the "transpose conjugate" of $\hat{A}$.) Next we observe that the operators $A(\mathbf{a})$ and $A(\mathbf{a})^\vee$ can be viewed as adjoint Hilbert-Schmidt operators on $\tilde{\mathcal{H}} = \mathcal{H}_1 \oplus \mathcal{H}_2$, with the inner product:

$$\langle(\mathbf{x},\mathbf{u}),(\mathbf{y},\mathbf{v})\rangle = \langle\mathbf{x},\mathbf{y}\rangle_1 + \langle\mathbf{u},\mathbf{v}\rangle_2.$$

Let $\tilde{\mathbf{e}}_{i,1} = (\mathbf{e}_{i,1}, 0), \tilde{\mathbf{e}}_{j,2} = (0, \mathbf{e}_{j,2})$ for $i \in [N_1], j \in [N_2]$. Define

$$\tilde{A}(\mathbf{a}) = \sum_{i=j=1}^{N_1,N_2} a_{ij}\tilde{\mathbf{e}}_{i,1}\tilde{\mathbf{e}}_{j,2}^\vee, \quad \tilde{A}(\mathbf{a})^\vee = \sum_{i=j=1}^{N_1,N_2}\bar{a}_{ij}\tilde{\mathbf{e}}_{j,2}\tilde{\mathbf{e}}_{i,1}^\vee$$

Then $\tilde{A}(\mathbf{a}), \tilde{A}(\mathbf{a})^\vee \in \mathrm{T}_2(\tilde{\mathcal{H}})$. Furthermore we have the following relations

$$\tilde{A}(\mathbf{a})\tilde{A}(\mathbf{a})^\vee \big| \mathcal{H}_1 = A(\mathbf{a})A(\mathbf{a})^\vee, \quad \tilde{A}(\mathbf{a})\tilde{A}(\mathbf{a})^\vee \big| \mathcal{H}_2 = 0,$$
$$\tilde{A}(\mathbf{a})^\vee\tilde{A}(\mathbf{a}) \big| \mathcal{H}_2 = A(\mathbf{a})^\vee A(\mathbf{a}), \quad \tilde{A}(\mathbf{a})^\vee\tilde{A}(\mathbf{a}) \big| \mathcal{H}_1 = 0.$$

Lemma A.4 in [14] yields that $A(\mathbf{a})A(\mathbf{a})^\vee \in \mathrm{T}_+(\mathcal{H}_1), A(\mathbf{a})^\vee A(\mathbf{a}) \in \mathrm{T}_+(\mathcal{H}_2)$, and the two operators have the same singular values. Thus the matrices $\hat{A}\hat{A}^\dagger, \hat{A}^\dagger\hat{A}$ represent the operators $A(\mathbf{a})A(\mathbf{a})^\vee, A(\mathbf{a})^\vee A(\mathbf{a})$ in the bases $\{\mathbf{e}_{i,1}\}, \{\mathbf{e}_{j,2}\}$ respectively.

Let $\mathbf{b} = \sum_{i=j=1}^{N_1,N_2} b_{ij}\mathbf{e}_{i,1} \otimes \mathbf{e}_{j,2} \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Denote $\hat{B} = [b_{ij}]_{i=j=1}^{N_1,N_2}$. Then $\langle \mathbf{a}, \mathbf{b}\rangle = \operatorname{Tr} \hat{A}\hat{B}^\dagger = \operatorname{Tr} \hat{B}^\dagger \hat{A}$.

Assume that $F \in \mathrm{T}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. We now discuss the notions of partial traces $\mathrm{Tr}_1(F) \in \mathrm{T}(\mathcal{H}_2)$ and $\mathrm{Tr}_2(F) \in \mathrm{T}(\mathcal{H}_1)$. Assume first that F is a rank one product operator: $(\mathbf{x} \otimes \mathbf{y})(\mathbf{u} \otimes \mathbf{v})^\vee$. Then

$$\mathrm{Tr}_1((\mathbf{x} \otimes \mathbf{y})(\mathbf{u} \otimes \mathbf{v})^\vee) = \langle \mathbf{x}, \mathbf{u}\rangle \mathbf{y}\mathbf{v}^\vee, \tag{2.6}$$
$$\mathrm{Tr}_2((\mathbf{x} \otimes \mathbf{y})(\mathbf{u} \otimes \mathbf{v})^\vee) = \langle \mathbf{y}, \mathbf{v}\rangle \mathbf{x}\mathbf{u}^\vee. \tag{2.7}$$

Hence

$$\|(\mathbf{x} \otimes \mathbf{y})(\mathbf{u} \otimes \mathbf{v})^\vee\|_1 = \|\mathbf{x}\|\|\mathbf{y}\|\|\mathbf{u}\|\|\mathbf{v}\|,$$
$$\|\mathrm{Tr}_1(\mathbf{x} \otimes \mathbf{y})(\mathbf{u} \otimes \mathbf{v})^\vee)\|_1 = |\langle \mathbf{x}, \mathbf{u}\rangle|\|\mathbf{y}\|\|\mathbf{v}\|,$$
$$\|\mathrm{Tr}_2((\mathbf{x} \otimes \mathbf{y})(\mathbf{u} \otimes \mathbf{v})^\vee)\|_1 = |\langle \mathbf{y}, \mathbf{v}\rangle|\|\mathbf{x}\|\|\mathbf{u}\|.$$

Lemma 2.1. *Assume that $\mathcal{H}_i$ is a separable Hilbert space of dimension N_i with a basis $\mathbf{e}_{j,i}, j \in [N_i]$ for $i \in [2]$. Denote $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Let $\mathbf{a}, \mathbf{b} \in \mathcal{H}$. Suppose that $\mathbf{a}$ has the representation* (2.4)*. Assume that $\mathbf{b}$ has a similar expansion and $\hat{A} = [a_{ij}], \hat{B} = [b_{ij}], i \in [N_1], j \in [N_2]$ are the representation matrices of $\mathbf{a}, \mathbf{b}$ respectively. Denote by C and D the following operators:*

$$\mathrm{Tr}_2\, \mathbf{a}\mathbf{b}^\vee = C = \sum_{i=p=1}^{N_1} c_{ip}\mathbf{e}_{i,1}\mathbf{e}_{p,1}^\vee, \mathrm{Tr}_1\, \mathbf{a}\mathbf{b}^\vee = D = \sum_{j=q=1}^{N_2} d_{jq}\mathbf{e}_{j,2}\mathbf{e}_{q,2}^\vee. \tag{2.8}$$

Then

$$\hat{C} = \hat{A}\hat{B}^\dagger = [c_{ip}]_{i=p=1}^{N_1}, \quad \hat{D} = \hat{A}^\top \overline{\hat{B}} = [d_{jq}]_{j=q=1}^{N_2}. \tag{2.9}$$

Furthermore $C \in \mathrm{T}(\mathcal{H}_1), D \in \mathrm{T}(\mathcal{H}_2)$ and the following inequalities and equalities hold

$$\max(\|\mathrm{Tr}_2\, \mathbf{a}\mathbf{b}^\vee\|_1, \|\mathrm{Tr}_1\, \mathbf{a}\mathbf{b}^\vee\|_1) \le \|\mathbf{a}\|\|\mathbf{b}\| = \|\mathbf{a}\mathbf{b}^\vee\|_1, \tag{2.10}$$

$$\langle (\mathrm{Tr}_2\, \mathbf{a}\mathbf{b}^\vee)\mathbf{x}, \mathbf{y}\rangle = \sum_{j=1}^{N_2} \langle \mathbf{x} \otimes \mathbf{e}_{j,2}, \mathbf{b}\rangle\langle \mathbf{a}, \mathbf{y} \otimes \mathbf{e}_{j,2}\rangle, \ \mathbf{x}, \mathbf{y} \in \mathcal{H}_1, \tag{2.11}$$

$$\langle (\mathrm{Tr}_1\, \mathbf{a}\mathbf{b}^\vee)\mathbf{u}, \mathbf{v}\rangle = \sum_{i=1}^{N_1} \langle \mathbf{e}_{i,1} \otimes \mathbf{u}, \mathbf{b}\rangle\langle \mathbf{a}, \mathbf{e}_{i,1} \otimes \mathbf{v}\rangle, \ \mathbf{u}, \mathbf{v} \in \mathcal{H}_2. \tag{2.12}$$

In particular

$$\operatorname{Tr} \mathbf{a}\mathbf{b}^\vee = \operatorname{Tr} \mathrm{Tr}_2\, \mathbf{a}\mathbf{b}^\vee = \operatorname{Tr} \mathrm{Tr}_1\, \mathbf{a}\mathbf{b}^\vee = \langle \mathbf{a}, \mathbf{b}\rangle. \tag{2.13}$$

Proof. Clearly $\mathbf{a}\mathbf{b}^\vee \in \mathrm{T}(\mathcal{H})$. Furthermore

$$\|\mathbf{a}\mathbf{b}^\vee\| = \|\mathbf{a}\|\|\mathbf{b}\| = (\sum_{i=j=1}^{N_1,N_2} |a_{ij}|^2)^{\frac{1}{2}}(\sum_{p=q=1}^{N_1,N_2} |b_{pq}|^2)^{\frac{1}{2}}.$$

Observe next that

$$\mathbf{a}\mathbf{b}^\vee = (\sum_{i=j=1}^{N_1,N_2} a_{ij}\mathbf{e}_{i,1}\otimes\mathbf{e}_{j,2})(\sum_{p=q=1}^{N_1,N_2} b_{pq}\mathbf{e}_{p,1}\otimes\mathbf{e}_{q,2})^\vee =$$
$$\sum_{i=j=p=q=1}^{N_1,N_2,N_1,N_2} a_{ij}\bar{b}_{pq}(\mathbf{e}_{i,1}\otimes\mathbf{e}_{j,2})(\mathbf{e}_{p,1}\otimes\mathbf{e}_{q,2})^\vee.$$

Use (2.6) and (2.7) to deduce that the operators $C = \mathrm{Tr}_2(\mathbf{a}\mathbf{b}^\vee)$ and $D = \mathrm{Tr}_1(\mathbf{a}\mathbf{b}^\vee)$, which represented by matrices $\hat{C}$ and $\hat{D}$ respectively, satisfy (2.9) and (2.11)-(2.12).

Let $\tilde{\mathcal{H}}$ and $\tilde{A}(\mathbf{a}), \tilde{A}(\mathbf{b}) \in \mathrm{T}_2(\tilde{\mathcal{H}})$ be defined as above. Then $\hat{C}$ and $\overline{\hat{D}}$ represent $\tilde{A}(\mathbf{a})\tilde{A}(\mathbf{b})^\vee\big|\mathcal{H}_1 \in \mathrm{T}(\mathcal{H}_1)$ and $A(\mathbf{a})^\vee A(\mathbf{b})\big|\mathcal{H}_2 \in \mathrm{T}(\mathcal{H}_2)$. This shows that C and D are in the trace class. Lemma A.4 in [14] yields that

$$\|\hat{C}\|_1 = \|\tilde{A}(\mathbf{a})\tilde{A}(\mathbf{b})^\vee\|_1 \le \|\tilde{A}(\mathbf{a})\|_2\|\tilde{A}(\mathbf{b})^\vee\|_2 = \|\hat{A}\|_2\|\hat{B}\|_2 = \|\mathbf{a}\|\|\mathbf{b}\|,$$
$$\|\hat{D}\|_1 = \|\tilde{A}(\mathbf{a})^\vee\tilde{A}(\mathbf{b})\|_1 \le \|\tilde{A}(\mathbf{a})^\vee\|_2\|\tilde{A}(\mathbf{b})\|_2 = \|\hat{A}\|_2\|\hat{B}\|_2 = \|\mathbf{a}\|\|\mathbf{b}\|.$$

This proves (2.10).

It is left to show (2.13). As $\sigma_1(\mathbf{a}\mathbf{b}^\vee) = \|\mathbf{a}\|\|\mathbf{b}\|$ and all other singular values of $\mathbf{a}\mathbf{b}^\vee$ are zero (2.3) yields that $\mathrm{Tr}\,\mathbf{a}\mathbf{b}^\vee = \langle\mathbf{a},\mathbf{b}\rangle$. Observe next

$$\mathrm{Tr}(\mathrm{Tr}_2\,\mathbf{a}\mathbf{b}^\vee) = \sum_{i=1}^{N_1}\langle(\mathrm{Tr}_2\,\mathbf{a}\mathbf{b}^\vee)\mathbf{e}_{i,1},\mathbf{e}_{i,1}\rangle =$$
$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\langle\mathbf{e}_{i,1}\otimes\mathbf{e}_{j,2},\mathbf{b}\rangle\langle\mathbf{a},\mathbf{e}_{i,1}\otimes\mathbf{e}_{j,2}\rangle = \langle\mathbf{a},\mathbf{b}\rangle.$$

The equaity $\mathrm{Tr}(\mathrm{Tr}_1\,\mathbf{a}\mathbf{b}^\vee) = \langle\mathbf{a},\mathbf{b}\rangle$ follows similarly. □

The following lemma is known, see Theorem 26.7 and its proof in [2], or the proof in [14].

Lemma 2.2. *Assume that* $F \in \mathrm{T}(\mathcal{H}_1\otimes\mathcal{H}_2)$. *Then*

(1) $\mathrm{Tr}_1(F) \in \mathrm{T}(\mathcal{H}_2), \mathrm{Tr}_2(F) \in \mathrm{T}(\mathcal{H}_1)$.
(2) $\|\mathrm{Tr}_1(F)\|_1, \|\mathrm{Tr}_2(F)\|_1 \le \|F\|_1$.
(3) $\mathrm{Tr}(\mathrm{Tr}_1 F) = \mathrm{Tr}(\mathrm{Tr}_2 F) = \mathrm{Tr}\, F$.
(4) Assume that $F \in \mathrm{T}_+(\mathcal{H})$. *Then* $\mathrm{Tr}_1(F) \in \mathrm{T}_+(\mathcal{H}_2), \mathrm{Tr}_2(F) \in \mathrm{T}_+(\mathcal{H}_1)$ *and*

$$\|F\|_1 = \mathrm{Tr}(F) = \mathrm{Tr}(\mathrm{Tr}_1(F)) = \|\mathrm{Tr}_1(F)\|_1 = \mathrm{Tr}(\mathrm{Tr}_2(F)) = \|\mathrm{Tr}_2(F)\|_1.$$

Recall that a sequence $\mathbf{a}_n, n \in \mathbb{N}$ in $\mathcal{H}$ is called weakly convergent to $\mathbf{a} \in \mathcal{H}$, denoted as $\mathbf{a}_n \stackrel{w}{\to} \mathbf{a}$, if $\lim_{n\to\infty}\langle\mathbf{a}_n,\mathbf{x}\rangle = \langle\mathbf{a},\mathbf{x}\rangle$ for all $\mathbf{x} \in \mathcal{H}$. As sequence of bounded operators $A_n \in \mathrm{B}(\mathcal{H}), n \in \mathbb{N}$ is called convergent in weak operator topology to $A \in \mathrm{B}(\mathcal{H})$, denoted as $A_n \stackrel{w.o.t.}{\to} A$, if $\lim_{n\to\infty}\langle A_n\mathbf{x},\mathbf{y}\rangle = \langle A\mathbf{x},\mathbf{y}\rangle$ for all $\mathbf{x},\mathbf{y} \in \mathcal{H}$.

Lemma 2.3. *Let* $\mathcal{H}_l$ *be a separable Hilbert space of dimension* $N_l \in \mathbb{N}\cup\{\infty\}$ *for* $l \in [2]$. *Set* $\mathcal{H} = \mathcal{H}_1\otimes\mathcal{H}_2$.

(1) Assume that $\mathbf{a}_n, \mathbf{b}_n, \in \mathcal{H}, n \in \mathbb{N}$, *and* $\mathbf{a}_n \stackrel{w}{\to} \mathbf{a}, \mathbf{b}_n \stackrel{w}{\to} \mathbf{b}$. *Then*

$$\mathbf{a}_n\mathbf{b}_n^\vee \stackrel{w.o.t.}{\to} \mathbf{a}\mathbf{b}^\vee \textit{ in } \mathrm{T}(\mathcal{H}), \tag{2.14}$$
$$\liminf \mathrm{Tr}\,\mathbf{a}_n\mathbf{a}_n^\vee \ge \mathrm{Tr}\,\mathbf{a}\mathbf{a}^\vee. \tag{2.15}$$

For each $\mathbf{x}_i \in \mathcal{H}_i$ *for* $i \in [2]$ *the following inequalities hold*

$$\liminf \langle (\mathrm{Tr}_1\, \mathbf{a}_n \mathbf{a}_n^\vee)\mathbf{x}_2, \mathbf{x}_2\rangle \ge \langle (\mathrm{Tr}_1\, \mathbf{a}\mathbf{a}^\vee)\mathbf{x}_2, \mathbf{x}_2\rangle, \tag{2.16}$$
$$\liminf \langle (\mathrm{Tr}_2\, \mathbf{a}_n \mathbf{a}_n^\vee)\mathbf{x}_1, \mathbf{x}_1\rangle \ge \langle (\mathrm{Tr}_2\, \mathbf{a}\mathbf{a}^\vee)\mathbf{x}_1, \mathbf{x}_1\rangle.$$

Assume that for $l \in [2]$ N_l *is finite. Then*

$$\mathrm{Tr}_l\, \mathbf{a}_n \mathbf{b}_n^\vee \overset{w.o.t.}{\to} \mathrm{Tr}_l\, \mathbf{a}\mathbf{b}^\vee. \tag{2.17}$$

(2) *Assume that the sequence* $\rho_n \in \mathrm{T}_+(\mathcal{H})$ *converges in weak operator topology to* $\rho \in \mathrm{T}(\mathcal{H})$. *Then* $\rho \in T_+(\mathcal{H})$ *and the following conditions hold:*

$$\liminf \mathrm{Tr}\, \rho_n \ge \mathrm{Tr}\, \rho, \tag{2.18}$$
$$\lim_{n\to\infty} \mathrm{Tr}\, \rho_n = \mathrm{Tr}\, \rho \iff \lim_{n\to\infty} \|\rho_n - \rho\|_1 = 0, \tag{2.19}$$
$$\liminf \langle (\mathrm{Tr}_1\, \rho_n)\mathbf{x}_2, \mathbf{x}_2\rangle \ge \langle (\mathrm{Tr}_1\, \rho)\mathbf{x}_2, \mathbf{x}_2\rangle, \tag{2.20}$$
$$\liminf \langle (\mathrm{Tr}_1\, \rho_n)\mathbf{x}_2, \mathbf{x}_2\rangle \ge \langle (\mathrm{Tr}_1\, \rho)\mathbf{x}_2, \mathbf{x}_2\rangle, \tag{2.21}$$
$$\liminf \langle (\mathrm{Tr}_2\, \rho_n)\mathbf{x}_1, \mathbf{x}_1\rangle \ge \langle (\mathrm{Tr}_2\, \rho)\mathbf{x}_1, \mathbf{x}_1\rangle. \tag{2.22}$$

If N_l *is finite then*

$$\mathrm{Tr}_l\, \rho_n \overset{w.o.t.}{\to} \mathrm{Tr}_l\, \rho. \tag{2.23}$$

Proof. (1) For each $\mathbf{u}, \mathbf{v} \in \mathcal{H}$ we have the equality $\langle (\mathbf{a}_n \mathbf{b}_n^\vee)\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{b}_n\rangle\langle \mathbf{a}_n, \mathbf{v}\rangle$, As $\mathbf{a}_n \overset{w}{\to} \mathbf{a}, \mathbf{b}_n \overset{w}{\to} \mathbf{b}$ we deduce (2.14). Recall that $\liminf \|\mathbf{a}_n\| \ge \|\mathbf{a}\|$. As $\mathrm{Tr}\, \mathbf{c}\mathbf{c}^\vee = \|\mathbf{c}\|^2$ for $\mathbf{c} \in \mathcal{H}$ we deduce (2.15).

Assume that N_2 is finite. We prove (2.17) for $l = 2$. Recall (2.11) for $\mathbf{a}_n, \mathbf{b}_n$:

$$\langle (\mathrm{Tr}_2\, \mathbf{a}_n \mathbf{b}_n^\vee)\mathbf{x}, \mathbf{y}\rangle = \sum_{j=1}^{N_2} \langle \mathbf{x} \otimes \mathbf{e}_{j,2}, \mathbf{b}_n\rangle\langle \mathbf{a}_n, \mathbf{y} \otimes \mathbf{e}_{j,2}\rangle,\ \mathbf{x}, \mathbf{y} \in \mathcal{H}_1$$

Letting $n \to \infty$ we get (2.11). Hence (2.17) holds for $l = 2$. Similar arguments apply if N_1 is finite.

We now show (2.16). Assume first that N_2 is finite. Then (2.17) yields the equality in (2.16). Assume that $N_2 = \infty$. Choose $N \in \mathbb{N}$ and let $L_{n,N}$ and L_N be the following finite rank operators in $\mathrm{T}(\mathcal{H}_1)$:

$$\langle L_{n,N}\mathbf{x}, \mathbf{y}\rangle = \sum_{j=1}^{N} \langle \mathbf{x} \otimes \mathbf{e}_{j,2}, \mathbf{a}_n\rangle\langle \mathbf{a}_n, \mathbf{y} \otimes \mathbf{e}_{j,2}\rangle,$$
$$\langle L_N\mathbf{x}, \mathbf{y}\rangle = \sum_{j=1}^{N} \langle \mathbf{x} \otimes \mathbf{e}_{j,2}, \mathbf{a}\rangle\langle \mathbf{a}, \mathbf{y} \otimes \mathbf{e}_{j,2}\rangle.$$

Clearly, the sequence $L_{n,N}, n \in \mathbb{N}$ converges in weak operator topology to L_N for each $N \in \mathbb{N}$. Observe next

$$\langle (\mathrm{Tr}_2\, \mathbf{a}_n \mathbf{a}_n^\vee)\mathbf{x}, \mathbf{x}\rangle = \sum_{j=1}^{\infty} |\langle \mathbf{a}_n, \mathbf{x} \otimes \mathbf{e}_{j,2}\rangle|^2 \ge \langle L_{n,N}\mathbf{x}, \mathbf{x}\rangle.$$

Hence

$$\liminf \langle (\mathrm{Tr}_2\, \mathbf{a}_n \mathbf{a}_n^\vee)\mathbf{x}, \mathbf{x}\rangle \ge \langle L_N\mathbf{x}, \mathbf{x}\rangle.$$

As $\lim_{N\to\infty} \langle L_N\mathbf{x}, \mathbf{x}\rangle = \langle (\mathrm{Tr}_2\, \mathbf{a}\mathbf{a}^\vee)\mathbf{x}, \mathbf{x}\rangle$ we deduce the second inequality in(2.16). Similarly we deduce the first inequality in (2.16).

(2) The claim that $\rho \in \mathrm{T}_+(\mathcal{H})$ and the inequality (2.18) follow from Lemma B.4 in [14] . Assume that the spectral decomposition of ρ_n is $\sum_{k=1}^{\infty} \sigma_k(\rho_n)\mathbf{a}_{k,n}\mathbf{a}_{k,n}^\vee$. Fix $\mathbf{x}_l \in \mathcal{H}_l$ for $l \in [2]$. We first choose a subsequence $n_p, p \in \mathbb{N}$ such that a particular $\liminf$ stated in part (2) of the lemma is achieved for this subsequence. Clearly $\rho_{n_p} \overset{w.o.t.}{\to} \rho$. Hence, without loss of generality we can assume that $n_p = p$ for $p \in \mathbb{N}$. We choose a subsequence $n_m, m \in \mathbb{N}$ such that

$$\lim_{m\to\infty} \sigma_k(\rho_{n_m}) = \sigma_k, \quad \mathbf{a}_{k,n_m} \overset{w}{\to} \mathbf{a}_k, \quad k \in \mathbb{N}.$$

As ρ_{n_m} converges weakly also to ρ we deduce that $\rho = \sum_{k=1}^{\infty} \sigma_n \mathbf{a}_k\mathbf{a}_k^\vee$ and $\|\mathbf{a}_k\| \le 1$ for $k \in \mathbb{N}$. Fix $\varepsilon > 0$. Then there exists $N = N(\varepsilon)$ such that $\sum_{k=N}^{\infty} \sigma_k < \varepsilon$. Furthermore there exists $k > K_2(\varepsilon)$ such that $\sigma_N(\rho_k) < \varepsilon$. Let

$$B_n = \sum_{k=1}^{N} \sigma_k(\rho_n)\mathbf{a}_{k,n}\mathbf{a}_{k,n}^\vee, \quad C_n = \sum_{k=N+1}^{\infty} \sigma_k(\rho_n)\mathbf{a}_{k,n}\mathbf{a}_{k,n}^\vee,$$
$$B = \sum_{k=1}^{N} \sigma_k\mathbf{a}_k\mathbf{a}_k^\vee, \quad C = \sum_{k=N+1}^{\infty} \sigma_k\mathbf{a}_k\mathbf{a}_k^\vee$$

Then

$$\rho_n = B_n + C_n,\ \rho = B + C,\ B_n, C_n, B, C \in T_+(\mathcal{H}),\ \|\rho - B\|_1 = \|C\|_1 < \varepsilon,$$
$$\mathrm{Tr}_l\,\rho_n \succeq \mathrm{Tr}_l\, B_n,\ \|\mathrm{Tr}_l\,\rho - \mathrm{Tr}_l\, B\|_1 = \|\mathrm{Tr}_l\, C\|_1 \le \|C\|_1 < \varepsilon,\ n \in \mathbb{N}, l \in [2].$$

For $l \in [2]$ let $\{l'\} = [2] \setminus \{l\}$. Part (1) yields that

$$\liminf\langle(\mathrm{Tr}_{l'}\,\rho_n)\mathbf{x}_l, \mathbf{x}_l\rangle \ge \liminf\langle(\mathrm{Tr}_{l'}\, B_n)\mathbf{x}_l, \mathbf{x}_l\rangle \ge$$
$$\langle(\mathrm{Tr}_{l'}\, B)\mathbf{x}_l, \mathbf{x}_l\rangle \ge \langle(\mathrm{Tr}_{l'}\,\rho)\mathbf{x}_l, \mathbf{x}_l\rangle - \varepsilon\|\mathbf{x}_l\|^2.$$

As $\varepsilon > 0$ can be chosen arbitrary small we deduce all the inequaities in part (2). The condition (2.19) is [10, Lemma 4.3], or [14, Lemma B5].

Assume that N_2 is finite. Then $\mathcal{H}$ is isometric to the direct sum of N_2 copies of $\mathcal{H}_1$. Where each copy $\mathcal{H}_{1,j}$ has the basis $\mathbf{e}_{i,1}\otimes\mathbf{e}_{j,2}$ for $i \in [N_1]$. Let $\rho_{n,j}; \mathcal{H}_{1,j} \to \mathcal{H}_{1,j}$ be the restriction of the sesquilinear form $\langle\rho_n\mathbf{u}, \mathbf{v}\rangle$, where $\mathbf{u} = \mathbf{x}\otimes\mathbf{e}_{j,2}, \mathbf{v} = \mathbf{y}\otimes\mathbf{e}_{j,2}$. Observe that $\mathrm{Tr}_2\,\rho_n = \sum_{j=1}^{N_2} \rho_{n,j}$. Define similarly $\rho^{(j)}$ for $j \in [N_2]$. Clearly, $\rho_{n,j} \overset{w.o.t.}{\to} \rho^{(j)}$ for $j \in [N_2]$. Hence $\mathrm{Tr}_2\,\rho_n \overset{w.o.t.}{\to} \mathrm{Tr}_2\,\rho = \sum_{j=1}^{N_2} \rho^{(j)}$. Similar results apply if N_1 is finite. □

We now give a simple example to show that in part (1) of Lemma 2.3 we may have strict inequalities.

Example 2.4. Assume that $N_1 = \infty$. Consider $\rho_n = (\mathbf{e}_n \otimes \mathbf{e}_1)(\mathbf{e}_n \otimes \mathbf{e}_1)^\vee, n \in \mathbb{N}$. Then $\mathbf{e}_n \otimes \mathbf{e}_1 \overset{w.o.t.}{\to} \mathbf{0}$. So $\rho_n \overset{w.o.t.}{\to} \rho = 0$. Clearly $\mathrm{Tr}_2(\rho_n) = \mathbf{e}_n\mathbf{e}_n^\vee \overset{w.o.t.}{\to} 0$, and $\mathrm{Tr}_1\,\rho_n = \mathbf{e}_1\mathbf{e}_1^\vee$. Thus $\mathrm{Tr}_1\,\rho_n$ does not converge weakly to $\mathrm{Tr}_1\,\rho$.

3. Proof of the main theorems

3.1. **Proof of Theorem 1.4.** As

$$\|\rho^{(n)}\|_1 = \mathrm{Tr}\,\rho^{(n)} = \mathrm{Tr}(\mathrm{Tr}_1\,\rho^{(n)}) = \mathrm{Tr}(\mathrm{Tr}_2\,\rho^{(n)}), \quad \|\rho_i\|_1 = \mathrm{Tr}\,\rho_i, i \in [2],$$

we deduce that $\mathrm{Tr}\,\rho_1 = \mathrm{Tr}\,\rho_2 = \lim_{n\to\infty} \mathrm{Tr}\,\rho^{(n)}$. The inequality (2.18) yields that $\mathrm{Tr}\,\rho_1 \ge \mathrm{Tr}\,\rho$. The condition (2.19) implies that $\lim_{n\to\infty} \|\rho^{(n)} - \rho\|_1 = 0$ if and only if $\mathrm{Tr}\,\rho_1 = \mathrm{Tr}\,\rho$. Assume to the contrary that $\mathrm{Tr}\,\rho_1 = \mathrm{Tr}\,\rho_2 > \mathrm{Tr}\,\rho$.

The next claims follow from the results in Appendix B in [14]. Recall that $\mathrm{T}(\mathcal{H}) \subset \mathrm{T}_2(\mathcal{H})$. Thus $\rho^{(n)}, n \in \mathbb{N}$ and ρ are in $\mathrm{T}_2(\mathcal{H})$. Hence $\rho^{(n)}, n \in \mathbb{N}$ converges in the weak topology to ρ in the Hilbert space $\mathrm{T}_2(\mathcal{H})$. Banach-Sacks theorem [1] yields that there exists a subsequence $n_j, j \in \mathbb{N}$ such that the Cesàro subsequence $\hat{\rho}_m = \frac{1}{m}\sum_{j=1}^m \rho^{(n_j)}, m \in \mathbb{N}$ converges in the norm $\|\cdot\|_2$ to ρ. It is straightforward to show that

$$\lim_{m\to\infty} \|\operatorname{Tr}_2 \hat{\rho}_m - \rho_1\|_1 + \|\operatorname{Tr}_1 \hat{\rho}_m - \rho_2\|_1 = 0.$$

The inequalities (2.22) and (2.18) yield that

$$\alpha_1 = \rho_1 - \operatorname{Tr}_2 \rho \in \mathrm{T}_+(\mathcal{H}_1), \quad \alpha_2 = \rho_2 - \operatorname{Tr}_1 \rho \in \mathrm{T}_+(\mathcal{H}_2).$$

Note that $\operatorname{Tr} \alpha_1 = \operatorname{Tr} \alpha_2 > 0$. Consider the spectral decompositions of α_1 and α_2:

$$\alpha_1 = \sum_{i=1}^\infty \sigma_{i,1}\mathbf{g}_i\mathbf{g}_i^\vee, \{\sigma_{i,1} \ge 0\} \searrow 0, \langle \mathbf{g}_i, \mathbf{g}_j\rangle = \delta_{ij}, i,j \in \mathbb{N}, \operatorname{Tr}\alpha_1 = \sum_{i=1}^\infty \sigma_{i,1},$$
$$\alpha_2 = \sum_{i=1}^\infty \sigma_{i,2}\mathbf{f}_i\mathbf{f}_i^\vee, \{\sigma_{i,2} \ge 0\} \searrow 0, \langle \mathbf{f}_i, \mathbf{f}_j\rangle = \delta_{ij}, i,j \in \mathbb{N}, \operatorname{Tr}\alpha_2 = \sum_{i=1}^\infty \sigma_{i,2}.$$

As $\operatorname{Tr}\alpha_1 = \operatorname{Tr}\alpha_2 > 0$ there exists $\delta > 0$, such that

$$\sigma_{1,1} > \delta, \ \sigma_{1,2} > \delta, \quad \delta > 0. \tag{3.1}$$

For $N \in \mathbb{N}$ let

$$\alpha_{N,1} = \sum_{i=1}^N \sigma_{i,1}\mathbf{g}_i\mathbf{g}_i^\vee, \ \tilde{\alpha}_{N,1} = \sum_{i=N+1}^\infty \sigma_{i,1}\mathbf{g}_i\mathbf{g}_i^\vee,$$
$$\alpha_{N,2} = \sum_{i=1}^N \sigma_{i,2}\mathbf{f}_i\mathbf{f}_i^\vee, \ \tilde{\alpha}_{N,2} = \sum_{i=N+1}^\infty \sigma_{i,2}\mathbf{f}_i\mathbf{f}_i^\vee.$$

Fix N big enough so that

$$\max(\|\tilde{\alpha}_{N,1}\|_1, \|\tilde{\alpha}_{N,2}\|_1) < \delta/10. \tag{3.2}$$

For simplicity of the exposition of the proof we consider the following most difficult case. First, α_1 and α_2 are not finite dimensional: $\sigma_{i,1}, \sigma_{i,2} > 0$ for all $i \in \mathbb{N}$. Second, let $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$ be the closure of subspaces spanned by $\mathbf{g}_i, i \in \mathbb{N}$ and $\mathbf{f}_i, i \in \mathbb{N}$ respectively. Let $\hat{\mathcal{H}}_i$ be the orthogonal complement of $\tilde{\mathcal{H}}_i$ in $\mathcal{H}_i$ for $i \in [2]$. Then $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$ are infinite dimensional with orthonormal bases $\hat{\mathbf{g}}_i, i \in \mathbb{N}$ and $\hat{\mathbf{f}}_i, i \in \mathbb{N}$ respectively. Thus $\mathbf{e}_{i,j}, i \in \mathbb{N}$ is an orthonormal basis for $\mathcal{H}_j$ for $j \in [2]$, where

$$\mathbf{e}_{2i-1,1} = \mathbf{g}_i, \ \mathbf{e}_{2i,1} = \hat{\mathbf{g}}_i, \quad \mathbf{e}_{2i-1,2} = \mathbf{f}_i, \ \mathbf{e}_{2i,2} = \hat{\mathbf{f}}_i, \text{ for } i \in \mathbb{N}, j \in [2]. \tag{3.3}$$

For $m \in \mathbb{N}$, let $P_{m,j}$ be the orthogonal projection in $\mathcal{H}_j$ on the subspace spanned by $\mathbf{e}_{i,j}, i \in [2m]$ for $j \in [2]$. Define $R_m = P_{m,1} \otimes P_{m,2}$ for $m \in \mathbb{N}$. Then $P_{m,1}, P_{m,2}, R_m$ converge to the identity operators in the strong operator topology in $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}$ respectively. Recall [13, Lemma 5]:

$$\lim_{m\to\infty} \|P_{m,1}\beta_1 P_{m,1} - \beta_1\|_1 + \|P_{m,2}\beta_2 P_{m,2} - \beta_2\|_1 + \|R_m \beta R_m - \beta\|_1 = 0$$

for all $\beta_i \in \mathrm{T}(\mathcal{H}_i), i \in [2]$ and $\beta \in \mathrm{T}(\mathcal{H})$.

Assume that we have the spectral decompositions

$$\hat\rho_n=\sum_{i=1}^\infty \lambda_{i,n}\mathbf{x}_{i,n}\mathbf{x}_{i,n}^\vee, \{\lambda_{i,n}\}\searrow 0, \langle \mathbf{x}_{i,n},\mathbf{x}_{j,n}\rangle=\delta_{ij}, \operatorname{Tr}\hat\rho_n=\sum_{i=1}^\infty\lambda_{i,n}, \tag{3.4}$$

$$\rho=\sum_{i=1}^\infty \lambda_i\mathbf{x}_i\mathbf{x}_i^\vee, \{\lambda_i\}\searrow 0, \langle \mathbf{x}_i,\mathbf{x}_j\rangle=\delta_{ij}, \operatorname{Tr}\rho=\sum_{i=1}^\infty\lambda_i.$$

Lemma B.6 of [14] yields that $\lim_{n\to\infty}\lambda_{i,n}=\lambda_i$ for each $i\in\mathbb{N}$. Furthermore, by passing to a subsequence of $\hat\rho_n$, we can assume that $\lim_{n\to\infty}\|\mathbf{x}_{i,n}-\mathbf{x}_i\|=0$ for each $\lambda_i>0$. Again, for simplicity of the exposition of the proof we will assume the most difficult case that $\lambda_i>0$ for each $i\in\mathbb{N}$.

Recall that $\lim_{m\to\infty}\|R_m\rho R_m-\rho\|_1=0$. Then there exists $m\in\mathbb{N}$ such that

$$\|R_m\rho R_m-\rho\|_1<\delta/10 \text{ and } m>N. \tag{3.5}$$

We now keep $m>N$ fixed. The inequality (2.2) yields

$$\sigma_i(R_m(\hat\rho_n-\rho)R_m)\le\|R_m\|^2\sigma_i(\hat\rho_n-\rho)=\sigma_i(\hat\rho_n-\rho), \text{ for } i\in\mathbb{N}\Rightarrow$$
$$\|R_m(\hat\rho_n-\rho)R_m)\|_2\le\|\hat\rho_n-\rho\|_2.$$

As $\lim_{n\to\infty}\|\hat\rho_n-\rho\|_2=0$ we deduce that there exists $M_1\in\mathbb{N}$ such that $\|R_m(\hat\rho_n-\rho)R_m)\|_2\le\delta/(20m)$ for $n>M_1$. Recall that $\operatorname{rank}R_m=4m^2$. Hence $\operatorname{rank}R_m(\hat\rho_n-\rho)R_m\le 4m^2$. Thus

$$\|R_m(\hat\rho_n-\rho)R_m)\|_1=\sum_{i=1}^{4m^2}\sigma_i(R_m(\hat\rho_n-\rho)R_m)\le$$
$$2m\left(\sum_{i=1}^{4m^2}\sigma_i^2(R_m(\hat\rho_n-\rho)R_m)\right)^{1/2}=2m\|R_m(\hat\rho_n-\rho)R_m)\|_2\le\delta/10.$$

Part (2) of Lemma 2.2 yields

$$\|\operatorname{Tr}_i R_m\hat\rho_nR_m-\operatorname{Tr}_i R_m\rho R_m\|_1\le\|R_m\hat\rho_nR_m-R_m\rho R_m\|_1\le\delta/10 \tag{3.6}$$

for $n>M_1$.

In addition, we have $\operatorname{Tr}_i\hat\rho_n$ converge in trace norm to ρ_{i+1}, where $\rho_3=\rho_1$. Thus there exists M_2, when $n>M_2$, we have

$$\|\operatorname{Tr}_i\hat\rho_n-\rho_{i+1}\|_1\le\delta/10, \text{ for } i\in[2].$$

Thus for $n>\max(M_1,M_2)$, we have

$$\|\operatorname{Tr}_i(\hat\rho_n-R_m\hat\rho_nR_m)-(\rho_{i+1}-\operatorname{Tr}_i(R_m\rho R_m))\|_1\le\delta/5. \tag{3.7}$$

Lemma 2.2 and (3.5) imply

$$\|\operatorname{Tr}_i(R_m\rho R_m)-\operatorname{Tr}_i\rho\|_1\le\|R_m\rho R_m-\rho\|_1<\delta/10 \text{ for } i\in[2]. \tag{3.8}$$

We use $\operatorname{Tr}_i\rho$ to replace the $\operatorname{Tr}_i(R_m\rho R_m)$ in (3.7) to get

$$\|\operatorname{Tr}_i(\hat\rho_n-R_m\hat\rho_nR_m)-(\rho_{i+1}-\operatorname{Tr}_i\rho)\|_1\le 3\delta/10.$$

Let Tr_0 stand for Tr_2. Recall that $\alpha_i=\rho_i-\operatorname{Tr}_{i-1}\rho=\alpha_{N,i}+\tilde\alpha_{N,i}$ for $i\in[2]$. The inequality (3.2) yields

$$\|\operatorname{Tr}_{i-1}(\hat\rho_n-R_m\hat\rho_nR_m)-\alpha_{N,i}\|_1\le 2\delta/5 \text{ for } i\in[2] \tag{3.9}$$

and $n>\max(M_1,M_2)$. We finally get the contradiction by showing that the above two inequalities are incompatible.

Recall the spectral decomposition of $\hat{\rho}_n$ given by (3.4). Using the bases of $\mathcal{H}_1, \mathcal{H}_2$ defined by (3.3), we can write

$$\mathbf{x}_{i,n} = \sum_{p,q=1}^{\infty} \mu_{p,q}^{i,n} \mathbf{e}_{p,1} \otimes \mathbf{e}_{q,2}.$$

So we have

$$\begin{aligned}
\lambda_{i,n}\mathbf{x}_{i,n}\mathbf{x}_{i,n}^{\vee} &= \lambda_{i,n}(\sum_{p,q=1}^{\infty} \mu_{p,q}^{i,n}\mathbf{e}_{p,1}\otimes\mathbf{e}_{q,2})(\sum_{r,s=1}^{\infty}\mu_{r,s}^{i,n}\mathbf{e}_{r,1}\otimes\mathbf{e}_{s,2})^{\vee} \\
&= \lambda_{i,n}\left(\sum_{p,q,r,s=1}^{\infty}\mu_{p,q}^{i,n}\bar{\mu}_{r,s}^{i,n}(\mathbf{e}_{p,1}\otimes\mathbf{e}_{q,2})(\mathbf{e}_{r,1}\otimes\mathbf{e}_{s,2})^{\vee}\right).
\end{aligned}$$

Hence

$$\hat{\rho}_n = \sum_{i=1}^{\infty}\lambda_{i,n}\left(\sum_{p,q,r,s=1}^{\infty}\mu_{p,q}^{i,n}\bar{\mu}_{r,s}^{i,n}(\mathbf{e}_{p,1}\otimes\mathbf{e}_{q,2})(\mathbf{e}_{r,1}\otimes\mathbf{e}_{s,2})^{\vee}\right),$$

$$\begin{aligned}
\hat{\rho}_n - R_m\hat{\rho}_nR_m &= \sum_{i=1}^{\infty}\lambda_{i,n}\left(\sum_{p,q,r,s=1}^{\infty}\mu_{p,q}^{i,n}\bar{\mu}_{r,s}^{i,n}(\mathbf{e}_{p,1}\otimes\mathbf{e}_{q,2})(\mathbf{e}_{r,1}\otimes\mathbf{e}_{s,2})^{\vee}\right) \\
&- \sum_{i=1}^{\infty}\lambda_{i,n}\left(\sum_{p,q,r,s=1}^{2m}\mu_{p,q}^{i,n}\bar{\mu}_{r,s}^{i,n}(\mathbf{e}_{p,1}\otimes\mathbf{e}_{q,2})(\mathbf{e}_{r,1}\otimes\mathbf{e}_{s,2})^{\vee}\right)
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathrm{Tr}_1(\mathbf{x}_{i,n}\mathbf{x}_{i,n}^{\vee}) &= \sum_{p,q,s=1}^{\infty}\mu_{p,q}^{i,n}\bar{\mu}_{p,s}^{i,n}\mathbf{e}_{q,2}\mathbf{e}_{s,2}^{\vee}, \\
\mathrm{Tr}_1\hat{\rho}_n &= \sum_{i=1}^{\infty}\lambda_{i,n}(\sum_{p,q,s=1}\mu_{p,q}^{i,n}\bar{\mu}_{p,s}^{i,n}\mathbf{e}_{q,2}\mathbf{e}_{s,2}^{\vee}) \\
&= \sum_{q,s=1}^{\infty}\left((\sum_{i=1}^{\infty}\sum_{p=1}^{\infty}\lambda_{i,n}\mu_{p,q}^{i,n}\bar{\mu}_{p,s}^{i,n})\mathbf{e}_{q,2}\mathbf{e}_{s,2}^{\vee}\right) \\
\mathrm{Tr}_1(\hat{\rho}_n - R_m\hat{\rho}_nR_m) &= \sum_{q,s=1}^{\infty}\left((\sum_{i=1}^{\infty}\sum_{p=1}^{\infty}\lambda_{i,n}\mu_{p,q}^{i,n}\bar{\mu}_{p,s}^{i,n})\mathbf{e}_{q,2}\mathbf{e}_{s,2}^{\vee}\right) \\
&- \sum_{q,s=1}^{2m}\left((\sum_{i=1}^{\infty}\sum_{p=1}^{2m}\lambda_{i,n}\mu_{p,q}^{i,n}\bar{\mu}_{p,s}^{i,n})\mathbf{e}_{q,2},\mathbf{e}_{s,2}^{\vee}\right)
\end{aligned}$$

Write down the diagonal elements of $\mathrm{Tr}_1(\hat{\rho}_n - R_m\hat{\rho}_nR_m)$:

$$\begin{aligned}
&\sum_{q=1}^{2m}(\sum_{i=1}^{\infty}\sum_{p=2m+1}^{\infty}\lambda_{i,n}\mu_{p,q}^{i,n}\bar{\mu}_{p,q}^{i,n})\mathbf{e}_{q,2}\mathbf{e}_{q,2}^{\vee} + \\
&\sum_{q=2m+1}^{\infty}(\sum_{i=1}^{\infty}\sum_{p=1}^{\infty}\lambda_{i,n}\mu_{p,q}^{i,n}\bar{\mu}_{p,q}^{i,n})\mathbf{e}_{q,2}\mathbf{e}_{q,2}^{\vee}.
\end{aligned} \tag{3.10}$$

As $m > N$ are fixed as mentioned above, and $n > \max(M_1, M_2)$, according to (3.9), we have

$$\|\mathrm{Tr}_1(\hat{\rho}_n - R_m\hat{\rho}_nR_m) - \alpha_{N,2}\|_1 \leq 2\delta/5.$$

Observe that the diagonal elements of $\mathrm{Tr}_1(\hat\rho_n - R_m\hat\rho_n R_m) - \alpha_{N,2}$ are:

$$\sum_{t=1}^{N}\big((\sum_{i=1}^{\infty}\sum_{p=2m+1}^{\infty}\lambda_{i,n}|\mu_{p,2t-1}^{i,n}|^2) - \sigma_{t,2}\big)\mathbf{e}_{2t-1,2}\mathbf{e}_{2t-1,2}^{\vee} +$$
$$\sum_{t=1}^{N}\big(\sum_{i=1}^{\infty}\sum_{p=2m+1}^{\infty}\lambda_{i,n}|\mu_{p,2t}^{i,n}|^2\big)\mathbf{e}_{2t,2}\mathbf{e}_{2t,2}^{\vee} +$$
$$\sum_{q=2N+1}^{2m}\big(\sum_{i=1}^{\infty}\sum_{p=2m+1}^{\infty}\lambda_{i,n}|\mu_{p,q}^{i,n}|^2\big)\mathbf{e}_{q,2}\mathbf{e}_{q,2}^{\vee} +$$
$$\sum_{q=2m+1}^{\infty}\big(\sum_{i=1}^{\infty}\sum_{p=1}^{\infty}\lambda_{i,n}|\mu_{p,q}^{i,n}|^2\big)\mathbf{e}_{q,2}\mathbf{e}_{q,2}^{\vee}.$$

Lemma A.3 in [14] yields that the absolute values of the diagonal elements of $\mathrm{Tr}_1(\hat\rho_n - R_m\hat\rho_n R_m) - \alpha_{N,2}$ are bounded by $\|\mathrm{Tr}_1(\hat\rho_n - R_m\hat\rho_n R_m) - \alpha_{N,2}\|_1 \le 2\delta/5$. As $\lambda_{i,n} \ge 0$ for $i, n \in \mathbb{N}$ we deduce

$$\sum_{t=1}^{N}|(\sum_{i=1}^{\infty}\sum_{p=2m+1}^{\infty}\lambda_{i,n}|\mu_{p,2t-1}^{i,n}|^2) - \sigma_{t,2}| + \sum_{t=1}^{N}\sum_{i=1}^{\infty}\sum_{p=2m+1}^{\infty}\lambda_{i,n}|\mu_{p,2t}^{i,n}|^2 +$$
$$\sum_{q=2N+1}^{2m}\sum_{i=1}^{\infty}\sum_{p=2m+1}^{\infty}\lambda_{i,n}|\mu_{p,q}^{i,n}|^2 + \sum_{q=2m+1}^{\infty}\sum_{i=1}^{\infty}\sum_{p=1}^{\infty}\lambda_{i,n}|\mu_{p,q}^{i,n}|^2 \le 2\delta/5.$$

In particular we deduce the following two inequalities:

$$|(\sum_{i=1}^{\infty}\sum_{p=2m+1}^{\infty}\lambda_{i,n}|\mu_{p,1}^{i,n}|^2) - \sigma_{1,2}| \le 2\delta/5$$

$$\sum_{q=2m+1}^{\infty}\sum_{i=1}^{\infty}\lambda_{i,n}|\mu_{1,q}^{i,n}|^2 \le 2\delta/5 \tag{3.11}$$

The inequality (3.1) and the first above inequality yield

$$\sum_{i=1}^{\infty}\sum_{p=2m+1}^{\infty}\lambda_{i,n}|\mu_{p,1}^{i,n}|^2 \ge \delta - 2\delta/5 = 3\delta/5 \tag{3.12}$$

Consider now similar inequaities for the diagonal entries of $\mathrm{Tr}_2(\hat\rho_n - R_m\hat\rho_n R_m) - \alpha_{N,1}$. Then the analogous inequality to (3.11) is

$$\sum_{p=2m+1}^{\infty}\sum_{i=1}^{\infty}\lambda_{i,n}|\mu_{p,1}^{i,n}|^2 \le 2\delta/5.$$

But this inequality contradicts the inequality (3.12).

The equalities $\mathrm{Tr}\,\rho_1 = \mathrm{Tr}\,\rho_2 = \lim_{n\to\infty}\mathrm{Tr}\,\rho^{(n)}$ establishes the last part of the theorem.

3.2. **Proof of Theorem 1.3.** We first observe:

Lemma 3.1. *Let $\mathcal{H}_1, \mathcal{H}_2$ be two separable Hilbert spaces. Assume that $\rho_i \in \mathrm{T}(\mathcal{H}_i)$ for $i \in [2]$. Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Then the function $f : \mathrm{T}(\mathcal{H}) \to [0, \infty)$ given by (1.4) is a convex Lipschitz function with the Lipschitz constant 2. Furthermore*

$$f(X) \ge 2\|X\|_1 - \|\rho_1\|_1 - \|\rho_2\|_1 \text{ for } X \in \mathrm{T}_+(\mathcal{H}). \tag{3.13}$$

Proof. Assume that $X_1, X_2 \in \mathrm{T}(\mathcal{H})$. We first show that f is a Lipschitz function with the Lipschitz constant 2. Then

$$\begin{aligned}
&|f(X_1) - f(X_2)| \\
&= |\|\operatorname{Tr}_2 X_1 - \rho_1\|_1 - \|\operatorname{Tr}_2 X_2 - \rho_1\|_1 + \|\operatorname{Tr}_1 X_1 - \rho_2\|_1 - \|\operatorname{Tr}_1 X_2 - \rho_2\|_1| \\
&\le |\|\operatorname{Tr}_2 X_1 - \rho_1\|_1 - \|\operatorname{Tr}_2 X_2 - \rho_1\|_1| + |\|\operatorname{Tr}_1 X_1 - \rho_2\|_1 - \|\operatorname{Tr}_1 X_2 - \rho_2\|_1| \\
&\le \|\operatorname{Tr}_2(X_1 - X_2)\|_1 + \|\operatorname{Tr}_1(X_1 - X_2)\|_1 \le 2\|X_1 - X_2\|_1.
\end{aligned}$$

We now show the convexity of f. Assume that $t \in (0,1)$. Let $X = tX_1 + (1-t)X_2$. Then

$$\begin{aligned}
f(X) &= \|\operatorname{Tr}_2(tX_1 + (1-t)X_2) - (t + (1-t))\rho_1\|_1 \\
&\quad + \|\operatorname{Tr}_1(tX_1 + (1-t)X_2) - (t + (1-t))\rho_2\|_1 \\
&\le t\|\operatorname{Tr}_2 X_1 - \rho_1\|_1 + (1-t)\|\operatorname{Tr}_2 X_2 - \rho_1\|_1 \\
&\quad + t\|\operatorname{Tr}_1 X_1 - \rho_2\|_1 + (1-t)\|\operatorname{Tr}_1 X_2 - \rho_2\|_1 \\
&= tf(X_1) + (1-t)f(X_2).
\end{aligned}$$

Assume that $X \in \mathrm{T}_+(\mathcal{H})$. Then $\operatorname{Tr}_j X \in \mathrm{T}_+(\mathcal{H}_{j+1})$ for $j \in [2]$, where $\mathcal{H}_3 = \mathcal{H}_1$. Hence $\|X\|_1 = \operatorname{Tr} X = \operatorname{Tr}(\operatorname{Tr}_j X) = \|\operatorname{Tr}_j X\|_1$ for $j \in [2]$. The triangle inequality yields

$$f(X) \ge \|\operatorname{Tr}_2 X\|_1 - \|\rho_1\|_1 + \|\operatorname{Tr}_1 X\|_1 - \|\rho_2\|_1 = 2\|X\|_1 - \|\rho_1\|_1 - \|\rho_2\|_1.$$

□

Lemma 3.2. *Let the assumptions of Lemma 3.1 hold. Assume that $\mathcal{X} \subseteq \mathcal{H}$ is a closed infinite dimensional subspace with an orthonormal basis $\mathbf{x}_i, i \in \mathbb{N}$. Let $\mathcal{X}_n$ be the subspace spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for $n \in \mathbb{N}$. Consider the infimum* (1.5)*. Then*

$$\mu_n(\rho_1, \rho_2) = \min\{f(X), X \in \mathrm{S}_+(\mathcal{X}_n), \|X\|_1 \le \|\rho_1\|_1 + \|\rho_2\|_1\}. \tag{3.14}$$

Furthermore, the sequence $\mu_n(\rho_1, \rho_2), n \in \mathbb{N}$ is nonincreasing.

Proof. Clearly $f(0) = \|\rho_1\|_1 + \|\rho_2\|_1$. Hence $\mu_n(\rho_1, \rho_2) \le f(0)$. Suppose that $X \in \mathrm{T}_+(\mathcal{H})$ and $\|X\|_1 > f(0)$. The inequality (3.13) yields that $f(X) \ge 2\|X\|_1 - f(0) > f(0)$. Hence it is enough to consider the infimum (1.5) restricted to $\{X \in \mathrm{S}_+(\mathcal{X}_n), \|X\|_1 \le f(0)\}$. This is a compact finite dimensional set. Hence the infimum is achieved. As $\mathcal{X}_n \subset \mathcal{X}_{n+1}$ we deduce that $\mu_{n+1}(\rho_1, \rho_2) \le \mu_n(\rho_1, \rho_2)$ for each $n \in \mathbb{N}$. □

Proof of Theorem 1.3. First assume that there exists $\rho \in \mathrm{T}_+(\mathcal{H})$ such that $\operatorname{Tr}_2 \rho = \rho_1, \operatorname{Tr}_1 \rho = \rho_2$ and $\operatorname{supp} \rho \subseteq \mathcal{X}$. As $\operatorname{Tr} \rho = \operatorname{Tr} \rho_1$ we deduce that $\rho \in \mathrm{S}_{+,1}(\mathcal{H})$. Next observe $\rho \in T_+(\mathcal{X})$. Let $P_n \in B(\mathcal{H})$ be the projection on span of $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Then $P_n \in B(\mathcal{X})$ and $P_n, n \in \mathbb{N}$ converges in the strong operator topology to $I_{\mathcal{X}}$. Lemma 5 in [13] yields that $\lim_{n\to\infty} \|P_n \rho P_n - \rho\|_1 = 0$ in $\mathrm{T}(\mathcal{X})$. As $\operatorname{supp} P_n \rho P_n \subseteq \mathcal{X}_n$ it follows that $P_n \rho P_n \in \mathrm{S}_+(\mathcal{X}_n)$ converges to ρ in norm in $T(\mathcal{H})$. Hence $\lim_{n\to\infty} f(P_n \rho P_n) = 0$. Clearly, $\mu_n(\rho_1, \rho_2) \le f(P_n \rho P_n)$. Hence $\lim_{n\to\infty} \mu_n(\rho_1, \rho_2) = 0$.

Second assume that $\lim_{n\to\infty} \mu_n(\rho_1, \rho_2) = 0$. Assume that $\rho^{(n)} \in \mathrm{T}_+(\mathcal{H})$, $\operatorname{supp} \rho^{(n)} \subseteq \mathcal{X}_n$ and $\mu_n(\rho_1, \rho_2) = f(\rho^{(n)})$. Clearly

$$\lim_{n\to\infty} \|\rho^{(n)}\|_1 = \lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} = \|\rho_1\| = \operatorname{Tr} \rho_1.$$

Thus the sequence $\rho^{(n)}, n \in \mathbb{N}$ is bounded. Hence, there exists a subsequence $\rho^{(n_k)}$ which converges in weak operator topology to ρ. Let $\mathbf{x} \in \mathcal{X}^\perp$. Then $\mathbf{x} \in \mathcal{X}_n^\perp$. Therefore $\rho^{(n)}\mathbf{x} = 0$ and $\langle \rho^{(n)}\mathbf{x}, \mathbf{y}\rangle = 0$ for each $\mathbf{y} \in \mathcal{H}$. As $\rho^{(n_k)} \overset{w.o.t.}{\to} \rho$ we deduce that $\langle \rho\mathbf{x}, \mathbf{y}\rangle = 0$ for each $\mathbf{y} \in \mathcal{H}$. Hence $\rho\mathbf{x} = \mathbf{0}$. Thus $\operatorname{supp}\rho \subseteq \mathcal{X}$. As $\lim_{k\to\infty} f(\rho^{(n_k)}) = 0$, Theorem 1.4 yields that $\lim_{k\to\infty} \|\rho^{(n_k)} - \rho\|_1 = 0$. Hence $\operatorname{Tr}_2 \rho = \rho_1$ and $\operatorname{Tr}_1 \rho = \rho_2$. □

4. An SDP solution when $\mathcal{X}$ is finite dimensional

The quantum Strassen problem can be easily generalized to a standard semidefinite problem in the finite dimensional case. The feasible set is bounded and contains a positive definite matrix. Hence we can solve this problem using interior-point methods [20]. Moreover, the strong duality for this SDP problems holds. In this section we show that we can extend this approach to separable infinite dimensional $\mathcal{H}_1$ and $\mathcal{H}_2$ provided that $\mathcal{X}$ is finite dimensional.

4.1. **Finite dimensional case.** Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ be a finite dimensional Hilbert space. Let $\mathcal{X} \subseteq \mathcal{H}$ be a closed subspace. Given two partial density operators $\rho_i \in \mathrm{S}_+(\mathrm{H}_i) \setminus \{0\}$, $i \in [2]$. We now state the following SDP problem:

$$\mu(\rho_1, \rho_2, \mathcal{X}) = \max\{\operatorname{Tr}(XP_{\mathcal{X}}),\ X \in \mathrm{S}_+(\mathcal{H}), \operatorname{Tr}_2 X \preceq \rho_1, \operatorname{Tr}_1 X \preceq \rho_2\}.$$

Note that the feasible set is convex and bounded, as $\operatorname{Tr} X \le \min(\operatorname{Tr}\rho_1, \operatorname{Tr}\rho_2)$. If $\operatorname{supp}\rho_i = \mathcal{H}_i$ for $i \in [2]$ then a feasible set contains a positive definite matrix. In other cases it is easy to show that it is enough to restrict the problem to $\mathcal{H}_i' = \operatorname{supp}\rho_i$ for $i \in [2]$ and $\mathcal{H}' = \mathcal{H}_1' \otimes \mathcal{H}_2'$. Then we can replace $\mathcal{X}$ by $\mathcal{X}' = \mathcal{X} \cap \mathcal{H}'$.

We write down its primal problem and dual problem.

Primal problem		Dual problem	
maximize:	$\langle A, X\rangle$,	minimize:	$\langle B, Y\rangle$
subject to:	$\Phi(X) \preceq B$;	subject to:	$\Phi^*(Y) \succeq A$;
	$X \in \mathrm{S}_+(\mathcal{H}_1 \otimes \mathcal{H}_2)$		$Y \in \mathrm{S}_+(\mathcal{H}_1 \oplus \mathcal{H}_2)$

Here

$$\Phi : \mathrm{S}_+(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathrm{S}_+(\mathcal{H}_1 \oplus \mathcal{H}_2), \quad \Phi^* : \mathrm{S}_+(\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathrm{S}_+(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

$$A = P_{\mathcal{X}}, \quad B = \begin{bmatrix} \rho_1 & \\ & \rho_2 \end{bmatrix},$$

$$\Phi(X) = \begin{bmatrix} \operatorname{Tr}_2(X) & \\ & \operatorname{Tr}_1(X) \end{bmatrix},$$

$$\Phi^*(Y) = \Phi^* \begin{bmatrix} Y_1 & \\ & Y_2 \end{bmatrix} = Y_1 \otimes I_2 + I_1 \otimes Y_2.$$

(Note that the above Φ is the restriction of Φ given by (1.1) to positive semidefinite matrices). It's easy to check the following equality:

$$\forall M, N, \langle \Phi(M), N\rangle = \langle M, \Phi^*(N)\rangle.$$

Moreover, the strong duality holds for this semidefinite program as we can check that the primal feasible set is not empty, (0 is an allowable point), and there exists an interior point in the dual feasible set.

- A primal feasible point: set $X = 0 \in \mathrm{S}_+(\mathcal{H}_1 \otimes \mathcal{H}_2), \operatorname{Tr}_1(X) \preceq \rho_2, \operatorname{Tr}_2(X) \preceq \rho_1$.

- A dual strict feasible point: set $Y = I_1 \oplus I_2 \in \mathrm{S}_+(\mathcal{H}_1 \oplus \mathcal{H}_2), \Phi^*(Y) = 2I_{12} \succ P_{\mathcal{X}}$.

Hence, the primal and dual problems have no duality gap and the bounded optimal solution of (4.1) can be computed by interior point methods [20].

Theorem 4.1. *Let $\rho_i \in S_{+,1}(\mathcal{H}_i), i \in [2]$. Assume that $\mathcal{X} \subset \mathcal{H}$. There exists $\rho \in S_{+,1}(\mathcal{H}), \operatorname{supp} \rho \subseteq \mathcal{X}$ such that $\operatorname{Tr}_2 \rho = \rho_1, \operatorname{Tr}_1 \rho = \rho_2$ if and only if $\mu(\rho_1, \rho_2, \mathcal{X}) = 1$.*

Proof. Assume that there exists $\rho \in S_{+,1}(\mathcal{H}), \operatorname{supp} \rho \subseteq \mathcal{X}$ such that $\operatorname{Tr}_2 \rho = \rho_1, \operatorname{Tr}_1 \rho = \rho_2$. We choose $X = \rho$, so $\operatorname{Tr}(\rho P_{\mathcal{X}}) = \operatorname{Tr}(\rho) = 1$ as $\operatorname{supp} \rho \subseteq \mathcal{X}$. For every feasible point X, $\operatorname{Tr}(XP_{\mathcal{X}}) \le \operatorname{Tr}(X) = \operatorname{Tr}(\operatorname{Tr}_2(X)) \le \operatorname{Tr}(\rho_1) = 1$. So $\mu(\rho_1, \rho_2, \mathcal{X}) = 1$.

Assume $\mu(\rho_1, \rho_2, \mathcal{X}) = 1$ and the maximum is reached by X_{max}. Then we have $1 = \operatorname{Tr}(X_{max} P_{\mathcal{X}}) \le \operatorname{Tr}(X_{max}) \le \operatorname{Tr}(\rho_1) = 1$, so $\operatorname{Tr}(X_{max} P_{\mathcal{X}}) = \operatorname{Tr}(X_{max})$, it means that $\operatorname{supp}(X_{max}) \subset \mathcal{X}$. From $\operatorname{Tr}_2 X \preceq \rho_1$ and $\operatorname{Tr}(\rho_1 - \operatorname{Tr}_2(X_{max})) = 0$, we derive $\rho_1 = \operatorname{Tr}_2(X_{max})$. In the same way, we can show $\rho_2 = \operatorname{Tr}_1(X_{max})$. □

According to Theorem 4.1, we can check the existence of quantum lifting by checking whether $\mu(\rho_1, \rho_2, \mathcal{X})$ is equal to 1. This can be done numerically by verifying if $\mu(\rho_1, \rho_2, \mathcal{X}) > 1 - \varepsilon$ for a given ε in polynomial time in the given data, see Nesterov and Nemirovsky [20].

4.2. **Infinite dimensional case.** In this subsection we assume that $\mathcal{X} \subset \mathcal{H}$ is finite dimensional.

4.2.1. *$\mathcal{H}_1$ is infinite dimensional and $\mathcal{H}_2$ is finite dimensional.*

Lemma 4.2. *Let $\mathcal{H}_1, \mathcal{H}_2$ be separable Hilbert spaces of dimensions $N_1 = \infty, N_2 < \infty$. Assume that $\mathcal{X} \subset \mathcal{H}$ is a finite dimensional subspace of dimension N. Then there exists a finite dimensional subspace $\mathcal{H}_1' \subset \mathcal{H}_1$ of dimension NN_2 at most such that $\mathcal{X} \subset \mathcal{H}' = \mathcal{H}_1' \otimes \mathcal{H}_2$.*

Proof. Assume that $\mathbf{e}_{i,1}, i \in \mathbb{N}$ is an orthonormal basis in $\mathcal{H}_1$, and $\mathcal{H}_2$ has an orthonormal basis $\{\mathbf{e}_{1,2}, \ldots, \mathbf{e}_{N_2,2}\}$. Assume that $\mathbf{x}_1, \ldots, \mathbf{x}_N$ is a basis in $\mathcal{X}$. Then

$$\mathbf{x}_l = \sum_{i=p=1}^{\infty, N_2} x_{ip,l} \mathbf{e}_{i,1} \otimes \mathbf{e}_{p,2}, \quad l \in [N].$$

Set $\mathbf{u}_{l,p} = \sum_{i=1}^{\infty} x_{ip,l} \mathbf{e}_{i,1}$. Then $\mathbf{x}_l = \sum_{p=1}^{N_2} \mathbf{u}_{l.p} \otimes \mathbf{e}_{p,2}$. Let $\mathcal{H}_1'$ be the subspace of $\mathcal{H}_1$ spanned by $\mathbf{u}_{l.p}$ for $l \in [N], p \in [N_2]$. Then $\dim \mathrm{H}_1' \le NN_2$ and $\mathcal{X} \subseteq \mathcal{H}_1' \otimes \mathcal{H}_2$. □

Thus, in this case the coupling problem is a finite dimensional problem.

4.2.2. *$\mathcal{H}_1$ and $\mathcal{H}_2$ are infinite dimensional.* Assume that $\mathcal{H}$ is an infinite dimensional separable Hilbert space. Let $\mathcal{X}$ be a closed subspace. Then $\mathrm{B}(\mathcal{X})$ is the subspace of all bounded operators in $L \in \mathrm{B}(\mathcal{H})$ such that $L(\mathcal{X}) \subseteq \mathcal{X}$ and $L(\mathcal{X}^\perp) = 0$. In particular, $L \in \mathrm{B}(\mathcal{H})$ has support in $\mathcal{X}$ if and only if $L \in \mathrm{B}(\mathcal{X})$.

We assume now that $\mathcal{X}$ is finite dimensional, and $N = \dim \mathcal{X}$. Then $\mathrm{B}(\mathcal{X})$ has complex dimension N^2. It can be identified with $\mathbb{C}^{N \times N}$ as follows. Fix an orthonormal basis $\mathbf{x}_1, \ldots, \mathbf{x}_N$ in $\mathcal{X}$. Then a basis in $\mathrm{B}(\mathcal{X})$ is $\mathbf{x}_i \mathbf{x}_j^\vee$ for $i, j \in [N]$. Thus $L \in \mathrm{B}(\mathcal{X})$ is of the form $L = \sum_{i=j=1}^{N} a_{ij} \mathbf{x}_i \mathbf{x}_j^\vee$. Hence L is one-to-one correspondence with $A = [a_{ij}] \in \mathbb{C}^{N \times N}$. Observe next that $L \in \mathrm{S}(\mathcal{X})$ if and only if A is Hermitian.

In what follows we need the following lemma:

Lemma 4.3. *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Assume that $\mathcal{X} \subset \mathcal{H}$ is a finite dimensional subspace of dimension N, and $\mathbf{x}_1, \ldots, \mathbf{x}_N$ is an orthonormal basis of $\mathcal{X}$. Let $Q_n \in \mathrm{K}(\mathcal{H}), n \in \mathbb{N}$ be a sequence of orthogonal projections such that $Q_n \to I$ in the strong operator topology. Set $\mathcal{X}_n = Q_n\mathcal{X}$.*

(1) *There exists $K \in \mathbb{N}$ such that $\dim \mathcal{X}_n = N$ for $n > K$.*
(2) *Let $\rho^n \in S_+(\mathcal{X}_n)$ and assume that $\operatorname{Tr} \rho^n \le c$ for $n > K$. Then there exists a subsequence ρ^{n_k} that converges in trace norm to $\rho \in S_+(\mathcal{X})$.*

Proof. First observe that since Q_n is an orthogonal projection we have the inequality $\|Q_n\mathbf{x}_i\| \le 1$ for $i \in [N]$ and $n \in \mathbb{N}$. As $\lim_{n\to\infty} \|Q_n\mathbf{x}_i - \mathbf{x}_i\| = 0$ for $i \in [N]$ we deduce that for a given $\varepsilon > 0$ there exists $K(\varepsilon)$ such that

$$1 - \varepsilon < \langle Q_n\mathbf{x}_i, Q_n\mathbf{x}_i\rangle \le 1, \quad |\langle Q_n\mathbf{x}_i, Q_n\mathbf{x}_j\rangle| < \varepsilon \text{ for } i, j \in [N] \text{ and } i \ne j.$$

Let $W_n = [\langle Q_n\mathbf{x}_i, Q_n\mathbf{x}_j\rangle] \in \mathbb{C}^{N\times N}$. Then W_n is Hermitian. We claim that W_n is positive definite for $\varepsilon < 1/N$. More precisely $\sigma_1(W_n - I_N) < N\varepsilon$. (This follows from Perron-Frobenius theorem, as the absolute value of each entry of $I - W_n$ is less than ε. See [12].) Let $\lambda_1(W_n) \ge \cdots \ge \lambda_N(W_n)$ be the eigenvalues of W_n. As $W_n - I_N$ is Hermitian it follows that $|\lambda_i(W_n - I_N)| \le N\varepsilon$.

(1) For $K = K(1/N)$, W is positive definite. Hence $Q_n\mathbf{x}_1, \ldots, Q_n\mathbf{x}_N$ are linearly independent for $n > K$.

(2) Assume that $n > K$. Denote by $W_n^{1/2}$ the unique positive definite matrix which is the square root of W_n. Note that the eigenvalues of $W_n^{1/2}$ satisfy also the inequality $|\lambda_i(W_n^{1/2} - I_N)| < N\varepsilon$. Hence $\lim_{n\to\infty} W_n^{1/2} = I_N$. Observe that $L \in B(\mathcal{X}_n)$ is of the form $\sum_{i=j=1}^N a_{ij}Q_n\mathbf{x}_i(Q_n\mathbf{x}_j)^\vee$. Furthermore $\rho \in S_+(\mathcal{X}_n)$ if and only if $A = [a_{ij}] \in \mathbb{C}^{N\times N}$ is Hermitian and positive semidefinite. However, the trace of L is not equal to the trace of A but to the trace of $W_n^{-1/2}AW_n^{-1/2}$ which is $\operatorname{Tr} W_n^{-1}A$. This follows from the observation that $\mathcal{X}_n$ has an orthonormal basis $(\mathbf{x}_1, \ldots, \mathbf{x}_N)W^{1/2}$. Note that

$$(1 - N\varepsilon)I_N \preceq W_n \preceq (1 + N\varepsilon)I_N \iff (1 + N\varepsilon)^{-1}I_N \preceq W_n \preceq (1 - N\varepsilon)^{-1}I_N.$$

Hence for $A \succeq 0$ we get

$$(1 + N\varepsilon)^{-1} \operatorname{Tr} A \le \operatorname{Tr} \rho \le (1 - N\varepsilon)^{-1} \operatorname{Tr} A$$

Assume that $\rho^n \in S_+(\mathcal{X}_n)$ is a sequence whose trace is bounded above. Let $\rho^n = \sum_{i=j=1}^N a_{ij,n}Q_n\mathbf{x}_i(Q_n\mathbf{x}_j)^\vee, n > K$. Set $A_n = [a_{ij,n}] \in \mathbb{C}^{N\times N}$. Then $A_n, n > K$ are positive semidefinite matrices with bounded traces. Therefore there exists a subsequence A_{n_k} which converges entrywise to $A = [a_{ij}]$. Set $\rho = \sum_{i=j=1}^N a_{ij}\mathbf{x}_i\mathbf{x}_j^\vee$. It now follows that $\lim_{k\to\infty} \|\rho^{n_k} - \rho\|_1 = 0$. □

Lemma 4.4. *Let $\mathcal{H}_1, \mathcal{H}_2$ be two separable Hilbert spaces with countable orthogonal bases $\mathbf{e}_{i,1}, \mathbf{e}_{i,2}$ for $i \in \mathbb{N}$ respectively. Set $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Assume that $\rho \in S_+(\mathcal{H}), \rho_i \in S_+(\mathcal{H}_i)$ are given and $\operatorname{Tr}_i \rho = \rho_i, i \in [2]$. Let $P_{n,i} \in S_+(\mathcal{H}_i)$ be the orthogonal projection on $\mathcal{H}_{i,n} = \operatorname{span}(\mathbf{e}_{1,i}, \ldots, \mathbf{e}_{n,i})$. For $n \in \mathbb{N}, i \in [2]$, $\rho_{i,n} = P_{n,i}\rho_i P_{n,i}$. Let $\rho^{(n)} = (P_{n,1} \otimes P_{n,2})\rho(P_{n,1} \otimes P_{n,2})$. Then we have $\operatorname{Tr}_2 \rho^{(n)} \preceq \rho_{1,n}, \operatorname{Tr}_1 \rho^{(n)} \preceq \rho_{2,n}$.*

Proof. Write

$$\rho = \sum_{p,q=1}^{\infty} \rho_{1,pq} \otimes \mathbf{e}_{p,2}\mathbf{e}_{q,2}^{\vee} = \sum_{i,j=1}^{\infty} \mathbf{e}_{i,1}\mathbf{e}_{j,2}^{\vee} \otimes \rho_{2,ij},$$

$$\rho_1 = \operatorname{Tr}_2 \rho = \sum_{p=1}^{\infty} \rho_{1,pp}, \quad \rho_2 = \operatorname{Tr}_1 \rho = \sum_{i=1}^{\infty} \rho_{2,ii}.$$

Where $\rho_{1,pq}$ is in a trace class operator on $\mathcal{H}_1$ and $\rho_{2,ij}$ is in a trace class operator on $\mathcal{H}_2$. Then

$$\begin{aligned}
\operatorname{Tr}_2 \rho^{(n)} &= \operatorname{Tr}_2 \left((P_{n,1} \otimes P_{n,2})(\sum_{p,q=1}^{\infty} \rho_{1,pq} \otimes \mathbf{e}_{p,2}\mathbf{e}_{q,2}^{\vee})(P_{n,1} \otimes P_{n,2}) \right) \\
&= \operatorname{Tr}_2 \left(\sum_{p,q=1}^{n} P_{n,1}\rho_{1,pq}P_{n,1} \otimes \mathbf{e}_{p,2}\mathbf{e}_{q,2}^{\vee} \right) \\
&= \sum_{p=1}^{n} P_{n,1}\rho_{1,pp}P_{n,1} \preceq \sum_{p=1}^{\infty} P_{n,1}\rho_{1,pp}P_{n,1} = \rho_{1,n}.
\end{aligned}$$

Similarly $\operatorname{Tr}_1 \rho^{(n)} \preceq \rho_{2,n}$. □

Theorem 4.5. *Let $\mathcal{H}_1, \mathcal{H}_2$ be two separable Hilbert spaces with countable orthogonal bases $\mathbf{e}_{i,1}, \mathbf{e}_{i,2}$ for $i \in \mathbb{N}$ respectively. Set $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Suppose $\mathcal{X} \subset \mathcal{H}$ is finite dimensional. Assume that $\rho_i \in \mathrm{S}_+(\mathcal{H}_i)$ are given and $\operatorname{Tr} \rho_1 = \operatorname{Tr} \rho_2 = 1$. Let $P_{n,i} \in \mathrm{S}_+(\mathcal{H}_i)$ be the orthogonal projection on $\mathcal{H}_{i,n} = \operatorname{span}(\mathbf{e}_{1,i}, \dots, \mathbf{e}_{n,i})$. For $n \in \mathbb{N}, i \in [2]$, set $\mathcal{X}_n = (P_{n,1} \otimes P_{n,2}\mathcal{X})$ and $\rho_{i,n} = P_{n,i}\rho_i P_{n,i}$.*

Consider the semidefinite programming problem

$$\begin{aligned}
&\mu_n(\rho_1, \rho_2, \mathcal{X}) = \\
&max\{\operatorname{Tr}(XP_{\mathcal{X}_n}); \qquad \operatorname{Tr}_2 X \preceq \rho_{1,n}, \operatorname{Tr}_1 X \preceq \rho_{2,n}, \\
&\qquad\qquad\qquad\qquad X \in (P_{n,1} \otimes P_{n,2})\mathrm{S}_+(\mathcal{H})(P_{n,1} \otimes P_{n,2})\}
\end{aligned}$$

Then the following statements are equivalent

(1) $\exists \rho \in \mathrm{S}_{+,1}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ satisfies

$$\operatorname{Tr}_1(\rho) = \rho_2, \operatorname{Tr}_2(\rho) = \rho_1, \operatorname{supp}(\rho) \subset \mathcal{X}.$$

(2) $\lim_{n\to\infty} \mu_n(\rho_1, \rho_2, \mathcal{X}) = 1$.

Proof. (1) $\Rightarrow$ (2) Assume that there exists an $\rho \in \mathrm{S}_{+,1}(\mathcal{H})$ such that $\operatorname{Tr}_2 \rho = \rho_1, \operatorname{Tr}_1 \rho = \rho_2$, $\operatorname{supp}(\rho) \subset \mathcal{X}$. Let $\rho^{(n)} = (P_{n,1} \otimes P_{n,2})\rho(P_{n,1} \otimes P_{n,2})$, $\rho^{(n)} \in (P_{n,1} \otimes P_{n,2})\mathrm{S}_+(\mathcal{H})(P_{n,1} \otimes P_{n,2})$. According to Lemma 4.4, we have $\operatorname{Tr}_2 \rho^{(n)} \preceq \rho_{1,n}, \operatorname{Tr}_1 \rho^{(n)} \preceq \rho_{2,n}$. Therefore, $\rho^{(n)}$ is a feasible solution of the maximal problem. Moreover, since $\operatorname{supp}(\rho) \subset \mathcal{X}$, we deduce that $\rho^{(n)}(\mathcal{H}) = (P_{n,1} \otimes P_{n,2})\rho(P_{n,1} \otimes P_{n,2})(\mathcal{H}) \subset \mathcal{X}_n$. As $\mathcal{X}_n$ is closed, and $\operatorname{supp}(\rho^{(n)})$ is the closure of $\rho^{(n)}(\mathcal{H})$, we have $\operatorname{supp}(\rho^{(n)}) \subset \mathcal{X}_n$. So we have

$$\mu_n(\rho_1, \rho_2, \mathcal{X}) \geq \operatorname{Tr}(\rho^{(n)} P_{\mathcal{X}_n}) = \operatorname{Tr}(\rho^{(n)}). \tag{4.1}$$

Since $P_{n,1} \otimes P_{n,2} \to I_1 \otimes I_2$ in the strong operator topology [13, Lemma 5] yields $\lim_{n\to\infty} \|\rho^{(n)} - \rho\|_1 = 0$. So $\lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} = \operatorname{Tr} \rho = 1$.

Since $P_{\mathcal{X}_n} \preceq I$, and $X \in S_+(\mathcal{H}), \operatorname{Tr}_2 X \preceq \rho_1$ we obtain

$$\operatorname{Tr} X P_{\mathcal{X}_n} = \operatorname{Tr} X^{1/2} X^{1/2} P_{\mathcal{X}_n} = \operatorname{Tr} X^{1/2} P_{\mathcal{X}_n} X^{1/2} \le \\ \operatorname{Tr} X^{1/2} I X^{1/2} = \operatorname{Tr} X = \operatorname{Tr}(\operatorname{Tr}_2 X) \le \operatorname{Tr} \rho_1 = 1.$$

Hence $\operatorname{Tr}(\rho^{(n)}) = \operatorname{Tr}(\rho^{(n)} P_{\mathcal{X}_n}) \le \mu_n(\rho_1, \rho_2, \mathcal{X}) \le 1$. By taking the limit on both sides we deduce $\lim_{n\to\infty} \mu_n(\rho_1, \rho_2, \mathcal{X}) = 1$.

(2) $\Rightarrow$ (1) Let $\varepsilon_n, n \in \mathbb{N}$ be a positive sequence converging to zero. Suppose that

$$\operatorname{Tr}(\rho^{(n)} P_{\mathcal{X}_n}) \ge \mu_n(\rho_1, \rho_2, \mathcal{X}) - \varepsilon_n,$$

and $\operatorname{Tr}_2 \rho^{(n)} \preceq \rho_{1,n}, \operatorname{Tr}_1 \rho^{(n)} \preceq \rho_{2,n}$,and $\rho^{(n)} \in (P_{n,1} \otimes P_{n,2}) S_+(\mathcal{H})(P_{n,1} \otimes P_{n,2}) \subset S_+(\mathcal{X}_n)$. According to Lemma 4.3(2), there exists n_k, such that $\rho^{(n_k)}$ converges in trace norm to $\rho \in S_+(\mathcal{X})$. Lemma 2.2 yields that $\operatorname{Tr}_i \rho^{(n_k)}$ converges to $\operatorname{Tr}_i \rho$ in trace norm for $i \in [2]$. By taking the limit of the following inequality

$$\mu_n(\rho_1, \rho_2, \mathcal{X}) - \varepsilon_n \le \operatorname{Tr}(P_{\mathcal{X}_{n_k}} \rho^{(n_k)}) \le \operatorname{Tr}(\rho^{(n_k)}) \le 1.$$

We have $\lim_{n\to\infty} \operatorname{Tr}(\rho^{(n_k)}) = 1$. As $\rho^{(n_k)}$ converges in trace norm to ρ, we deduce that $\operatorname{Tr}(\rho) = 1$.

For each n_k, we have $\operatorname{Tr}_i(\rho^{(n_k)}) \preceq \rho_{j,n_k}$, where $\{i, j\} = [2]$. Lemma 5 in [13] yields that $\lim_{k\to\infty} \rho_{j,n_k} = \rho_j$ for $j \in [2]$. Hence $\operatorname{Tr}_i \rho \preceq \rho_j$ for $\{i, j\} = [2]$. Furthermore, $\operatorname{Tr}(\operatorname{Tr}_i \rho) = \operatorname{Tr} \rho_1 = \operatorname{Tr} \rho_2 = 1$. Hence $\rho_1 = \operatorname{Tr}_2 \rho$ and $\rho_2 = \operatorname{Tr}_1 \rho$. □

5. Continuity of the Hausdorff metric

Let Φ be given by (1.1). Lemma 2.2 yields that Φ is a bounded linear operator satisfying $\|\Phi\| \le 2$. Denote $\Sigma = \Phi(\mathrm{T}_+(\mathcal{H}_1 \otimes \mathcal{H}_2))$. Note that $(\rho_1, \rho_2) \in \Sigma$ if and only if and only if $\rho_i \in \mathrm{T}_+(\mathcal{H}_i)$ and $\operatorname{Tr} \rho_1 = \operatorname{Tr} \rho_2$.

Proposition 5.1. *Assume that $(\rho_1, \rho_2) \in \Sigma$. Then the set $\mathcal{M}(\rho_1, \rho_2)$ given by (1.3) is a nonempty, convex, compact, metric set with respect to the distance induced by the norm in $\mathrm{T}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. That is for each sequence $\gamma_m \in \mathcal{M}(\rho_1, \rho_2), m \in \mathbb{N}$ there exists a subsequence γ_{m_k} which converges in norm to $\gamma \in \mathcal{M}(\rho_1, \rho_2)$.*

Proof. Clearly $\mathcal{M}(0, 0) = \{0\}$ and the proposition is trivial in this case. Assume that $\operatorname{Tr} \rho_1 = \operatorname{Tr} \rho_2 > 0$. Then $\frac{1}{\operatorname{Tr} \rho_1} \rho_1 \otimes \rho_2 \in \mathcal{M}(\rho_1, \rho_2)$. Clearly $\mathcal{M}(\rho_1, \rho_2)$ is a convex metric space. Note that $\|\gamma\|_1 = \operatorname{Tr} \rho_1$ for each $\gamma \in \mathcal{M}(\rho_1, \rho_2)$. Hence $\mathcal{M}(\rho_1, \rho_2)$ is a bounded set. Assume that $\gamma_m \in \mathcal{M}(\rho_1, \rho_2), m \in \mathbb{N}$. Then there exists a subsequence γ_{m_k} which converges in the weak operator topology to γ. Clearly $\operatorname{Tr}_2 \gamma_{m_k} = \rho_1, \operatorname{Tr}_1 \gamma_{m_k} = \rho_2$. Theorem 1.4 yields that $\lim_{m\to\infty} \|\gamma_{m_k} - \gamma\|_1 = 0$. Hence $\gamma \in \mathcal{M}(\rho_1, \rho_2)$. □

Observe that $\mathrm{T}_+(\mathcal{H}_1 \otimes \mathcal{H}_2)$ fibers over Σ: $\mathrm{T}_+(\mathcal{H}_1 \otimes \mathcal{H}_2) = \cup_{(\rho_1,\rho_2)\in\Sigma} \mathcal{M}(\rho_1, \rho_2)$. We define the distance between two fibers using the Hausdorff metric. The distance from $\beta \in \mathrm{T}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ to $\mathcal{M}(\rho_1, \rho_2)$ is defined as

$$\operatorname{dist}(\beta, \mathcal{M}(\rho_1, \rho_2)) = \inf\{\|\beta - \gamma\|_1, \gamma \in \mathcal{M}(\rho_1, \rho_2)\}.$$

Since $\mathcal{M}(\rho_1, \rho_2)$ is compact it follows that there exists $\gamma(\beta) \in \mathcal{M}(\rho_1, \rho_2)$ such that $\operatorname{dist}(\beta, \mathcal{M}(\rho_1, \rho_2)) = \|\beta - \gamma(\beta)\|_1$. Assume that $(\sigma_1, \sigma_2) \in \Sigma$. Then the semidistance between $\mathcal{M}(\sigma_1, \sigma_2)$ and $\mathcal{M}(\rho_1, \rho_2)$ and is given as

$$\operatorname{sd}(\mathcal{M}(\sigma_1, \sigma_2), \mathcal{M}(\rho_1, \rho_2)) = \\ \sup\{\operatorname{dist}(\beta, \mathcal{M}(\rho_1, \rho_2)), \beta \in \mathcal{M}(\sigma_1, \sigma_2))\}.$$

Since $\mathcal{M}(\rho_1,\rho_2)$ and $\mathcal{M}(\sigma_1,\sigma_2)$ are compact it follows

$$\mathrm{sd}(\mathcal{M}(\sigma_1,\sigma_2),\mathcal{M}(\rho_1,\rho_2))=\|\beta-\gamma\|_1 \text{ for some } \beta\in\mathcal{M}(\sigma_1,\sigma_2),\gamma\in\mathcal{M}(\rho_1,\rho_2).$$

Recall that the Hausdorff distance between $\mathcal{M}(\sigma_1,\sigma_2)$ and $\mathcal{M}(\rho_1,\rho_2)$ is given by

$$\mathrm{hd}(\mathcal{M}(\sigma_1,\sigma_2),\mathcal{M}(\rho_1,\rho_2))=\\ \max(\mathrm{sd}(\mathcal{M}(\sigma_1,\sigma_2),\mathcal{M}(\rho_1,\rho_2)),\mathrm{sd}(\mathcal{M}(\rho_1,\rho_2),\mathcal{M}(\sigma_1,\sigma_2))).$$

Theorem 5.2. *The Hausdorff distance on the fibers over Σ is a complete metric. Furthermore the sequence $\mathcal{M}(\rho_{1,m},\rho_{2,m}),m\in\mathbb{N}$ converges to $\mathcal{M}(\rho_1,\rho_2)$ in Hausdorff metric if and only if the sequence $(\rho_{1,m},\rho_{2,m}),m\in\mathbb{N}$ converges in norm to (ρ_1,ρ_2).*

Proof. Since each $\mathcal{M}(\rho_1,\rho_2)$ is compact it follows that

$$\mathrm{hd}(\mathcal{M}(\sigma_1,\sigma_2),\mathcal{M}(\rho_1,\rho_2))=0 \iff \mathcal{M}(\sigma_1,\sigma_2)=\mathcal{M}(\rho_1,\rho_2) \iff \\ (\sigma_1,\sigma_2)=(\rho_1,\rho_2).$$

As $\mathcal{M}(\sigma_1,\sigma_2)$ and $\mathcal{M}(\rho_1,\rho_2)$ are compact there exist $\beta\in\mathcal{M}(\sigma_1,\sigma_2)$ and $\gamma\in\mathcal{M}(\rho_1,\rho_2)$ such that $\mathrm{hd}(\mathcal{M}(\sigma_1,\sigma_2),\mathcal{M}(\rho_1,\rho_2))=\|\beta-\gamma\|_1$. Lemma 2.2 yields that $\|\sigma_1-\rho_1\|_1+\|\sigma_2-\rho_2\|_1\le 2\mathrm{hd}(\mathcal{M}(\sigma_1,\sigma_2),\mathcal{M}(\rho_1,\rho_2))$.

Assume that the sequence $\mathcal{M}(\rho_{1,m},\rho_{2,m}),m\in\mathbb{N}$ is a Cauchy sequence in the Hausdorff metric. Hence the sequence $(\rho_{1,m},\rho_{2,,m})$, $m\in\mathbb{N}$ is a Cauchy sequence in Σ. Therefore there exists $(\rho_1,\rho_2)\in\Sigma$ such that $\lim_{m\to\infty}\|\rho_{1,m}-\rho_1\|_1+\|\rho_{2,m}-\rho_2\|_1=0$.

We now show that the sequence $\mathcal{M}(\rho_{1,m},\rho_{2,m}),m\in\mathbb{N}$ converges to $\mathcal{M}(\rho_1,\rho_2)$ in the Hausdorff metric. Since the sequence $(\rho_{1,m},\rho_{2,m})$ is bounded, and each $\mathcal{M}(\rho_{1,m},\rho_{2,m})$ is compact, it is straightforward to show using Theorem 1.4 that the sequence $\mathrm{sd}(\mathcal{M}(\rho_{1,m},\rho_{2,m}),\mathcal{M}(\rho_1,\rho_2))$ converges to zero. It is left to show that

$$\mathrm{sd}(\mathcal{M}(\rho_1,\rho_2),\mathcal{M}(\rho_{1,m},\rho_{2,m}))=\mathrm{dist}(\gamma_m,\mathcal{M}(\rho_{1,m},\rho_{2,m}))\to 0,\quad \gamma_m\in\mathcal{M}(\rho_1,\rho_2).$$

Assume to the contrary that the above condition does not hold. Then there exists $\delta>0$ and a subsequence $\{m_k\},k\in\mathbb{N}$ such that $\mathrm{dist}(\gamma_{m_k},\mathcal{M}(\rho_{1,m_k},\rho_{2,m_k}))\ge 2\delta$. As $\mathcal{M}(\rho_1,\rho_2)$ is compact there exists a subsequence $\{m_{k_l}\},l\in\mathbb{N}$ and $\gamma\in\mathcal{M}(\rho_1,\rho_2)$ such that $\lim_{l\to\infty}\|\gamma_{m_{k_l}}-\gamma\|_1=0$. Hence we can assume that

$$\mathrm{dist}(\gamma,\mathcal{M}(\rho_{1,m_{k_l}},\rho_{2,m_{k_l}}))\ge\delta$$

for all $l\in\mathbb{N}$. Without a loss of generality we assume that $m_{k_l}=l$ for $l\in\mathbb{N}$. We will contradict this statement.

Firstly, we assume that $\rho_{j,m},\rho_j\succ 0$ and all their eigenvalues are simple. Then there exists orthonormal bases $\{\mathbf{e}_{n,j,m}\},\{\mathbf{e}_{n,j}\},n\in\mathbb{N}$ of $\mathcal{H}_j$ such that

$$\rho_{j,m}=\sum_{i_j=1}^{\infty}\lambda_{i_j,j,m}\mathbf{e}_{i_j,j,m}\otimes\mathbf{e}_{i_j,j,m}^\vee,\ \lambda_{i_j,j,m}>\lambda_{i_j+1,j,m}>0,$$

$$\rho_j=\sum_{i_j=1}^{\infty}\lambda_{i_j,j}\mathbf{e}_{i_j,j}\otimes\mathbf{e}_{i_j,j}^\vee,\ \lambda_{i_j,j}>\lambda_{i_j+1,j}>0.$$

Let $P_{n,j,m}$ and $P_{n,j}$ be the orthogonal projections of $\mathcal{H}_j$ on

$$\mathcal{H}_{n,j,m}=\mathrm{span}(\mathbf{e}_{1,j,m},\dots,\mathbf{e}_{n,j,m}) \text{ and } \mathcal{H}_{n,j}=\mathrm{span}(\mathbf{e}_{1,j},\dots,\mathbf{e}_{n,j,m})$$

respectively. Define $\rho_{j,m}^{(n)} = P_{n,j,m}\rho_{j,m}P_{n,j,m}, \rho_j^{(n)} = P_{n,j,m}\rho_j P_{n,j,m}$. As

$$\lim_{m\to\infty} \|\rho_{j,m} - \rho_j\|_1 = 0,$$

it follows that $|\lambda_{i_j,j,m} - \lambda_{i_j,j}| \to 0, \|\mathbf{e}_{i_j,j,m} - \mathbf{e}_{i_j,j}\| \to 0, i_j \to \infty$, after we choose the phases (signs) of $\mathbf{e}_{i_j,i,m}$([14] Lemma B.6). Hence for each $n \in \mathbb{N}$

$$\lim_{m\to\infty} \|P_{n,j,m} - P_{n,j}\|_1 = 0, \tag{5.1}$$

$$\lim_{m\to\infty} \|\rho_{j,m}^{(n)} - \rho_j^{(n)}\|_1 = 0 \Rightarrow \lim \|(\rho_{j,m} - \rho_{j,m}^{(n)}) - (\rho_j - \rho_j^{(n)})\|_1 = 0.$$

Let $\gamma_{n_1,n_2,m} = P_{n_1,1,m}\otimes P_{n_2,2,m}\gamma P_{n_1,1,m}\otimes P_{n_2,2,m}$ and $\gamma_{n_1,n_2} = P_{n_1,1}\otimes P_{n_2,2}\gamma P_{n_1,1}\otimes P_{n_2,2}$. Then $\lim_{m\to\infty} \|\gamma_{n_1,n_2,m} - \gamma_{n_1,n_2}\|_1 = 0$, and $\lim_{n_1,n_2\to\infty} \|\gamma - \gamma_{n_1,n_2}\|_1 = 0$.

The arguments of the proof of Lemma 4.4 yields that $\operatorname{Tr}_2 \gamma_{n_1,n_2,m} \preceq \rho_{1,m}^{(n_1)}$, and $\operatorname{Tr}_1 \gamma_{n_1,n_2,m} \preceq \rho_{2,m}^{(n_2)}$. Hence $\rho_{j,m} - \operatorname{Tr}_{j'} \gamma_{n_1,n_2,m} \succeq 0$, where $\{j, j'\} = [2]$. Define

$$\sigma_{n_1,n_2,m} = \gamma_{n_1,n_2,m} + \frac{1}{\operatorname{Tr}(\rho_{1,m} - \operatorname{Tr}_2 \gamma_{n_1,n_2,m})}(\rho_{1,m} - \operatorname{Tr}_2 \gamma_{n_1,n_2,m}) \otimes (\rho_{2,m} - \operatorname{Tr}_1 \gamma_{n_1,n_2,m}).$$

Then $\sigma_{n_1,n_2,m} \in \mathcal{M}(\rho_{1,m}, \rho_{2,m})$ and

$$\|\sigma_{n_1,n_2,m} - \gamma\|_1 \le \|\gamma_{n_1,n_2,m} - \gamma_{n_1,n_2}\|_1 + \|\gamma_{n_1,n_2} - \gamma\|_1 + \|\rho_{1,m} - \rho_1\|_1 + \|\rho_1 - \operatorname{Tr}_2 \gamma_{n_1,n_2}\|_1 + \|\operatorname{Tr}_2 \gamma_{n_1,n_2} - \operatorname{Tr}_2 \gamma_{n_1,n_2,m}\|_1.$$

Recall that (Lemma 2.2)

$$\|\rho_1 - \operatorname{Tr}_2 \gamma_{n_1,n_2}\|_1 \le \|\gamma - \gamma_{n_1,n_2}\|_1,\ \|\operatorname{Tr}_2 \gamma_{n_1,n_2} - \operatorname{Tr}_2 \gamma_{n_1,n_2,m}\|_1 \le \|\gamma_{n_1,n_2} - \gamma_{n_1,n_2,m}\|_1.$$

Use (5.1) a choice of $n_1, n_2 \gg 1$ and corresponding $m \gg 1$ such that

$$\operatorname{dist}(\gamma, \mathcal{M}(\rho_{1,m}, \rho_{2,m})) \le \|\gamma - \sigma_{n_1,n_2,m}\|_1 < \delta.$$

This inequality contradicts our assumption and proves that $\mathcal{M}(\rho_{1,m}, \rho_{2,m})$ converges to $\mathcal{M}(\rho_1, \rho_2)$ in the Hausdorff metric.

We discuss briefly how to modify the above arguments to general $(\rho_{1,m}, \rho_{2,m})$ and (ρ_1, ρ_2). For ρ_1, ρ_2 with simple eigenvalues we don't need to modify anything as $\lambda_{n_j,j,m} > \lambda_{n_j+1,j,m}$ for fixed n_j and $m > N_j(n_j)$. Let us consider now the case where ρ_1 and ρ_2 are positive definite but may have multiple eigenvalues. Each eigenvalue must have a finite multiplicity. Suppose that $\lambda_{n_j,j} > \lambda_{n_j+1,j}$. Then $\lambda_{n_j,j,m} > \lambda_{n_j+1,j,m}$ for $m > N_j(n_j)$. As $P_{n_j,j}$ is well defined it follows that (5.1) holds.

Denote by $\mathcal{H}'_j$ the closure of the range of ρ_j. It is straightforward to show using Lemma 2.1 that $\mathcal{M}(\rho_1, \rho_2) \subset \mathrm{T}_+(\mathcal{H}'_1 \otimes \mathcal{H}'_2)$. Then $P_{n,j}$ are the corresponding projections in $\mathcal{H}'_j$. Then for $\lambda_{n_j,j} > \lambda_{n_j+1,j}$, (5.1) holds.

Finally, the last part of the theorem that the sequence $\mathcal{M}(\rho_{1,m}, \rho_{2,m}), m \in \mathbb{N}$ converges to $\mathcal{M}(\rho_1, \rho_2)$ in Hausdorff metric if the sequence $(\rho_{1,m}, \rho_{2,m}), m \in \mathbb{N}$ converges in norm to (ρ_1, ρ_2) follows straightforward from the above arguments. □

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Department of Mathematics and Computer Science, University of Illinois at Chicago, Chicago, Illinois, 60607-7045, USA
E-mail address: friedlan@uic.edu

University of Chinese Academy of Sciences, Beijing 100049, China, and University of Technology Sydney, NSW, Australia
E-mail address: gejingtong@amss.ac.cn

KLMM, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China, University of Chinese Academy of Sciences, Beijing 100049, China
E-mail address: lzhi@mmrc.iss.ac.cn