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EXTENSIONS OF S-LEMMA FOR NONCOMMUTATIVE POLYNOMIALS

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ABSTRACT. We consider the problem of extending the classical S-lemma from the commutative case to noncommutative cases. Precisely, we extend the S-lemma to three kinds of noncommutative polynomials: noncommutative polynomials whose coefficients are real numbers, noncommutative matrix-valued polynomials, and hereditary noncommutative matrix-valued polynomials. Different from the commutative case, the S-lemma for noncommutative polynomials can be extended to the case involving multiple quadratic constraints. Some examples are given to demonstrate the relations between these newly derived conditions.

KEYWORDS: *S-lemma, noncommutative polynomial, linear matrix inequality, positive semidefinite matrix, completely positive linear map.*

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1. INTRODUCTION

1.1. RELATED WORK AND MOTIVATION. Quadratic inequalities arise in many areas of theoretical and applied mathematics. In the field of optimization, quadratically constrained quadratic programming (QCQP) appears in various disciplines [15] and includes some important subclasses, like the trust region problem [40], Max-Cut problem, $0-1$ quadratic programming problem and box-constrained quadratic programming problem [33]. QCQP has been widely studied in the literature [1, 14, 27, 43, 52, 57] and is known to be NP-hard in general [49]. However, when there is just one constraint, a QCQP problem can be reformulated as a semidefinite programming problem by applying the celebrated S-lemma [5].

The classical S-lemma answers the question that when a quadratic inequality is a consequence of another quadratic inequality. More specifically, let f, g : $\mathbb{R}^m \to \mathbb{R}$ be quadratic polynomials in m variables, $\mathbb{R}$ denote the set of real numbers, and suppose there is a real vector $\hat{X} \in \mathbb{R}^m$ such that $g(\hat{X}) > 0$. The following two statements are equivalent.

(1) For all $X \in \mathbb{R}^m$, if $g(X) \geq 0$, then $f(X) \geq 0$.
(2) There is a nonnegative real number λ such that $f(X) - \lambda g(X) \geq 0$ for all $X \in \mathbb{R}^m$.

There are numerous applications of S-lemma in quadratic optimization [12, 27, 35, 52], control theory [2, 6, 41], signal processing [34], statistics [26], etc. As shown in the nice survey on S-lemma by Pólik and Terlaky [45], there are many different approaches for proving the S-lemma. In [55, 56], Yakubovich used the convexity result in [13] to prove the S-lemma. Pólik and Terlaky [45] provided a modern proof that is similar to the one given in the book by Ben-Tal and Nemirovski [2] and they also extended the proof to a non-homogeneous case. An elementary proof of the S-lemma could be derived based on a lemma given by Yuan [58].

Many problems in systems and control theory [51] are described by signal flow diagrams and can naturally be converted to inequalities involving polynomials in matrices, which lead to noncommutative polynomials. Optimization problems with polynomial constraints in noncommuting variables also arise naturally in quantum theory and quantum information science [3, 11, 16, 38, 39, 44]. Therefore, it would be interesting to know whether the classical S-lemma can be extended to noncommutative polynomials.

Characterizations of polynomials that are positive on a semialgebraic set is a fundamental problem in real algebraic geometry and have various applications in polynomial optimization [4, 30, 31, 32, 42, 46, 50]. Some of these results have been extended to noncommutative cases [7, 9, 17, 18, 19, 21, 22, 23, 24, 25, 28, 29, 37, 44].

Helton [18] and McCullough [36] proved a surprising result that positive noncommutative polynomials are sums of squares (SOS), which is known to be false for commutative polynomials. The first explicit such example was given by Motzkin in 1967

$$\text{Motzkin}(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^3y^3z^3,$$

which is nonnegative but can not be written as a sum of squares.

Helton and McCullough showed that a noncommutative polynomial which is positive on a bounded semi-algebraic set of operators has a weighted sum of squares representation [24, Theorem 1.2]. It can certainly be used to answer the question:

“ When a noncommutative quadratic inequality is a consequence of another noncommutative quadratic inequality?"

There are well-developed packages for solving sums of squares problems for noncommuting polynomials [8, 53]. However, it should be noted that Theorem 1.2 in [24] requires that the positivity domain defined by noncommutative inequalities is bounded. Moreover, one also needs to solve a sequence of SOS relaxations with respect to fixed degrees to find the SOS representation. Sharp degree bounds are given in [9] when the positivity domain is a polydisc or a ball. Helton, Klep and Volčič showed a Positivstellensatz for hereditary quadratic polynomials, by adding a slack variable [17]. On the other hand, we wish to seek a noncommutative version of S-lemma which requires no boundedness condition on the domain defined by noncommutative quadratic inequalities and no slack variables.

Helton, Klep and McCullough gave a linear Positivestellensatz for characterizing the matricial linear matrix inequality (LMI) domination problems [20]. They showed that a linear pencil inequality in noncommutative variables is a consequence of other linear pencil inequalities if and only if there exists a completely positive map between the coefficients of these two linear pencil inequalities. This result was generalized by Zalar to solve the linear operator inequality (LOI) domination problems [59]. Stimulated by the excellent work of Helton, Klep and McCullough [20], we aim to extend the classical S-lemma to the noncommutative case by introducing a nonzero completely positive map which can be seen as a generalization of the nonnegative real number λ in the classical S-lemma.

1.2. SUMMARY OF THE MAIN RESULTS. The symbol $\mathbb{N}^+$ denotes the set of positive natural numbers. For $n \in \mathbb{N}^+$, $\mathbb{R}^{n\times n}$ (resp. $\mathbb{SR}^n$) stands for the set of $n \times n$ real matrices (resp. symmetric matrices). For $m, n \in \mathbb{N}^+$, the symbol $(\mathbb{R}^{n\times n})^m$ (resp. $(\mathbb{SR}^n)^m$) denotes the vector space consisting of m-dimensional vectors of $n \times n$ real matrices (resp. symmetric matrices).

The main part of the paper is devoted to extending the S-lemma for noncommutative polynomials with matrix coefficients. Let $f(x)$ be a noncommutative matrix-valued homogeneous quadratic symmetric polynomial:

$$f(x) = \sum_{i=1,j=1}^{m} A_{ij} x_i x_j,$$

where $A_{ij} = A_{ji}^T, A_{ij} \in \mathbb{R}^{q\times q}$ for all i, j.

For $n \in \mathbb{N}^+$, let $\mathbb{1}_n$ represent the identity map from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{n\times n}$. Inspired by Choi's characterization of a completely positive map via a positive semidefinite Choi matrix (Theorem 2.1), we generalize the condition in the classical S-lemma of existing a nonnegative number λ such that $f(X) - \lambda g(X) \geq 0$ for all $X \in \mathbb{R}^m$ to the existence of a completely positive linear mapping $\phi : \mathbb{R}^{q\times q} \to \mathbb{R}^{q\times q}$ such that

$$f(X) - (\phi \otimes \mathbb{1}_n) g(X) \succeq 0$$

for all $X \in (\mathbb{SR}^n)^m, n \in \mathbb{N}^+$. The main results of this paper are stated below.

THEOREM 1.1. *Let $f(x)$ and $g(x)$ be noncommutative matrix-valued homogeneous quadratic symmetric polynomials,*

$$f(x) = \sum_{i=1,j=1}^{m} A_{ij} x_i x_j, \quad g(x) = \sum_{i=1,j=1}^{m} B_{ij} x_i x_j,$$

where $A_{ij}, B_{ij} \in \mathbb{R}^{q\times q}$ and $A_{ij} = A_{ji}^T$, $B_{ij} = B_{ji}^T$ for all $i, j = 1, \ldots, m$. Suppose that there is an $\hat{X} \in (\mathbb{SR}^{\hat{n}})^m$ for some $\hat{n} \in \mathbb{N}^+$, such that $g(\hat{X}) \succ 0$.

Then the following two statements are equivalent:

(1) *For all* $X \in (\mathbb{SR}^n)^m$, $n > q$, *if* $(\mathrm{Id}_q \otimes P)g(X)(\mathrm{Id}_q \otimes P) \succeq 0$, *then* $(\mathrm{Id}_q \otimes P)f(X)(\mathrm{Id}_q \otimes P) \succeq 0$, *where* $P : \mathbb{R}^n \to \mathbb{R}^n$ *is the projection to the last q coordinates, i.e.,* $P = \mathrm{diag}(0, \mathrm{Id}_q)$.
(2) *There is a nonzero completely positive linear mapping* $\phi : \mathbb{R}^{q\times q} \to \mathbb{R}^{q\times q}$ *such that* $f(X) - (\phi \otimes \mathbb{1}_n)g(X) \succeq 0$ *for all* $X \in (\mathbb{SR}^n)^m, n \in \mathbb{N}^+$.

Interestingly, it is straightforward to extend Theorem 1.1 to the case involving multiple quadratic constraints (Remark 3.3), which is clearly different from the S-lemma for the commutative polynomials.

Hereditary noncommutative matrix-valued polynomials are matrix-valued polynomials in noncommuting letters $\{x_1, x_2, \ldots, x_m, x_1^T, x_2^T, \ldots, x_m^T\}$, and the letters $x_1^T, x_2^T, \ldots, x_m^T$ always appear on the right side of the monomials.

The following theorem is for a special case of hereditary noncommutative matrix-valued polynomials. Its proof can be adjusted from the proof of Theorem 1.1.

THEOREM 1.2. *Let* $f(x), g(x)$ *be hereditary noncommutative matrix-valued homogeneous polynomials,*

$$f(x) = \sum_{i=1,j=1}^{m} A_{ij}x_i x_j^T, \quad g(x) = \sum_{i=1,j=1}^{m} B_{ij}x_i x_j^T,$$

where $A_{ij}, B_{ij} \in \mathbb{R}^{q\times q}$ *and* $A_{ij} = A_{ji}^T$, $B_{ij} = B_{ji}^T$ *for all* $i, j = 1, \ldots, m$. *Suppose that there is an* $\hat{X} \in (\mathbb{R}^{\hat{n}\times\hat{n}})^m$ *for some* $\hat{n} \in \mathbb{N}^+$, *such that* $g(\hat{X}) \succ 0$. *Then the following two statements are equivalent:*

(1) *For all* $X \in (\mathbb{R}^{n\times n})^m$, $n \in \mathbb{N}^+$, *if* $g(X) \succeq 0$, *then* $f(X) \succeq 0$.
(2) *There is a nonzero completely positive linear mapping* $\phi : \mathbb{R}^{q\times q} \to \mathbb{R}^{q\times q}$, *such that* $f(X) - (\phi \otimes \mathbb{1}_n)g(X) \succeq 0$ *for all* $X \in (\mathbb{R}^{n\times n})^m$, $n \in \mathbb{N}^+$.

According to the Kraus representations [54, Theorem 2.22], a completely positive map $\phi : \mathbb{R}^{q\times q} \to \mathbb{R}^{q\times q}$ can be written as

$$\phi(A) = \sum_{j=1}^{\mu} V_j^T A V_j, \ \forall\, A \in \mathbb{R}^{q\times q},$$

for some matrices $V_j \in \mathbb{R}^{q\times q}$, $j = 1, \ldots, \mu$. Therefore, by Theorem 2.2, it is easy to show that the condition (2) in Theorem 1.1 and 1.2 are equivalent to showing that $f(x)$ has a weighted SOS representation

$$f(x) = \sum_{j=1}^{\mu} V_j^T g(x) V_j + r(x)^T r(x),$$

for some matrices $V_j \in \mathbb{R}^{q\times q}$, $j = 1, \ldots, \mu$, and noncommutative matrix-valued homogeneous linear polynomial $r(x)$.

1.3. ORGANIZATION OF THE PAPER. Some preliminary concepts and results are given in Section 2. Section 3 is devoted to proving the main results Theorems 1.1 and 1.2. In Section 4, we present some other variants of S-Lemma. Finally, in Section 5, we discuss some problems arising in extending S-lemma to noncommutative matrix-valued nonhomogeneous polynomials.

2. PRELIMINARIES

2.1. NONCOMMUTATIVE MATRIX-VALUED SYMMETRIC POLYNOMIALS. In the paper, we deal with noncommutative matrix-valued polynomials. Different from the commutative polynomials, the variables and coefficients are all matrices. As the matrix multiplication is not commutative, the monomials are words in noncommuting variables.

The polynomial p we consider in this paper has the following form:

$$p = \sum_{\omega \in \mathcal{W}_m} p_\omega \omega, \tag{2.1}$$

where $p_\omega \in \mathbb{R}^{q\times q}$, $q \in \mathbb{N}^+$, only finitely many p_ω are nonzero, and $\mathcal{W}_m$ is a set of words generated by the set of noncommuting letters $\{x_1, x_2, \ldots, x_m\}$.

When we evaluate a polynomial p at $X \in (\mathbb{SR}^n)^m$, we let

$$p(X) = \sum_{\omega \in \mathcal{W}_m} p_\omega \otimes \omega(X),$$

where $\omega(X) = X_{i_1} X_{i_2} \cdots X_{i_k}$ if $\omega = x_{i_1} x_{i_2} \cdots x_{i_k}$, and the symbol $\otimes$ denotes the Kronecker product

$$N \otimes M = [N_{ij} M]_{lk},$$

for finite dimensional matrices $N = (N_{ij}), M = (M_{lk})$. Matrices N and M can have different dimensions. We define the evaluation of a tuple $X \in (\mathbb{SR}^n)^m$ on the empty word as $\mathrm{Id_n}$, where $\mathrm{Id_n}$ denotes the identity matrix in $\mathbb{R}^{n\times n}$.

Define the transpose of a polynomial p as

$$p^T = \sum_{\omega \in \mathcal{W}_m} p_\omega^T \omega^T,$$

where $\omega^T = x_{i_k} \cdots x_{i_2} x_{i_1}$ for the word $\omega = x_{i_1} x_{i_2} \cdots x_{i_k}$. If $p = p^T$, we say p is symmetric. *Polynomials considered in this paper are always assumed to be symmetric.*

In particular, for noncommutative matrix-valued homogeneous quadratic polynomials

$$f(x) = \sum_{i=1,j=1}^{m} A_{ij} x_i x_j \quad \text{and} \quad g(x) = \sum_{i=1,j=1}^{m} B_{ij} x_i x_j,$$

where $A_{ij} = A_{ji}^T, B_{ij} = B_{ji}^T,\ A_{ij}, B_{ij} \in \mathbb{R}^{q\times q}$, the evaluations of f and g at $X \in (\mathbb{SR}^n)^m$ are

$$f(X) = \sum_{i=1,j=1}^{m} A_{ij} \otimes X_i X_j \quad \text{and} \quad g(X) = \sum_{i=1,j=1}^{m} B_{ij} \otimes X_i X_j.$$

If we restrict the coefficients (2.1) being real numbers, i.e., $q = 1$, then we get a usual noncommutative scalar-valued polynomial.

2.2. COMPLETELY POSITIVE LINEAR MAP. A real number can be seen as a linear map from $\mathbb{R}$ to $\mathbb{R}$, and if the number is positive, the linear map translates a positive real number to another positive real number. Similarly, we can define positive linear maps and completely positive linear maps between real vector spaces of higher dimensions.

For $s, t \in \mathbb{N}^+$, a linear map $\phi : \mathbb{R}^{s\times s} \to \mathbb{R}^{t\times t}$, can be represented by a matrix in $\mathbb{R}^{st\times st}$

$$\mathbf{J}(\phi) = \sum_{a,b=1}^{s} \phi(E_{ab}) \otimes E_{ab} = \begin{pmatrix} J_{11} & \cdots & J_{1t} \\ \vdots & \ddots & \vdots \\ J_{t1} & \cdots & J_{tt} \end{pmatrix}, \tag{2.2}$$

where $E_{ab} \in \mathbb{R}^{s\times s}$ is the matrix whose (a, b)-th entry is 1 and all others entries are 0, and

$$J_{ij} = [\phi(E_{ab})_{ij}]_{a,b=1,\ldots,s} \in \mathbb{R}^{s\times s},$$

with $\phi(E_{ab})_{ij}$ being the (i, j)-th entry of $\phi(E_{ab})$.

The matrix $\mathbf{J}(\phi)$ is called the Choi matrix of ϕ [10]. It is easy to verify that for any $M \in \mathbb{R}^{s\times s}$,

$$\phi(M) = \begin{pmatrix} \langle J_{11}, M\rangle & \cdots & \langle J_{1t}, M\rangle \\ \vdots & \ddots & \vdots \\ \langle J_{t1}, M\rangle & \cdots & \langle J_{tt}, M\rangle \end{pmatrix}.$$

In this paper, $\langle\cdot,\cdot\rangle$ stands for the Frobenius inner product of matrices.

We say that the linear map ϕ is positive, if for every positive semidefinite matrix $M \in \mathbb{SR}^{s\times s}, M \succeq 0$, its image under the map ϕ is also positive semidefinite, i.e., $\phi(M) \succeq 0$. Recall that $\mathbb{1}_n$ represents the identity map from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{n\times n}$. We say ϕ is completely positive, if for all $n \in \mathbb{N}^+$, the linear map $\phi \otimes \mathbb{1}_n$ is a positive linear map from $\mathbb{R}^{sn\times sn}$ to $\mathbb{R}^{tn\times tn}$, where

$$\phi \otimes \mathbb{1}_n(M \otimes N) = \phi(M) \otimes \mathbb{1}_n(N),\ M \in \mathbb{R}^{s\times s},\ N \in \mathbb{R}^{n\times n}.$$

THEOREM 2.1. [10] *The linear map $\phi : \mathbb{R}^{s\times s} \to \mathbb{R}^{t\times t}$ where $s, t \in \mathbb{N}^+$ is completely positive, if and only if the Choi matrix $\mathbf{J}(\phi)$ is positive semidefinite.*

There is a one-to-one correspondence between the set of all completely positive maps from $\mathbb{R}^{s\times s}$ to $\mathbb{R}^{t\times t}$ and the set of positive semidefinite matrices in $\mathbb{R}^{st\times st}$.

ϕ^* represents the adjoint operator of ϕ and ϕ^* is completely positive if and only if ϕ is completely positive.

2.3. POSITIVITY OF NONCOMMUTATIVE MATRIX-VALUED HOMOGENEOUS QUADRATIC POLYNOMIALS. For a commutative polynomial $h(x) = x^T Hx$ with $H \in \mathbb{SR}^m$ and $x = [x_1, x_2, \ldots, x_m]^T$, we know $h(X) \geq 0$ for all $X \in \mathbb{R}^m$ if and only if $H \succeq 0$. It is very interesting to see that this property can be extended to noncommutative polynomials. The following result is a special case of [36, Theorem 0.2].

THEOREM 2.2. *Let $f(x) = \sum_{i=1,j=1}^m A_{ij}x_ix_j$ be a noncommutative matrix-valued homogeneous quadratic polynomial, where the matrices $A_{ij} = A_{ji}^T \in \mathbb{R}^{q\times q}$ for all $i, j = 1, \ldots, m$. Define the coefficient matrix*

$$\mathcal{A} = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix}. \tag{2.3}$$

Then $f(X)$ is positive semidefinite for all $X \in (\mathbb{SR}^n)^m$, $n \in \mathbb{N}^+$, if and only if $\mathcal{A}$ is positive semidefinite.

Proof. By [36, Theorem 0.2], $f(X)$ is positive semidefinite for all $X \in (\mathbb{SR}^n)^m$, $n \in \mathbb{N}^+$, if and only if there exists a noncommutative matrix-valued homogeneous linear polynomial $U(x) = \sum_{i=1}^m U_ix_i$ such that $f(x) = U(x)U(x)^T$, which is equivalent to the decomposition

$$\mathcal{A} = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix} = \begin{pmatrix} U_1 \\ \vdots \\ U_m \end{pmatrix} \begin{pmatrix} U_1^T & \cdots & U_m^T \end{pmatrix} \succeq 0.$$

∎

We give in Appendix A an alternative proof of Theorem 2.2 based on the Choi matrix of a certain linear map.

2.4. NONCOMMUTATIVE POLYNOMIALS WITH SCALAR COEFFICIENTS. S-lemma for noncommutative polynomials with scalar coefficients can be adapted easily from the classical S-lemma for commutative polynomials.

PROPOSITION 2.3. *Let $f(x), g(x)$ be noncommutative homogeneous quadratic polynomials,*

$$f(x) = \sum_{i=1,j=1}^m a_{ij}x_ix_j, \quad g(x) = \sum_{i=1,j=1}^m b_{ij}x_ix_j,$$

where $a_{i,j}, b_{i,j} \in \mathbb{R}$ and $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$ for all $i, j = 1, \ldots, m$. Suppose that there is an $\hat{X} \in (\mathbb{SR}^{\hat{n}})^m$ for some $\hat{n} \in \mathbb{N}^+$, such that $g(\hat{X}) \succ 0$. Then the following three statements are equivalent:

(1) *For all* $X \in \mathbb{R}^m$, *if* $g(X) \geq 0$, *then* $f(X) \geq 0$.
(2) *For all* $X \in (\mathbb{SR}^n)^m$, $n \in \mathbb{N}^+$, *if* $g(X) \succeq 0$, *then* $f(X) \succeq 0$.
(3) *There is a nonnegative real number* λ *such that* $f(X) - \lambda g(X) \succeq 0$ *for all* $X \in (\mathbb{SR}^n)^m$, $n \in \mathbb{N}^+$.

Proof. For the given polynomials f and g, their corresponding coefficient matrices are

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mm} \end{pmatrix}.$$

(3)$\Longrightarrow$(2)$\Longrightarrow$(1): The implications are obvious.

(1)$\Longrightarrow$(3): Assume that for all $X \in \mathbb{R}^m$, $g(X) = X^T B X \leq 0$. Then we know $B \preceq 0$ and hence

$$g(X) = X^T (B \otimes \mathrm{Id}_n) X \preceq 0,$$

for all $X \in (\mathbb{SR}^n)^m$, $n \in \mathbb{N}^+$, which contradicts to the condition that there is an $\hat{X} \in (\mathbb{SR}^{\hat{n}})^m$ for some $\hat{n} \in \mathbb{N}^+$, such that $g(\hat{X}) \succ 0$. Hence, there always exists an $\tilde{X} \in \mathbb{R}^m$ such that $g(\tilde{X}) > 0$. According to the classical S-lemma, it follows that there exists a positive real number λ such that $f(X) - \lambda g(X) \geq 0$ for all $X \in \mathbb{R}^m$, in particular, $A - \lambda B \succeq 0$. Then we know

$$f(X) - \lambda g(X) = X^T ((A - \lambda B) \otimes \mathrm{Id}_n) X \succeq 0,$$

for all $X \in (\mathbb{SR}^n)^m$, $n \in \mathbb{N}^+$. ∎

3. S-LEMMA FOR NONCOMMUTATIVE POLYNOMIALS

In this section, we will prove the S-lemma for noncommutative matrix-valued polynomials in Theorems 1.1 and 1.2.

3.1. S-LEMMA FOR NONCOMMUTATIVE MATRIX-VALUED POLYNOMIALS.

Let $f(x)$ and $g(x)$ be noncommutative matrix-valued homogeneous quadratic polynomials,

$$f(x) = \sum_{i=1,j=1}^{m} A_{ij} x_i x_j, \quad g(x) = \sum_{i=1,j=1}^{m} B_{ij} x_i x_j,$$

where $A_{ij}, B_{ij} \in \mathbb{R}^{q \times q}$ and $A_{ij} = A_{ji}^T$, $B_{ij} = B_{ji}^T$ for all $i, j = 1, \ldots, m$. Define

$$\mathcal{A} = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} B_{11} & \cdots & B_{1m} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mm} \end{pmatrix}.$$

Let us prove the main result Theorem 1.1 about extended S-lemma for noncommutative matrix-valued homogeneous quadratic polynomials.

Proof of Theorem 1.1. Assume that condition (2) is satisfied. Let $P : \mathbb{R}^n \to \mathbb{R}^n$ be the projection to the last q coordinates. For all $X \in (\mathbb{SR}^n)^m$, $n, m \in \mathbb{N}^+$, we have

$$\begin{aligned}
&\sum_{i=1,j=1}^{m} A_{ij} \otimes X_i X_j - \sum_{i=1,j=1}^{m} \phi(B_{ij}) \otimes X_i X_j \succeq 0 \\
\Longrightarrow &\sum_{i=1,j=1}^{m} A_{ij} \otimes X_i X_j - \sum_{i=1,j=1}^{m} (\phi \otimes \mathbb{1}_n)(B_{ij} \otimes X_i X_j) \succeq 0 \\
\Longrightarrow &(\mathrm{Id}_q \otimes P) \left(\sum_{i=1,j=1}^{m} A_{ij} \otimes X_i X_j \right) (\mathrm{Id}_q \otimes P) \\
&- (\mathrm{Id}_q \otimes P) \left(\sum_{i=1,j=1}^{m} (\phi \otimes \mathbb{1}_n)(B_{ij} \otimes X_i X_j) \right) (\mathrm{Id}_q \otimes P) \succeq 0 \\
\Longrightarrow &(\mathrm{Id}_q \otimes P) \left(\sum_{i=1,j=1}^{m} A_{ij} \otimes X_i X_j \right) (\mathrm{Id}_q \otimes P) \\
&- \sum_{i=1,j=1}^{m} (\mathrm{Id}_q \phi(B_{ij}) \mathrm{Id}_q) \otimes (P \mathrm{Id}_n X_i X_j \mathrm{Id}_n P) \succeq 0 \qquad (3.1) \\
\Longrightarrow &(\mathrm{Id}_q \otimes P) \left(\sum_{i=1,j=1}^{m} A_{ij} \otimes X_i X_j \right) (\mathrm{Id}_q \otimes P) \\
&- \sum_{i=1,j=1}^{m} (\phi(\mathrm{Id}_q B_{ij} \mathrm{Id}_q)) \otimes (\mathrm{Id}_n P X_i X_j P \mathrm{Id}_n) \succeq 0 \\
\Longrightarrow &(\mathrm{Id}_q \otimes P) \left(\sum_{i=1,j=1}^{m} A_{ij} \otimes X_i X_j \right) (\mathrm{Id}_q \otimes P) \\
&- (\phi \otimes \mathbb{1}_n) \left((\mathrm{Id}_q \otimes P) \left(\sum_{i=1,j=1}^{m} B_{ij} \otimes X_i X_j \right) (\mathrm{Id}_q \otimes P) \right) \succeq 0. \\
\Longrightarrow &(\mathrm{Id}_q \otimes P) f(X) (\mathrm{Id}_q \otimes P) \succeq (\phi \otimes \mathbb{1}_n) \left((\mathrm{Id}_q \otimes P) g(X) (\mathrm{Id}_q \otimes P) \right).
\end{aligned}$$

As ϕ is a completely positive linear map, if $(\mathrm{Id}_q \otimes P) g(X) (\mathrm{Id}_q \otimes P) \succeq 0$, then $(\phi \otimes \mathbb{1}_n)(\mathrm{Id}_q \otimes P) g(X) (\mathrm{Id}_q \otimes P) \succeq 0$. It follows that $(\mathrm{Id}_q \otimes P) f(X) (\mathrm{Id}_q \otimes P) \succeq 0$. Condition (1) is established.

Now we assume that condition (2) is false, our aim is to show that condition (1) is also false. For a fixed linear map $\phi : \mathbb{R}^{q \times q} \to \mathbb{R}^{q \times q}$, let

$$\phi g(x) = \sum_{i=1,j=1}^{m} \phi(B_{ij}) x_i x_j. \qquad (3.2)$$

Consider the set

$$\{ f(x) - \phi g(x) \mid \phi : \mathbb{R}^{q \times q} \to \mathbb{R}^{q \times q} \text{ is a completely positive linear map} \}.$$

Clearly, $f(x)-\phi g(x)$ is a homogeneous quadratic polynomial and its coefficient matrix has the following form

$$\begin{pmatrix} A_{11}-\phi(B_{11}) & \cdots & A_{1m}-\phi(B_{1m}) \\ \vdots & \ddots & \vdots \\ A_{m1}-\phi(B_{m1}) & \cdots & A_{mm}-\phi(B_{mm}) \end{pmatrix} = \mathcal{A}-(\mathbb{1}_m\otimes\phi)\mathcal{B}.$$

The set

$$\mathcal{D}=\{\mathcal{A}-(\mathbb{1}_m\otimes\phi)\mathcal{B}\mid \phi:\mathbb{R}^{q\times q}\to\mathbb{R}^{q\times q} \text{ is a completely positive linear map}\}$$

is a closed translated convex cone for the matrix $\mathcal{A}$ in $\mathbb{SR}^{mq}$. Let $\mathcal{C}$ denote the positive semidefinite cone in $\mathbb{SR}^{mq}$. Since condition (2) is false, according to Theorem 2.2, the coefficient matrix of $f(x)-\phi g(x)$ can not be positive semidefinite. Hence, we have $\mathcal{C}\cap\mathcal{D}=\emptyset$.

Let us define

$$K=\{\mathbf{J}(\phi)\mid \phi:\mathbb{R}^{q\times q}\to\mathbb{R}^{q\times q} \text{ is a completely positive linear map}, \|\mathbf{J}(\phi)\|=1\},$$

where $\mathbf{J}(\phi)$ is defined by (2.2) and $\|\cdot\|$ stands for the operator norm of the matrix. The set K is compact. For any completely positive linear map $\phi:\mathbb{R}^{q\times q}\to\mathbb{R}^{q\times q}$ with $\|\mathbf{J}(\phi)\|=1$, define

$$\mathcal{D}_{\mathbf{J}(\phi)}=\{\mathcal{A}-\lambda(\mathbb{1}_m\otimes\phi)\mathcal{B}\mid\lambda\geq 0\},$$

and

$$k(\mathbf{J}(\phi))=\inf\{\|\mathcal{M}_1-\mathcal{M}_2\|\mid \mathcal{M}_1\in\mathcal{C},\ \mathcal{M}_2\in\mathcal{D}_{\mathbf{J}(\phi)}\}.$$

Then, $k(\mathbf{J}(\phi))$ can be seen as a continuous function on K. Since K is compact, there is a completely positive linear map ϕ^0 and $\mathbf{J}(\phi^0)\in K$, such that $k(\mathbf{J}(\phi^0))=\min_{\mathbf{J}(\phi)\in K}k(\mathbf{J}(\phi))$.

For the completely positive linear map ϕ^0, we have

$$\begin{aligned}(\phi^0\otimes\mathbb{1}_{\hat{n}})g(\hat{X}) &= \sum_{i=1,j=1}^{m}\phi^0(B_{ij})\otimes\hat{X}_i\hat{X}_j\succeq 0,\\ (\phi^0\otimes\mathbb{1}_{\hat{n}})g(\hat{X}) &= \sum_{i=1,j=1}^{m}\phi^0(B_{ij})\otimes\hat{X}_i\hat{X}_j\neq 0.\end{aligned}\tag{3.3}$$

Now we show that $(\mathbb{1}_m\otimes\phi^0)\mathcal{B}$ has a positive eigenvalue. If not, we would have

$$(\mathbb{1}_m\otimes\phi^0)\mathcal{B}\preceq 0.$$

Then $(\phi^0\otimes\mathbb{1}_n)g(X)\preceq 0$ for all $X\in(\mathbb{SR}^n)^m$, which contradicts (3.3). Define

$$d(\lambda)=\inf\{\|\mathcal{M}_1-\mathcal{A}+\lambda(\mathbb{1}_m\otimes\phi^0)\mathcal{B}\|\mid\mathcal{M}_1\in\mathcal{C}\}.$$

The condition that $(\mathbb{1}_m\otimes\phi^0)\mathcal{B}$ has a positive eigenvalue ensures that

$$\lim_{\lambda\to+\infty}d(\lambda)=+\infty,$$

then there exists a positive number c such that

$$\inf_{\lambda\in[0,+\infty)} d(\lambda) = \inf_{\lambda\in[0,c]} d(\lambda).$$

As $[0,c]$ is compact and $d(\lambda)$ is continuous, there exists a λ_0 such that

$$d(\lambda_0) = \inf_{\lambda\in[0,+\infty)} d(\lambda) = k(\mathbf{J}(\phi^0)).$$

Because $\mathcal{C}\cap\mathcal{D}=\emptyset$, we have

$$\inf\{\|M_1 - M_2\| | M_1\in\mathcal{C},\ M_2\in\mathcal{D}\} = \|\mathcal{M}_1 - \mathcal{A} + \lambda_0(\mathbb{1}_m\otimes\phi^0)\mathcal{B}\| > 0.$$

By the separation theorem [47, Theorem 11.4], there is a matrix $M^s\in\mathbb{R}^{(mq)\times(mq)}$, such that

$$\langle M_1, M^s\rangle \geq a_0 > \langle M_2, M^s\rangle, \quad \forall M_1\in\mathcal{C},\ \forall M_2\in\mathcal{D}.$$

It is clear by the self-duality of the cone $\mathcal{C}$ in the Frobenius inner product that $M^s\succeq 0$ and $a_0 = 0$. Then we have

$$\langle\mathcal{A}, M^s\rangle < 0 \text{ and } \langle(\mathbb{1}_m\otimes\phi)\mathcal{B}, M^s\rangle \geq 0, \tag{3.4}$$

for every completely positive linear map $\phi:\mathbb{R}^{q\times q}\to\mathbb{R}^{q\times q}$.

Let $\{e_1, e_2, \ldots, e_q\}$ be the standard orthogonal basis of $\mathbb{R}^q$, and $E=\sum_{i=1}^q e_i\otimes e_i$. The matrix M^s can be written in the following form

$$M^s = \begin{pmatrix} M^s_{11} & \cdots & M^s_{1m} \\ \vdots & \ddots & \vdots \\ M^s_{m1} & \cdots & M^s_{mm}\end{pmatrix}, \tag{3.5}$$

where each $M^s_{ij}\in\mathbb{R}^{q\times q}$. The condition (3.4) can be written in the following form:

$$\langle\mathcal{A}, M^s\rangle = \sum_{i=1,j=1}^m \langle A_{ij}, M^s_{ij}\rangle = E^T\left(\sum_{i=1,j=1}^m A_{ij}\otimes M^s_{ij}\right)E < 0. \tag{3.6}$$

Moreover, we have

$$\begin{aligned}
\langle(\mathbb{1}_m\otimes\phi)\mathcal{B}, M^s\rangle &= \sum_{i=1,j=1}^m \langle\phi(B_{ij}), M^s_{ij}\rangle \\
&= \sum_{i=1,j=1}^m \sum_{a=1,b=1}^q \langle\phi(B_{ij}), E_{ab}\rangle\langle M^s_{ij}, E_{ab}\rangle \\
&= \left\langle \sum_{i=1,j=1}^m \phi(B_{ij})\otimes M^s_{ij}, \sum_{a=1,b=1}^q E_{ab}\otimes E_{ab}\right\rangle \\
&= \left\langle \sum_{i=1,j=1}^m B_{ij}\otimes M^s_{ij}, \sum_{a=1,b=1}^q \phi^*(E_{ab})\otimes E_{ab}\right\rangle \\
&= \left\langle \sum_{i=1,j=1}^m B_{ij}\otimes M^s_{ij}, \mathbf{J}(\phi^*)\right\rangle \geq 0,
\end{aligned}$$

where $E_{ab} \in \mathbb{R}^{q\times q}$ are matrices whose (a,b)-th entry is 1 and all others are 0. According to Theorem 2.1, the set

$$\left\{\mathbf{J}(\phi^*) \mid \phi : \mathbb{R}^{q\times q} \to \mathbb{R}^{q\times q} \text{ is completely positive}\right\}$$

is equivalent to the positive semidefinite cone in $\mathbb{SR}^{q^2}$. It follows that

$$\sum_{i=1,j=1}^{m} B_{ij} \otimes M^s_{ij} \succeq 0. \tag{3.7}$$

In order to show that the condition (1) in Theorem 1.1 is not satisfied, we need to translate the inequality conditions (3.6) and (3.7) into the evaluations of f and g at some matrix vector $X \in (\mathbb{SR}^q)^m$. This requires some work, since the positive semidefinite matrix $M^s = (M^s_{ij}) \in (\mathbb{SR}^q)^m$ may not belong to the set

$$\mathcal{X} = \{YY^T \mid Y \in (\mathbb{SR}^q)^m\},$$

which is a strict subset of the positive semidefinite cone $\mathcal{C} \subset \mathbb{SR}^{mq}$, and thus we can not ensure that there always exists an $X \in (\mathbb{SR}^q)^m$ such that

$$f(X) = \sum_{i=1,j=1}^{m} A_{ij} \otimes M^s_{ij} \quad \text{and} \quad g(X) = \sum_{i=1,j=1}^{m} B_{ij} \otimes M^s_{ij}.$$

This is the main reason why we introduce a projection (3.10) to construct an evaluation point.

Since M^s defined in (3.5) is a positive semidefinite matrix, it has the decomposition

$$\begin{aligned} M^s &= \sum_{k=1}^{r} v_k v_k^T,\ v_k \in \mathbb{R}^{mq},\ r = \operatorname{rank}(M^s), \\ v_k &= \begin{pmatrix} v_k^1 \\ \vdots \\ v_k^m \end{pmatrix},\ v_k^l \in \mathbb{R}^q,\ 1 \le l \le m,\ k = 1,\dots,r. \end{aligned} \tag{3.8}$$

We define $X^M := (X^M_1, \dots, X^M_m) \in (\mathbb{R}^{(r+q)\times(r+q)})^m$, where for each $i = 1,\dots,m$,

$$X^M_i = \begin{pmatrix} & & & (v_1^i)^T \\ & 0 & & \vdots \\ & & & (v_r^i)^T \\ v_1^i & \cdots & v_r^i & 0 \end{pmatrix}, \tag{3.9}$$

and $P^M : \mathbb{R}^{(r+q)} \to \mathbb{R}^{(r+q)}$ is the projection to the last q coordinates:

$$P^M = \begin{pmatrix} 0 & \\ & \mathrm{Id}_q \end{pmatrix}. \tag{3.10}$$

Then the condition (3.7) can be used to show

$$\begin{aligned}(\mathrm{Id}_q \otimes P^M)g(X^M)(\mathrm{Id}_q \otimes P^M) &= \sum_{i=1,j=1}^{m} B_{ij} \otimes P^M X_i^M X_j^M P^M \\ &= \sum_{i=1,j=1}^{m} B_{ij} \otimes M_{ij}^s \succeq 0.\end{aligned}$$

On the other hand, the condition (3.6) can be used to show

$$\begin{aligned}E^T(\mathrm{Id}_q \otimes P^M)f(X^M)(\mathrm{Id}_q \otimes P^M)E &= E^T \left(\sum_{i=1,j=1}^{m} A_{ij} \otimes P^M X_i^M X_j^M P^M \right) E \\ &= E^T \left(\sum_{i=1,j=1}^{m} A_{ij} \otimes M_{ij}^s \right) E < 0.\end{aligned}$$

Therefore, we have

$$(\mathrm{Id}_q \otimes P^M)f(X^M)(\mathrm{Id}_q \otimes P^M) \not\succeq 0.$$

Hence, condition (1) in Theorem 1.1 is false. ∎

COROLLARY 3.1. *Under the same assumption in Theorem 1.1, the statements* (1) *and* (2) *in Theorem 1.1 are also equivalent to the following two conditions:*

(3) *For all* $X \in (\mathbb{SR}^n)^m$*, all orthogonal projection matrices* $P \in \mathbb{R}^{n\times n}$*,* $n \in \mathbb{N}^+$*, if* $(\mathrm{Id}_q \otimes P)g(X)(\mathrm{Id}_q \otimes P) \succeq 0$*, then* $(\mathrm{Id}_q \otimes P)f(X)(\mathrm{Id}_q \otimes P) \succeq 0$*.*
(4) *For all* $X \in (\mathbb{SR}^n)^m$*, all matrices* $Q \in \mathbb{R}^{n\times \ell}$*,* $\ell, n \in \mathbb{N}^+$*, if* $(\mathrm{Id}_q \otimes Q^T)g(X)(\mathrm{Id}_q \otimes Q) \succeq 0$*, then* $(\mathrm{Id}_q \otimes Q^T)f(X)(\mathrm{Id}_q \otimes Q) \succeq 0$*.*

Proof. From the proof of (2) $\Rightarrow$ (1), we can see that (2) $\Rightarrow$ (4) also holds by modifying derivations in (3.1).

$$\begin{aligned}&\sum_{i=1,j=1}^{m} A_{ij} \otimes X_i X_j - \sum_{i=1,j=1}^{m} \phi(B_{ij}) \otimes X_i X_j \succeq 0 \\ \Longrightarrow &\sum_{i=1,j=1}^{m} A_{ij} \otimes X_i X_j - \sum_{i=1,j=1}^{m} (\phi \otimes \mathbb{1}_n)(B_{ij} \otimes X_i X_j) \succeq 0 \\ \Longrightarrow &(\mathrm{Id}_q \otimes Q^T) \left(\sum_{i=1,j=1}^{m} A_{ij} \otimes X_i X_j \right) (\mathrm{Id}_q \otimes Q) \\ &- (\mathrm{Id}_q \otimes Q^T) \left(\sum_{i=1,j=1}^{m} (\phi \otimes \mathbb{1}_n)(B_{ij} \otimes X_i X_j) \right) (\mathrm{Id}_q \otimes Q) \succeq 0\end{aligned}$$

$$
\begin{aligned}
&\Longrightarrow (\mathrm{Id}_q \otimes Q^T)\left(\sum_{i=1,j=1}^{m} A_{ij} \otimes X_i X_j\right)(\mathrm{Id}_q \otimes Q) \\
&\quad - \sum_{i=1,j=1}^{m} (\mathrm{Id}_q \phi(B_{ij}) \mathrm{Id}_q) \otimes (Q^T \mathrm{Id}_n X_i X_j \mathrm{Id}_n Q) \succeq 0 \\
&\Longrightarrow (\mathrm{Id}_q \otimes Q^T)\left(\sum_{i=1,j=1}^{m} A_{ij} \otimes X_i X_j\right)(\mathrm{Id}_q \otimes Q) \\
&\quad - \sum_{i=1,j=1}^{m} (\phi(\mathrm{Id}_q B_{ij} \mathrm{Id}_q)) \otimes (\mathrm{Id}_l Q^T X_i X_j Q \mathrm{Id}_l) \succeq 0 \\
&\Longrightarrow (\mathrm{Id}_q \otimes Q^T)\left(\sum_{i=1,j=1}^{m} A_{ij} \otimes X_i X_j\right)(\mathrm{Id}_q \otimes Q) \\
&\quad - (\phi \otimes \mathbb{1}_l)\left((\mathrm{Id}_q \otimes Q^T)\left(\sum_{i=1,j=1}^{m} B_{ij} \otimes X_i X_j\right)(\mathrm{Id}_q \otimes Q)\right) \succeq 0. \\
&\Longrightarrow (\mathrm{Id}_q \otimes Q^T) f(X) (\mathrm{Id}_q \otimes Q) \succeq (\phi \otimes \mathbb{1}_l)\left((\mathrm{Id}_q \otimes Q^T) g(X) (\mathrm{Id}_q \otimes Q)\right).
\end{aligned}
$$

As ϕ is a completely positive linear map, if $(\mathrm{Id}_q \otimes Q^T)g(X)(\mathrm{Id}_q \otimes Q) \succeq 0$, then $(\phi \otimes \mathbb{1}_\ell)(\mathrm{Id}_q \otimes Q^T)g(X)(\mathrm{Id}_q \otimes Q) \succeq 0$. It follows that $(\mathrm{Id}_q \otimes Q^T)f(X)(\mathrm{Id}_q \otimes Q) \succeq 0$. Condition (4) is established.

The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (1) are obvious. ∎

REMARK 3.2. Theorem 1.1 is still true when the dimension q_f of the coefficients of the polynomial f is not equal to the dimension q_g of the coefficients of the polynomial g. In fact, if $q_f < q_g$, we can always add zeros to the coefficients of f to make $q_f = q_g$. Consider the case when $q_f > q_g$. Suppose that k is the smallest positive integer satisfying $q_f \leq kq_g$. Define a new polynomial $\tilde{g} = \overset{k}{\oplus} g$. Then Theorem 1.1 is still valid after replacing g by $\tilde{g}$.

REMARK 3.3. Suppose that we are given multiple noncommutative matrix-valued homogeneous quadratic polynomials

$$g_1(x), \ldots, g_L(x),$$

and there is an $\hat{X} \in (\mathbb{S}\mathbb{R}^{\hat{n}})^m$ for some $\hat{n} \in \mathbb{N}^+$, such that $g_l(\hat{X}) \succ 0$, $l = 1, \ldots, L$. Let

$$
\overset{L}{\oplus} f(x) = \begin{pmatrix} f(x) & & \\ & \ddots & \\ & & f(x) \end{pmatrix}, \quad \overset{L}{\underset{l=1}{\oplus}} g_l(x) = \begin{pmatrix} g_1(x) & & \\ & \ddots & \\ & & g_L(x) \end{pmatrix}. \tag{3.11}
$$

Applying Theorem 1.1 to each diagonal entry of $\overset{L}{\oplus} f(x)$ and $\overset{L}{\underset{l=1}{\oplus}} g_l(x)$, we can prove the equivalence between two statements below:

(1) For all $X \in (\mathbb{SR}^n)^m$, $n > q$, if $(\mathrm{Id}_q \otimes P)g_l(X)(\mathrm{Id}_q \otimes P) \succeq 0$, $l = 1, \dots, L$, then $(\mathrm{Id}_q \otimes P)f(X)(\mathrm{Id}_q \otimes P) \succeq 0$, where $P : \mathbb{R}^n \to \mathbb{R}^n$ is the projection to the last q coordinates.

(2) There is a nonzero completely positive linear mapping $\phi : \mathbb{R}^{Lq\times Lq} \to \mathbb{R}^{Lq\times Lq}$ such that $\overset{L}{\oplus} f(X) - (\phi \otimes \mathbb{1}_n) \overset{L}{\underset{l=1}{\oplus}} g_l(X) \succeq 0$ for all $X \in (\mathbb{SR}^n)^m, n \in \mathbb{N}^+$.

3.2. S-LEMMA FOR HEREDITARY NONCOMMUTATIVE MATRIX-VALUED POLYNOMIALS. Now let $f(x)$ and $g(x)$ be hereditary noncommutative matrix-valued homogeneous polynomials with the following form

$$f(x) = \sum_{i=1,j=1}^{m} A_{ij}x_ix_j^T,\ g(x) = \sum_{i=1,j=1}^{m} B_{ij}x_ix_j^T,$$

where $A_{i,j}, B_{i,j} \in \mathbb{R}^{q\times q}$ and $A_{i,j} = A_{j,i}^T$, $B_{i,j} = B_{j,i}^T$ for all $i, j = 1, ..., m$. In other words, we do not assume that the matrix variables for f and g are symmetric. Now we prove Theorem 1.2 in which the statement (1) is simpler than its counterpart from Theorem 1.1, i.e., there is no need to apply projections in the hereditary case.

Proof of Theorem 1.2. The implication $(2) \Rightarrow (1)$ is obvious.

Let us show the implication $(1) \Rightarrow (2)$. Assume that condition (2) in Theorem 1.2 is false. Similar to the discussion in the proof of Theorem 1.1, we can find a separation matrix $M^s \succeq 0$ which satisfies the condition (3.4) and has the decomposition (3.8).

Since we do not require the variable X_i^M to be symmetric, instead of constructing X_i^M as in (3.9), we let

$$X_i^M = \begin{pmatrix} v_1^i & \cdots & v_r^i \end{pmatrix}.$$

Letting $n = \max\{r, q\}$, we add zero rows or columns into $X_i^M \in \mathbb{R}^{q\times r}$ to make it a square matrix in $\mathbb{R}^{n\times n}$. Without loss of generality, we assume that $r > q$, and define new matrices $\tilde{X}_i^M \in \mathbb{R}^{n\times n}$ for $i = 1, \dots, m$,

$$\tilde{X}_i^M = \begin{pmatrix} v_1^i & \cdots & v_r^i \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Let $\tilde{X}^M := (\tilde{X}_1^M, \dots, \tilde{X}_m^M) \in (\mathbb{R}^{n\times n})^m$. We can translate the inequality conditions (3.6) and (3.7) into the evaluations of f and g at $\tilde{X}^M \in (\mathbb{R}^{n\times n})^m$. In particular, we have

$$g(\tilde{X}^M) = \sum_{i=1,j=1}^{m} B_{ij} \otimes \begin{pmatrix} M_{ij}^s & 0 \\ 0 & 0 \end{pmatrix} \succeq 0.$$

Let $\{e_1, e_2, \ldots, e_q\}$ be the standard orthogonal basis of $\mathbb{R}^q$, $\{f_1, f_2, \ldots, f_n\}$ be the standard orthogonal basis of $\mathbb{R}^n$, and $E' = \sum_{i=1}^{q} e_i \otimes f_i$, we have

$$E'^T f(\tilde{X}^M) E' = E'^T \left(\sum_{i=1,j=1}^{m} A_{ij} \otimes \begin{pmatrix} M_{ij}^s & 0 \\ 0 & 0 \end{pmatrix} \right) E' < 0.$$

Therefore, condition (1) in Theorem 1.2 is also false. ∎

4. OTHER VARIANTS OF S-LEMMA IN NONCOMMUTATIVE CASES

Different from commutative polynomials, there are many ways to extend the S-lemma for (noncommutative matrix-valued) polynomials with matrix evaluations. Comparing with the condition (1) in Theorem 1.1, we consider the following condition which is a more direct extension of the classical S-lemma:

(1′) For all $X \in (\mathbb{SR}^n)^m$, $n \in \mathbb{N}^+$, if $g(X) \succeq 0$, then $f(X) \succeq 0$.

REMARK 4.1. It is straightforward to verify that condition (2) in Theorem 1.1 implies (1′). Therefore, under the assumption that there is an $\hat{X} \in (\mathbb{R}^{\hat{n}\times\hat{n}})^m$ for some $\hat{n} \in \mathbb{N}^+$, such that $g(\hat{X}) \succ 0$, the condition (1) in Theorem 1.1 implies the condition (1′), but it is unknown if it is true the other way around.

As illustrated by the following example, without the assumption of the existence of $\hat{X}$ such that $g(\hat{X}) \succ 0$, (1′) can not imply the condition (1) in Theorem 1.1.

EXAMPLE 4.2. We construct two noncommutative matrix-valued polynomials

$$f = \begin{pmatrix} x_1x_2 + x_2x_1 - x_2x_2 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$g = \begin{pmatrix} x_1x_1 - x_2x_2 & 0 \\ 0 & x_1x_2 + x_2x_1 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_1x_2 - x_2x_1 \\ x_2x_1 - x_1x_2 & 0 \end{pmatrix}.$$

For any $X \in (\mathbb{SR}^n)^2$, if $g(X) \succeq 0$, then we have $X_1X_2 - X_2X_1 = 0$. So X_1, X_2 have the same eigenspaces. Let $X = (X_1, X_2)$, where

$$X_1 = \sum_{i=1}^{r} \lambda_i v_i v_i^T,\ X_2 = \sum_{i=1}^{r} \mu_i v_i v_i^T \quad \text{(spectral decomposition)}.$$

Assuming that $g(X) \succeq 0$, we have

$$(\lambda_i)^2 - (\mu_i)^2 \geq 0 \text{ and } \lambda_i\mu_i \geq 0 \text{ for all } i = 1, \ldots, r,$$

$$X_1X_2 + X_2X_1 - X_2X_2 = \sum_{i=1}^{r}(2\lambda_i\mu_i - \mu_i^2)v_iv_i^T \succeq 0.$$

Then $f(X)$ is positive semidefinite. Hence f and g satisfy the condition $(1')$.

On the other hand, let

$$X_1^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_2^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \mathrm{Id}_4.$$

It is straightforward to verify that

$$(\mathrm{Id}_4 \otimes P)g(X^0)(\mathrm{Id}_4 \otimes P) = \begin{pmatrix} P(X_1^0X_1^0 - X_2^0X_2^0)P & 0 \\ 0 & P(X_1^0X_2^0 + X_2^0X_1^0)P \end{pmatrix}$$
$$\oplus \begin{pmatrix} 0 & P(X_1^0X_2^0 - X_2^0X_1^0)P \\ P(X_2^0X_1^0 - X_1^0X_2^0)P & 0 \end{pmatrix}.$$

The top left corner matrix is a positive semidefinite matrix

$$P(X_1^0X_1^0 - X_2^0X_2^0)P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \succeq 0.$$

The other submatrices are all zero matrices

$$P(X_1^0X_2^0 + X_2^0X_1^0)P = \pm P(X_1^0X_2^0 - X_2^0X_1^0)P = 0.$$

Therefore, we have

$$(\mathrm{Id}_4 \otimes P)g(X^0)(\mathrm{Id}_4 \otimes P) \succeq 0.$$

However, we have

$$(\mathrm{Id}_4 \otimes P)f(X^0)(\mathrm{Id}_4 \otimes P) = \begin{pmatrix} P(X_1^0X_2^0 + X_2^0X_1^0 - X_2^0X_2^0)P & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The top left corner matrix is negative semidefinite

$$P(X_1^0X_2^0 + X_2^0X_1^0 - X_2^0X_2^0)P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \preceq 0.$$

Therefore, we have

$$(\mathrm{Id}_4 \otimes P)f(X^0)(\mathrm{Id}_4 \otimes P) \preceq 0,$$

and thus the condition (1) in Theorem 1.1 is false for the given f and g.

With the assumption that there is an $\hat{X} \in (\mathbb{SR}^{\hat{n}})^m$ for some $\hat{n} \in \mathbb{N}^+$, such that $g(\hat{X}) \succ 0$, whether or not $(1')$ can imply the condition (1) in Theorem 1.1 is an interesting problem and we wish to investigate it in future.

Furthermore, one can also consider the following condition:

$(1'')$ For all $X \in (\mathbb{SR}^n)^m$, $n \in \mathbb{N}^+$, given a vector $v \in \mathbb{R}^{qn}$, if $v^T g(X) v \geq 0$ then $v^T f(X) v \geq 0$.

The following example shows that the condition $(1'')$ is strictly stronger than the condition (1) in Theorem 1.1 and the condition $(1')$.

EXAMPLE 4.3. We are given the noncommutative matrix-valued polynomials

$$f = \begin{pmatrix} x_1x_1 & 0 \\ 0 & x_1x_1 - x_2x_2 \end{pmatrix},$$

and

$$g = \begin{pmatrix} x_1x_1 - x_2x_2 & 0 \\ 0 & x_1x_1 \end{pmatrix}.$$

Let us define a linear map ϕ_2 from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$

$$\phi_2 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}.$$

It is easy to verify that ϕ_2 is a completely positive linear map. We have

$$f(X) - (\phi_2 g)(X) = 0 \text{ for all } X \in \mathbb{SR}^n,\ n \in \mathbb{N}^+.$$

The condition (2) in Theorem 1.1 is satisfied. Therefore, condition (1) in Theorem 1.1 and condition $(1')$ are satisfied too. However, for

$$X^0 = [1, 2]^T,\ v = [0, 1]^T,$$

we have

$$v^T g(X^0) v = 1 > 0, \text{ but } v^T f(X^0) v = -3 < 0.$$

Therefore, the condition $(1'')$ above is not satisfied.

5. SOME DISCUSSIONS

In this paper, we established several variants of the S-lemma for noncommutative matrix-valued homogeneous quadratic polynomials. In contrast to the commutative case, the extension of the S-lemma from homogeneous to nonhomogeneous polynomials remains an open problem.

In the commutative case, it is straightforward to convert a nonhomogeneous polynomial to a homogeneous one by introducing a new variable. However, this process becomes more complicated in noncommutative cases. First of all, due to the noncommutativity of variables, the homogenization of a noncommutative polynomial is not unique.

EXAMPLE 5.1. For the noncommutative matrix-valued nonhomogeneous quadratic polynomial

$$f(x) = \begin{pmatrix} x^2 & x \\ x & 1 \end{pmatrix},$$

we have two different choices of homogenization:

$$h_1(x_0, x) = \begin{pmatrix} x^2 & xx_0 \\ x_0x & x_0^2 \end{pmatrix} \quad \text{and} \quad h_2(x_0, x) = \begin{pmatrix} x^2 & x_0x \\ xx_0 & x_0^2 \end{pmatrix}.$$

For all $X \in \mathbb{SR}^n$, $n \in \mathbb{N}^+$, it holds that

$$f(X) = h_1(\mathrm{Id_n}, X) = h_2(\mathrm{Id_n}, X).$$

The coefficient matrices of $h_1(X_0, X)$ and $h_2(X_0, X)$ satisfy the following conditions:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \not\succeq 0.$$

By Theorem 2.2, we know that $h_1(X_0, X)$ is positive semidefinite for all $X_0, X \in \mathbb{SR}^n$, $n \in \mathbb{N}^+$, while $h_2(X_0, X)$ is not positive semidefinite for all $X_0, X \in \mathbb{SR}^n$, $n \in \mathbb{N}^+$.

Unlike the commutative case proved in [55, 56], it is unclear how to derive S-lemma for nonhomogeneous quadratic polynomials from homogeneous ones. In particular, for a general nonhomogeneous quadratic polynomial f and its homogenization h, we have

$$h(X_0, X) \neq X_0 f(X_0^{-\frac{1}{2}} X X_0^{-\frac{1}{2}}) X_0, \quad X \in (\mathbb{SR}^n)^m, \ X_0 \in \mathbb{SR}^n \text{ is invertible.}$$

Thus, the S-lemma for general non-homogeneous quadratic polynomials in noncommutative cases are still unknown and left for future research.

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Appendix A. An alternative proof of Theorem 2.2

We give below a different proof of Theorem 2.2 based on the Choi matrix.

Proof of Theorem 2.2. Let $\mathcal{A}^{'}$ represent the canonical shufle of $\mathcal{A}$, i.e.,

$$\mathcal{A}^{'} = [\mathcal{A}^{'}_{ij}]_{i,j=1,\ldots,q} \quad \text{where} \quad \mathcal{A}^{'}_{ij} = [\langle A_{kj}, E_{ij}\rangle]_{k,l=1,\ldots,m}.$$

Then, $\mathcal{A} \succeq 0$ if and only if $\mathcal{A}^{'} \succeq 0$ [48, Proposition 1]. Using the matrix $\mathcal{A}^{'}$ as the Choi matrix, define a linear map

$$\begin{aligned} \psi_f : \mathbb{R}^{m\times m} &\to \mathbb{R}^{q\times q} \\ M &\mapsto \begin{pmatrix} \langle \mathcal{A}^{'}_{11}, M\rangle & \cdots & \langle \mathcal{A}^{'}_{1q}, M\rangle \\ \vdots & \ddots & \vdots \\ \langle \mathcal{A}^{'}_{q1}, M\rangle & \cdots & \langle \mathcal{A}^{'}_{qq}, M\rangle \end{pmatrix}. \end{aligned}$$

According to Theorem 2.1, the linear map ψ_f is completely positive. Hence, $\psi_f \otimes \mathbb{1}_n$ is a positive linear map for all $n \in \mathbb{N}^+$. It is essential to notice that

$$f(X) = \psi_f \otimes \mathbb{1}_n \begin{pmatrix} X_1X_1 & \cdots & X_1X_m \\ \vdots & \ddots & \vdots \\ X_mX_1 & \cdots & X_mX_m \end{pmatrix}$$

for all $X \in (\mathbb{SR}^n)^m$, $n \in \mathbb{N}^+$. Then we know that $f(X)$ is positive semidefinite for all $X \in (\mathbb{SR}^n)^m$, $n \in \mathbb{N}^+$.

On the other hand, we define $f'(X) = \sum_{i=1,j=1}^m X_iX_j \otimes A_{ij}$. It is obvious that for any $X \in (\mathbb{SR}^n)^m$, $n \in \mathbb{N}^+$,

$$f(X) \succeq 0 \Leftrightarrow f'(X) \succeq 0.$$

Let $X^0 := (X_1^0, \ldots, X_m^0) \in (\mathbb{SR}^{(m+1)})^m$, where each X_i^0 is the matrix whose $(1, i+1)$-th entry and $(i+1, 1)$-th entry are 1 and all others are 0, i.e.,

$$X_i^0 := \begin{array}{c c} & (i+1)\text{-th} \\ (i+1)\text{-th} & \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \end{array}.$$

It is easy to check that

$$f'(X^0) = \begin{pmatrix} \sum_{i=1}^m A_{ii} & & & \\ & A_{11} & \cdots & A_{1m} \\ & \vdots & \ddots & \vdots \\ & A_{m1} & \cdots & A_{mm} \end{pmatrix}.$$

By assumption $f'(X^0) \succeq 0$, and hence we have $\mathcal{A} \succeq 0$.

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