SIAM J. OPTIM.
Vol. 25, No. 1, pp. 377–395

SEMIDEFINITE REPRESENTATIONS OF NONCOMPACT CONVEX SETS*

FENG GUO†, CHU WANG‡, AND LIHONG ZHI‡

Abstract. We consider the problem of the semidefinite representation of a class of noncompact basic semialgebraic sets. We introduce the conditions of pointedness and closedness at infinity of a semialgebraic set and show that under these conditions our modified hierarchies of nested theta bodies and Lasserre's relaxations converge to the closure of the convex hull of S. Moreover, if the Putinar-Prestel's Bounded Degree Representation (PP-BDR) property is satisfied, our theta body and Lasserre's relaxation are exact when the order is large enough; if the PP-BDR property does not hold, our hierarchies converge uniformly to the closure of the convex hull of S restricted to every fixed ball centered at the origin. We illustrate through a set of examples that the conditions of pointedness and closedness are essential to ensure the convergence. Finally, we provide some strategies to deal with cases where the conditions of pointedness and closedness are violated.

Key words. convex sets, semidefinite representations, theta bodies, sums of squares, moment matrices

AMS subject classifications. 11E25, 52A27, 90C22, 14P10

DOI. 10.1137/140952247

1. Introduction.

Consider the basic semialgebraic set

$$S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\},$$

where $g_i(X) \in \mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n],\ i = 1, \ldots, m$. The convex hull of S is denoted by $\mathbf{co}(S)$ and its closure is denoted by $\mathbf{cl}(\mathbf{co}(S))$. Characterizing $\mathbf{cl}(\mathbf{co}(S))$ is an important issue raised in [1, 8, 9, 14]. There is a considerable amount of interesting work by many people. For instance, using the same variables appearing in S, Rostalski and Sturmfels [19] exploit projective varieties to explicitly find the polynomials that describe the boundary of $\mathbf{co}(S)$ when S is a compact real algebraic variety; by introducing more variables, theta bodies [4] and Lasserre's relaxations [10] have been given to compute $\mathbf{cl}(\mathbf{co}(S))$ approximately or exactly when S is a compact semialgebraic set. In this paper, we aim to extend work in [4, 10] and provide sufficient conditions such that the modified hierarchies of theta bodies and Lasserre's relaxations of a noncompact semialgebraic set S can still converge to the closure of the convex hull of S.

Let $\tilde{g}_1, \ldots, \tilde{g}_m$ be homogenized polynomials of $g_1, \ldots, g_m$, respectively. We lift S to a cone of $\widetilde{S}^{\mathrm{o}} \subseteq \mathbb{R}^{n+1}$,

$$\widetilde{S}^{\mathrm{o}} := \{\tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{g}_1(\tilde{x}) \geq 0, \ldots, \tilde{g}_m(\tilde{x}) \geq 0,\ x_0 > 0\}.$$

*Received by the editors January 10, 2014; accepted for publication (in revised form) November 24, 2014; published electronically February 5, 2015. This work was supported by National Institute for Mathematical Sciences 2014 Thematic Program on Applied Algebraic Geometry in Daejeon, South Korea.

http://www.siam.org/journals/siopt/25-1/95224.html

†School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, People's Republic of China (fguo@dlut.edu.cn). This author was supported by the Fundamental Research Funds for the Central Universities, the Chinese National Natural Science Foundation under grants 11401074.

‡KLMM, Academy of Mathematics and Systems Science, CAS, Beijing, 100190, People's Republic of China (cwang@mmrc.iss.ac.cn, lzhi@mmrc.iss.ac.cn). These authors were supported by NKBRPC 2011CB302400, the Chinese National Natural Science Foundation under grants 91118001, 60821002/F02.

Let $\widetilde{X} := (X_0, X_1, \ldots, X_n)$. Denote $\mathcal{Q}_k(\widetilde{G})$ as the kth quadratic module generated by

$$\widetilde{G} := \left\{\tilde{g}_1, \ldots, \tilde{g}_m,\ X_0,\ \|\widetilde{X}\|_2^2 - 1,\ 1 - \|\widetilde{X}\|_2^2\right\},$$

and

$$\widetilde{S} := \left\{\tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{g}_1(\tilde{x}) \geq 0, \ldots, \tilde{g}_m(\tilde{x}) \geq 0,\ x_0 \geq 0,\ \|\tilde{x}\|_2^2 = 1\right\}.$$

Denote $\mathbb{P}[\widetilde{X}]_1 := (\mathbb{R}[\widetilde{X}]_1 \backslash \mathbb{R}) \cup \{0\}$, where $\mathbb{R}[\widetilde{X}]_1$ is the set of linear polynomials in $\mathbb{R}[\widetilde{X}]$. We construct the hierarchy of theta bodies $\widetilde{\mathrm{TH}}_k(\widetilde{G})$,

$$\widetilde{\mathrm{TH}}_k(\widetilde{G}) := \left\{x \in \mathbb{R}^n \mid \tilde{l}(1, x) \geq 0 \quad \forall\, \tilde{l} \in \mathcal{Q}_k(\widetilde{G}) \cap \mathbb{P}[\widetilde{X}]_1\right\},$$

and Lasserre's relaxations $\widetilde{\Omega}_k(\widetilde{G})$,

$$\widetilde{\Omega}_k(\widetilde{G}) := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists y \in \mathbb{R}^{\tilde{s}(2k)} \text{ s.t. } \mathscr{L}_y(X_0) = 1, \\ \mathscr{L}_y(X_i) = x_i,\ i = 1, \ldots, n, \\ M_{k-1}(X_0 y) \succeq 0,\ M_{k-1}((\|\widetilde{X}\|_2^2 - 1)y) = 0, \\ M_k(y) \succeq 0,\ M_{k-k_j}(\tilde{g}_j y) \succeq 0,\ j = 1, \ldots, m, \end{array} \right\},$$

where $\tilde{s}(k) = \binom{n+k+1}{n+1}$ and $k_j = \lceil \deg g_j/2 \rceil$ for every $k \in \mathbb{N}$.

Our contribution. Consider a noncompact basic semialgebraic set S.

- Assuming that S is closed at ∞ [13] and the convex cone $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is closed and pointed (equivalently, it contains no lines through the origin):
  - We prove that the hierarchies of $\widetilde{\mathrm{TH}}_k(\widetilde{G})$ and $\mathbf{cl}(\widetilde{\Omega}_k(\widetilde{G}))$ defined above converge to $\mathbf{cl}(\mathbf{co}(S))$ asymptotically. If $\mathcal{Q}_k(\widetilde{G})$ is closed, then $\widetilde{\mathrm{TH}}_k(\widetilde{G}) = \mathbf{cl}(\widetilde{\Omega}_k(\widetilde{G}))$ for $k \in \mathbb{N}$.
  - If the Putinar-Prestel's Bounded Degree Representation (PP-BDR) [10] holds for $\widetilde{G}$ with order k', then we conclude that $\mathbf{cl}(\mathbf{co}(S)) = \widetilde{\mathrm{TH}}_{k'}(\widetilde{G}) = \mathbf{cl}(\widetilde{\Omega}_{k'}(\widetilde{G}))$. If the PP-BDR property does not hold, then for every $\epsilon > 0$, $\mathbf{cl}(\widetilde{\Omega}_k(\widetilde{G}))$ converges uniformly to $\mathbf{cl}(\mathbf{co}(S))$ restricted to every fixed ball centered at the origin.
- We show that the conditions of closedness and pointedness are essential to guarantee the convergence of the constructed hierarchies.
  - We observe that the condition of closedness of S at ∞ depends on the generators of S and in many cases, we can force S to become closed at ∞ by adding a redundant linear polynomial obtained by the property of pointedness of $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$.
  - If $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is not pointed, then we divide S into 2^n parts along each axis. If S is closed at ∞ and each part satisfies PP-BDR property, we can compute modified Lasserre's relaxations for each one and then glue them together properly.

Structure of the paper. We provide in section 2 some preliminaries about convex sets and cones. We also recall some known results about theta bodies [4] and Lasserre's relaxations [10] for compact semialgebraic sets. An example is given to show that for a noncompact semialgebraic set S, the sequence defined in (2.3) or (2.5) does not converge to $\mathbf{cl}(\mathbf{co}(S))$. In section 3, when S is a noncompact semialgebraic set, we provide sufficient conditions for guaranteeing the convergence of modified theta bodies and Lasserre's relaxations for computing $\mathbf{cl}(\mathbf{co}(S))$. Some examples are also given to illustrate our method. More discussions on these sufficient conditions are given in section 4.

2. Preliminaries. In this section we present some preliminaries needed in the rest of this paper.

2.1. Convex sets and cones. The symbol $\mathbb{R}$ denotes the set of real numbers. For $x \in \mathbb{R}^n$, $\|x\|_2$ denotes the standard Euclidean norm of x. A subset $C \subseteq \mathbb{R}^n$ is *convex* if for any $u, v \in C$ and any θ with $0 \leq \theta \leq 1$, we have $\theta u + (1-\theta)v \in C$. For any subset $W \subseteq \mathbb{R}^n$, denote $\mathbf{ri}\,(W)$, $\mathbf{cl}(W)$, and $\mathbf{co}(W)$ as the relative interior, closure, and convex hull of W, respectively. A subset $K \subseteq \mathbb{R}^n$ is a *cone* if it is closed under positive scalar multiplication. The *dual cone* of K is

$$K^* = \{c \in \mathbb{R}^n \mid \langle c, x\rangle \geq 0 \quad \forall x \in K\}.$$

In particular, $(\mathbb{R}^n)^* = \mathbb{R}^n$ and $L^* = L^\perp$ for any subspace $L \subseteq \mathbb{R}^n$. A cone K need not be convex, but its dual cone K^* is always convex and closed. The second dual K^{**} is the closure of the convex hull of K. Hence, if K is a closed convex cone, then $K^{**} = K$.

PROPOSITION 2.1 (see [17, Corollary 16.4.2]). *Let K_1 and $K_2 \subseteq \mathbb{R}^n$ be two closed convex cones, then*

$$(K_1 + K_2)^* = K_1^* \cap K_2^* \quad \text{and} \quad (K_1 \cap K_2)^* = \mathbf{cl}\,(K_1^* + K_2^*)\,. \tag{2.1}$$

In particular, for any subspace $L \subseteq \mathbb{R}^n$,

$$(K_1 \cap L)^* = \mathbf{cl}\left(K_1^* + L^\perp\right).$$

DEFINITION 2.2. *A closed convex cone K is pointed if $K \cap -K = \{0\}$, i.e., K contains no lines through the origin.*

It is well known that the convex hull of a compact set in $\mathbb{R}^n$ is closed. However, it is generally not true for a noncompact set. For example, let

$$V := \{(x_1, x_2) \in \mathbb{R}^2 \mid \ x_1 x_2 = 1,\ x_2 \geq 0\} \cup \{(0,0)\},$$

then

$$\mathbf{co}\,(V) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0,\ x_2 > 0\} \cup \{(0,0)\},$$

which is not closed.

THEOREM 2.3. *Let K be a closed cone. The following assertions are equivalent*:
1. $\mathbf{co}(K)$ *is closed and pointed;*
2. $\mathbf{co}(K)$ *contains no lines through the origin;*
3. *there exists a vector $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ such that $\langle c, x\rangle > 0$ for all $x \in \mathbf{co}(K)\backslash\{0\}$.*

Proof. For the proof, see Theorem 11 and Lemma 2 in [2]. ☐

Remark 2.4. Although the pointedness is defined on closed convex sets, by Theorem 2.3, it is safe to say that $\mathbf{co}(K)$ is pointed for a closed cone K if $\mathbf{co}(K) \cap -\mathbf{co}(K) = \{0\}$.

2.2. Quadratic modules and moment matrices. Let $\mathbb{N}$ denote the set of nonnegative integers and we set $\mathbb{N}^n_k := \{\alpha \in \mathbb{N}^n \mid |\alpha| = \sum_{i=1}^n \alpha_i \leq k\}$ for $k \in \mathbb{N}$. The symbol $\mathbb{R}[X]$ denotes the ring of multivariate polynomials in variables $(X_1, \ldots, X_n)$ with real coefficients. For $\alpha \in \mathbb{N}^n$, X^α denotes the monomial $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ whose degree is $|\alpha| := \sum_{i=1}^n \alpha_i$. The symbol $\mathbb{R}[X]_k$ denotes the set of real polynomials of degree at most k.

For any $p(X) \in \mathbb{R}[X]_k$, let $\mathbf{p}$ denote its column vector of coefficients in the canonical monomial basis of $\mathbb{R}[X]_k$. A polynomial $p(X) \in \mathbb{R}[X]$ is said to be a *sum of squares of polynomials* if it can be written as $p(X) = \sum_{i=1}^{s} u_i(X)^2$ for some $u_1(X), \ldots, u_s(X) \in \mathbb{R}[X]$. The symbol Σ^2 denotes the set of polynomials that are sums of squares.

Let $G := \{g_1, \ldots, g_m\}$ be a set of polynomials that define the semialgebraic set S. We denote

$$\mathcal{Q}(G) := \left\{ \sum_{j=0}^{m} \sigma_j g_j \;\middle|\; g_0 = 1,\ \sigma_j \in \Sigma^2 \right\}$$

as the *quadratic module* generated by G and its kth *quadratic module*

$$\mathcal{Q}_k(G) := \left\{ \sum_{j=0}^{m} \sigma_j g_j \;\middle|\; g_0 = 1,\ \sigma_j \in \Sigma^2,\ \deg(\sigma_j g_j) \le 2k \right\}.$$

It is clear that $p(x) \ge 0$ for any $p \in \mathcal{Q}(G)$ and $x \in S$.

Definition 2.5. *We say $\mathcal{Q}(G)$ satisfies the Archimedean condition if there exists $\psi \in \mathcal{Q}(G)$ such that the inequality $\psi(x) \ge 0$ defines a compact set in $\mathbb{R}^n$.*

Note that the Archimedean condition implies S is compact but the inverse is not necessarily true. However, for any compact set S we can always force the associated quadratic module to satisfy the condition by adding a redundant constraint $M - \|x\|_2^2$ for sufficiently large M.

Theorem 2.6 (see [16, Putinar's Positivstellensatz]). *Suppose that $\mathcal{Q}(G)$ satisfies the Archimedean condition. If a polynomial $p \in \mathbb{R}[X]$ is positive on S, then $p \in \mathcal{Q}_k(G)$ for some $k \in \mathbb{N}$.*

Definition 2.7 (see [10, Definition 3] PP-BDR of affine polynomials). *One says that PP-BDR of affine polynomials holds for G if there exists $k \in \mathbb{N}$ such that if p is affine and positive on S, then $p \in \mathcal{Q}_k(G)$, except perhaps on a set of vectors $\mathbf{p} \in \mathbb{R}^n$ with Lebesgue measure zero. Call k its order.*

Let $y := (y_\alpha)_{\alpha \in \mathbb{N}^n_{2k}}$ be a *truncated moment sequence* of degree $2k$. Its associated kth *moment matrix* is the matrix $M_k(y)$ indexed by $\mathbb{N}^n_k$, with (α, β)th entry $y_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{N}^n_k$. Given a polynomial $p(X) = \sum_\alpha p_\alpha X^\alpha$, for $k \ge d_p = \lceil \deg(p)/2 \rceil$, the $(k - d_p)$th *localizing moment matrix* $M_{k-d_p}(py)$ is defined as the moment matrix of the *shifted vector* $((py)_\alpha)_{\alpha \in \mathbb{N}^n_{2(k-d_p)}}$ with $(py)_\alpha = \sum_\beta p_\beta y_{\alpha+\beta}$. $\mathscr{M}_{2k}$ denotes the space of all truncated moment sequences with degree at most $2k$. For any $y \in \mathscr{M}_{2k}$, the Riesz functional $\mathscr{L}_y$ on $\mathbb{R}[X]_{2k}$ is defined by

$$\mathscr{L}_y\left(\sum_\alpha q_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n} \right) := \sum_\alpha q_\alpha y_\alpha \quad \forall q(X) \in \mathbb{R}[X]_{2k}.$$

From the definition of the localizing moment matrix $M_{k-d_p}(py)$, it is easy to check that

$$\mathbf{q}^T M_{k-d_p}(py)\mathbf{q} = \mathscr{L}_y(p(X)q(X)^2) \quad \forall q(X) \in \mathbb{R}[X]_{k-d_p}. \tag{2.2}$$

2.3. Lasserre's relaxations and theta bodies. For a compact basic semialgebraic set $S \subseteq \mathbb{R}^n$, Lasserre investigated the semidefinite representations of $\mathbf{co}(S)$ in [10]. Let $s(k) := \binom{n+k}{n}$ and $k_j := \lceil \deg g_j/2 \rceil$ for $j = 1, \ldots, m$. For any $k \in \mathbb{N}$, define

$$\Omega_k(G) := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists y \in \mathbb{R}^{s(2k)} \text{ s.t. } \mathscr{L}_y(1) = 1, \\ \mathscr{L}_y(X_i) = x_i,\ i = 1, \ldots, n,\ M_k(y) \succeq 0, \\ M_{k-k_j}(g_j y) \succeq 0,\ j = 1, \ldots, m, \end{array} \right\}. \tag{2.3}$$

It has been proved in [10, Theorems 2 and 6] that

1. if PP-BDR property holds for G with order k, then $\mathbf{co}(S) = \Omega_k(G)$;
2. if we assume $\mathcal{Q}(G)$ is Archimedean, then for every fixed $\epsilon > 0$, there is $k_\epsilon \in \mathbb{N}$ such that $\mathbf{co}(S) \subseteq \Omega_{k_\epsilon}(G) \subset \mathbf{co}(S) + \epsilon\mathbf{B}_1$, where $\mathbf{B}_1 := \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$.

Another hierarchy of semidefinite relaxations of convex hulls closely related to $\{\Omega_k(G)\}$ is called *theta bodies* defined on real varieties [4, 5], which can be extended to semialgebraic sets. Let $\mathbb{R}[X]_1$ denote the subset of all linear polynomials in $\mathbb{R}[X]$; we have

$$\mathbf{cl}\,(\mathbf{co}\,(S)) = \bigcap_{p \in \mathbb{R}[X]_1, p|_S \geq 0} \{x \in \mathbb{R}^n \mid p(x) \geq 0\}. \tag{2.4}$$

Define the kth theta body of G as

$$\mathrm{TH}_k(G) := \{x \in \mathbb{R}^n \mid p(x) \geq 0 \quad \forall p \in \mathcal{Q}_k(G) \cap \mathbb{R}[X]_1\}. \tag{2.5}$$

Clearly, we have

$$\mathrm{TH}_1(G) \supseteq \mathrm{TH}_2(G) \supseteq \cdots \supseteq \mathrm{TH}_k(G) \supseteq \mathrm{TH}_{k+1}(G) \supseteq \cdots \supseteq \mathbf{cl}\,(\mathbf{co}\,(S))\,.$$

When $\mathcal{Q}(G)$ is Archimedean, by Putinar's Positivstellensatz, (2.4), and (2.5), we have immediately

$$\mathbf{cl}\,(\mathbf{co}\,(S)) = \bigcap_{k=1}^{\infty} \mathrm{TH}_k(G). \tag{2.6}$$

Theorem 2.8. *If $\mathcal{Q}_k(G)$ is closed, then* $\mathrm{TH}_k(G) = \mathbf{cl}(\Omega_k(G))$.

Proof. The proof is similar to the one given in [4, Theorem 2.8] for S being a real variety. ∎

Remark 2.9. When S has a nonempty interior then $\mathcal{Q}_k(G)$ is closed for every $k \in \mathbb{N}$ [15, 20]. Hence, $\mathrm{TH}_k(G) = \mathbf{cl}(\Omega_k(G))$ for any $k \in \mathbb{N}$.

The assumption of the Archimedean condition plays an essential role in the hierarchies of Lasserre's relaxations (2.3) and theta bodies (2.5). However, for a noncompact semialgebraic set S, the Archimedean condition is violated. We cannot guarantee that the sequence defined in (2.3) or (2.5) converges to $\mathbf{cl}(\mathbf{co}(S))$. This can be observed from the following example.

Example 2.10. Consider the basic semialgebraic set

$$S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_1^2 - x_2^3 \geq 0\}. \tag{2.7}$$

As shown in Figure 1, S defines the gray shadow below the right half of the cusp. Let $G := \{X_1,\ X_1^2 - X_2^3\}$. It is clear that $\mathbf{cl}(\mathbf{co}(S)) = S$. We show a tangent line $l(X_1, X_2) := 1 + 2X_1 - 3X_2 = 0$ of S at $(1, 1)$ in Figure 1. For every $c_1X_1 + c_2X_2 + c_0 \in \mathcal{Q}_k(G) \cap \mathbb{R}[X]_1, c_0, c_1, c_2 \in \mathbb{R}$, we have

$$c_1X_1 + c_2X_2 + c_0 = \sigma_0(X_1, X_2) + \sigma_1(X_1, X_2)X_1 + \sigma_2(X_1, X_2)(X_1^2 - X_2^3),$$

where $\sigma_0, \sigma_1, \sigma_2 \in \Sigma^2$. Substituting $X_1 = 0$, we have

$$c_2X_2 + c_0 = \sigma_0(0, X_2) - X_2^3\sigma_2(0, X_2).$$

Since the highest degree terms in $\sigma_0(0, X_2)$ and $-X_2^3\sigma_2(0, X_2)$ cannot cancel each other out, we have $\sigma_2(0, X_2) = 0$ and $\sigma_0(0, X_2)$ is a constant. This implies $c_2 = 0$ and

$$\mathrm{TH}_k(G) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0\}$$

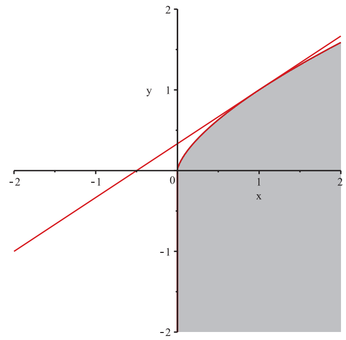

FIG. 1. *The semialgebraic set S in Example* 2.10 *and the tangent line l.*

for all $k \in \mathbb{N}$. Hence, theta bodies $\mathrm{TH}_k(G)$ defined in (2.5) cannot converge to $\mathbf{cl}(\mathbf{co}(S))$. Moreover, since S has a nonempty interior, by Remark 2.9, $\mathcal{Q}_k(G)$ is closed for every $k \in \mathbb{N}$. By Theorem 2.8, we have $\mathrm{TH}_k(G) = \mathbf{cl}(\Omega_k(G))$ for $k \in \mathbb{N}$. Hence, the hierarchy of Lasserre's relaxations $\Omega_k(G)$ defined in (2.3) cannot converge to $\mathbf{cl}(\mathbf{co}(S))$, either. The main reason that $\mathrm{TH}_k(G)$ does not converge to $\mathbf{co}(S)$ is that none of the tangent lines of S, except $X_1 = 0$, can be approximated by polynomials in $\mathcal{Q}_k(G) \cap \mathbb{R}[X]_1,\ k \in \mathbb{N}$. In particular, $l_\epsilon := l + \epsilon \notin \mathcal{Q}_k(G)$ for any $\epsilon > 0,\ k \in \mathbb{N}$.

Remark 2.11. Because the semialgebraic sets and projected spectrahedra in all examples in this paper are unbounded, they are shown in figures after being truncated properly.

In the next section, we show how to overcome the difficulty in semidefinite representations of convex hulls of noncompact semialgebraic sets.

3. Semidefinite representations of noncompact convex sets.

In this section, we study how to modify theta bodies and Lasserre's relaxations for computing $\mathbf{cl}(\mathbf{co}(S))$ when S is a noncompact semialgebraic set. The main idea is to lift S to a cone of $\widetilde{S}^{\mathrm{o}}$ in $\mathbb{R}^{n+1}$ via homogenization, a technique which has been used in [3, 11] for dealing with noncompact semialgebraic sets, and show that the modified theta bodies and Lasserre's relaxations converge to $\mathbf{cl}(\mathbf{co}(S))$ when S is closed at ∞ and $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is closed and pointed. Some examples are given to illustrate that the conditions of pointedness and closedness are essential to ensure the convergence.

3.1. Nested and closed convex approximations of $\mathbf{cl}(\mathbf{co}(S))$.

Consider a polynomial $f(X) \in \mathbb{R}[X]$ and its homogenization $\tilde{f}(\widetilde{X}) \in \mathbb{R}[\widetilde{X}]$, where $\widetilde{X} = (X_0, X_1, \ldots, X_n)$ and $\tilde{f}(\widetilde{X}) = X_0^d f(X/X_0)$, $d = \deg(f)$. For a given semialgebraic set

$$S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}, \tag{3.1}$$

define

$$\begin{aligned} \widetilde{S}^{\mathrm{o}} &:= \{\tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{g}_1(\tilde{x}) \geq 0,\ \ldots,\ \tilde{g}_m(\tilde{x}) \geq 0,\ x_0 > 0\}, \\ \widetilde{S}^{\mathrm{c}} &:= \{\tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{g}_1(\tilde{x}) \geq 0,\ \ldots,\ \tilde{g}_m(\tilde{x}) \geq 0,\ x_0 \geq 0\}, \\ \widetilde{S} &:= \{\tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{g}_1(\tilde{x}) \geq 0,\ \ldots,\ \tilde{g}_m(\tilde{x}) \geq 0,\ x_0 \geq 0,\ \|\tilde{x}\|_2^2 = 1\}. \end{aligned} \tag{3.2}$$

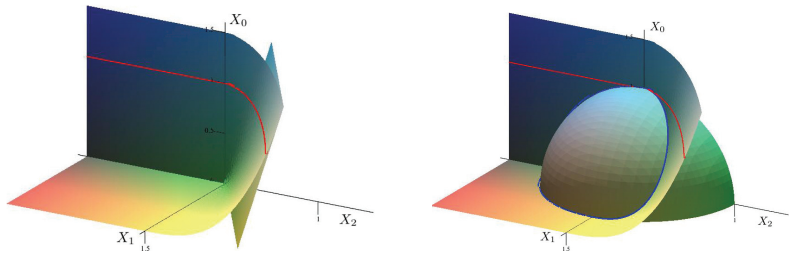

FIG. 2. *Left: the cone $\widetilde{S}^{c}$ and hyperplane $\tilde{l}$ generated by S and l, respectively; Right: the compact patch $\widetilde{S}$ (cut by the blue line) on the unit sphere, in Example* 2.10.

Note that $\widetilde{S}^{o}$ depends only on S, while $\widetilde{S}^{c}$ and $\widetilde{S}$ depend not only on S but also on the choice of the inequalities $g_1(x) \geq 0, \ldots, g_m(x) \geq 0$.

PROPOSITION 3.1 (see [6, Proposition 2.1]). *$f(x) \geq 0$ on S if and only if $\tilde{f}(\tilde{x}) \geq 0$ on $\mathbf{cl}(\widetilde{S}^{o})$.*

COROLLARY 3.2. *For any $f \in \mathbb{R}[X]_1$, $f(x) \geq 0$ on $\mathbf{cl}(\mathbf{co}(S))$ if and only if $\tilde{f}(\tilde{x}) \geq 0$ on $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{o}))$.*

Proof. Since $f(X)$ is linear, $f(x) \geq 0$ on $\mathbf{cl}(\mathbf{co}(S))$ if and only if $f(x) \geq 0$ on S, and $\tilde{f}(\tilde{x}) \geq 0$ on $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{o}))$ if and only if $\tilde{f}(\tilde{x}) \geq 0$ on $\mathbf{cl}(\widetilde{S}^{o})$. The conclusion follows from Proposition 3.1. ▯

Example 2.10 *continued*. By the definitions (2.7) and (3.2), we have

$$\begin{aligned}
\widetilde{S}^{o} &= \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1 \geq 0,\ x_0 x_1^2 - x_2^3 \geq 0,\ x_0 > 0\}, \\
\widetilde{S}^{c} &= \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1 \geq 0,\ x_0 x_1^2 - x_2^3 \geq 0,\ x_0 \geq 0\}, \\
\widetilde{S} &= \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1 \geq 0,\ x_0 x_1^2 - x_2^3 \geq 0,\ x_0 \geq 0,\ \|\tilde{x}\|_2^2 = 1\}.
\end{aligned}$$

In Figure 2 (left), we show the cone $\widetilde{S}^{c}$ in $\mathbb{R}^3$ and the hyperplane $\tilde{l}(X_0, X_1, X_2) := X_0 + 2X_1 - 3X_2 = 0$ generated by l. It can be seen that $\tilde{l}$ is still tangent to $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{o}))$ which illustrates Corollary 3.2. The compact patch $\widetilde{S}$ on the unit sphere is shown in Figure 2 (right).

DEFINITION 3.3 (see [13]). *S is closed at ∞ if $\mathbf{cl}(\widetilde{S}^{o}) = \widetilde{S}^{c}$.*

Remark 3.4. It is shown in [6] that the closedness at ∞ is a generic property. Namely, if we consider the space of coefficients of generators g_i's of all possible sets S of form (3.1) in the canonical monomial basis of $\mathbb{R}[X]_d$, where n, m, and d of the highest degree of g_i's are fixed, then the coefficients of g_i's of those sets S which are not closed at ∞ are in a Zariski closed set of the space.

Since $\widetilde{S}^{c}$ and $\widetilde{S}$ depend also on the inequalities $g_1(x) \geq 0, \ldots, g_m(x) \geq 0$, the closedness at ∞ of S depends not only on S but also on its generators. See the following example.

Example 3.5. Consider the set $S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 - x_1^2 \geq 0\}$. We have

$$\begin{aligned}
\widetilde{S}^{o} &= \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0 x_2 - x_1^2 \geq 0,\ x_0 > 0\}, \\
\widetilde{S}^{c} &= \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0 x_2 - x_1^2 \geq 0,\ x_0 \geq 0\}, \\
\widetilde{S} &= \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0 x_2 - x_1^2 \geq 0,\ x_0 \geq 0,\ \|\tilde{x}\|_2 = 1\}.
\end{aligned}$$

Since $\widetilde{S}^{c} \backslash \mathbf{cl}(\widetilde{S}^{o}) = \{(0, 0, x_2) \in \mathbb{R}^3 \mid x_2 < 0\} \neq \emptyset$, S is not closed at ∞. However, let $S' := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 - x_1^2 \geq 0, 1 + x_2 \geq 0\}$; it is easy to check that $S = S'$, but S' is closed at ∞. It will be shown in subsection 4.1 that for a special class of semialgebraic sets which are not closed at ∞, we can always add a redundant linear generator to S such that it is closed at ∞.

Assumption 3.6. (i) S is closed at ∞; (ii) The convex cone $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is closed and pointed.

Let $\mathbb{P}[\widetilde{X}]_1$ be the set of homogeneous polynomials of degree one in $\mathbb{R}[\widetilde{X}]$ plus the zero polynomial and

$$\widetilde{G} := \{\tilde{g}_1, \ldots, \tilde{g}_m,\ X_0,\ \|\widetilde{X}\|_2^2 - 1,\ 1 - \|\widetilde{X}\|_2^2\}. \tag{3.3}$$

DEFINITION 3.7. *The modified kth theta body of noncompact S* (3.1) *is*

$$\widetilde{\mathrm{TH}}_k(\widetilde{G}) := \{x \in \mathbb{R}^n \mid \tilde{l}(1,x) \geq 0 \quad \forall\, \tilde{l} \in \mathcal{Q}_k(\widetilde{G}) \cap \mathbb{P}[\widetilde{X}]_1\}. \tag{3.4}$$

THEOREM 3.8. *Let $S \subseteq \mathbb{R}^n$ be the semialgebraic set defined as in* (3.1). *Suppose that Assumption* 3.6 *is satisfied, then $\mathbf{cl}(\mathbf{co}(S)) \subseteq \widetilde{\mathrm{TH}}_k(\widetilde{G})$ for every $k \in \mathbb{N}$ and*

$$\mathbf{cl}\,(\mathbf{co}\,(S)) = \bigcap_{k=1}^{\infty} \widetilde{\mathrm{TH}}_k(\widetilde{G}). \tag{3.5}$$

Proof. We first show $\mathbf{cl}(\mathbf{co}(S)) \subseteq \widetilde{\mathrm{TH}}_k(\widetilde{G})$ for every $k \in \mathbb{N}$. Fix an $\tilde{l} \in \mathcal{Q}_k(\widetilde{G}) \cap \mathbb{P}[\widetilde{X}]_1$, then we have $\tilde{l}(\tilde{x}) \geq 0$ on $\widetilde{S}$. Since $\tilde{l}$ is homogeneous, we have $\tilde{l}(\tilde{x}) \geq 0$ on $\widetilde{S}^{\mathrm{c}}$. Since $\widetilde{S}^{\mathrm{o}} \subseteq \widetilde{S}^{\mathrm{c}}$, we have $\tilde{l}(\tilde{x}) \geq 0$ on $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$. By Corollary 3.2, we have $\tilde{l}(1,x) = l(x) \geq 0$ on $\mathbf{cl}(\mathbf{co}(S))$, which implies that $\mathbf{cl}(\mathbf{co}(S))$ is included in $\widetilde{\mathrm{TH}}_k(\widetilde{G})$ for every $k \in \mathbb{N}$. Thus, the modified theta bodies defined in (3.4) form a hierarchy of closed convex approximations of $\mathbf{co}(S)$ as follows:

$$\widetilde{\mathrm{TH}}_1(\widetilde{G}) \supseteq \widetilde{\mathrm{TH}}_2(\widetilde{G}) \supseteq \cdots \supseteq \widetilde{\mathrm{TH}}_k(\widetilde{G}) \supseteq \widetilde{\mathrm{TH}}_{k+1}(\widetilde{G}) \supseteq \cdots \supseteq \mathbf{cl}\,(\mathbf{co}\,(S))\,. \tag{3.6}$$

We now verify that this hierarchy converges to $\mathbf{cl}(\mathbf{co}(S))$ asymptotically. Assume $u \notin \mathbf{cl}(\mathbf{co}(S))$, we show that $u \notin \widetilde{\mathrm{TH}}_k(\widetilde{G})$ for some $k \in \mathbb{N}$. Since $\mathbf{cl}(\mathbf{co}(S))$ is closed and convex, by the hyperplane separation theorem, there exists a vector $(f_0, \mathbf{f}) \in \mathbb{R}^{n+1}$ that satisfies

$$\langle \mathbf{f}, u\rangle < f_0 \quad \text{and} \quad \langle \mathbf{f}, x\rangle > f_0 \quad \text{on} \quad \mathbf{cl}\,(\mathbf{co}\,(S))\,.$$

Let $\tilde{f}(\widetilde{X}) := \sum_{i=1}^n f_i X_i - f_0 X_0 \in \mathbb{R}[\widetilde{X}]$, then

$$\tilde{f}(1,u) < 0 \quad \text{and} \quad \tilde{f}(1,x) = f(x) > 0 \quad \text{on} \quad \mathbf{cl}\,(\mathbf{co}\,(S))\,.$$

By Corollary 3.2, we have

$$\tilde{f}(\tilde{x}) \geq 0 \quad \forall\, x \in \mathbf{co}\left(\mathbf{cl}\left(\widetilde{S}^{\mathrm{o}}\right)\right). \tag{3.7}$$

Since $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is closed and pointed, by Theorem 2.3, there exists a polynomial $\tilde{h}(\widetilde{X}) = \sum_{i=0}^n h_i X_i \in \mathbb{P}[\widetilde{X}]_1$ such that $\tilde{h}(\tilde{x}) > 0$ on $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))\backslash\{0\}$. We choose a small $\epsilon > 0$ such that $(\tilde{f} + \epsilon\tilde{h})(1,u) < 0$ and rename $\tilde{f} + \epsilon\tilde{h}$ as $\tilde{f}$, then

$$\tilde{f}(1,u) < 0 \quad \text{and} \quad \tilde{f}(\tilde{x}) > 0 \quad \text{on} \quad \mathbf{cl}(\widetilde{S}^{\mathrm{o}})\backslash\{0\}. \tag{3.8}$$

We have assumed that S is closed at ∞, $\mathbf{cl}(\widetilde{S}^{\mathrm{o}}) \cap \{\tilde{x} \in \mathbb{R}^{n+1} \mid \|\tilde{x}\|_2 = 1\} = \widetilde{S}$, hence

$$\tilde{f}(1,u) < 0 \quad \text{and} \quad \tilde{f}(\tilde{x}) > 0 \quad \text{on} \quad \widetilde{S}. \tag{3.9}$$

Since $\mathcal{Q}(\widetilde{G})$ is Archimedean, by Putinar's Positivstellensatz, there exists a $k' \in \mathbb{N}$ such that $\tilde{f} \in \mathcal{Q}_{k'}(\widetilde{G}) \cap \mathbb{P}[\widetilde{X}]_1$. Since $\tilde{f}(1,u) < 0$, we have $u \notin \widetilde{\mathrm{TH}}_{k'}(\widetilde{G})$. This implies

$$\bigcap_{k=1}^{\infty} \widetilde{\mathrm{TH}}_k(\widetilde{G}) \subseteq \mathbf{cl}\left(\mathbf{co}\left(S\right)\right). \tag{3.10}$$

Finally, by (3.6) and (3.10), we can conclude $\mathbf{cl}(\mathbf{co}(S)) = \bigcap_{k=1}^{\infty} \widetilde{\mathrm{TH}}_k(\widetilde{G})$. ∎

Example 2.10 *continued.* For every $(0, u_1, u_2) \in \widetilde{S}^{\mathrm{c}} \backslash \widetilde{S}^{\mathrm{o}}$, let

$$u^{(\epsilon)} := \left(\epsilon,\ u_1,\ \sqrt[3]{\epsilon u_1^2 + u_2^3}\right).$$

Then $\{u^{(\epsilon)}\}_{\epsilon>0} \subseteq \widetilde{S}^{\mathrm{o}}$ and $\lim_{\epsilon\to 0} u^{(\epsilon)} = (0, u_1, u_2)$. Hence, we have $\widetilde{S}^{\mathrm{c}} \backslash \widetilde{S}^{\mathrm{o}} \subseteq \mathbf{cl}(\widetilde{S}^{\mathrm{o}})$ and S is closed at ∞. Moreover, it can be verified that

$$\tilde{g}(X_0, X_1, X_2) := 2X_0 + 2X_1 - 3X_2$$

is positive on $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))\backslash\{0\}$ which implies $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is pointed by Theorem 2.3. Hence, Assumption 3.6 holds for S. Let $\epsilon > 0$ tend to 0, $\tilde{l}$ can be approximated by the $\tilde{l} + \epsilon\tilde{g}$ which are positive on $\widetilde{S}$. Moreover, since $\mathcal{Q}(\widetilde{G})$ is Archimedean, by Putinar's Positivstellensatz, $\tilde{l} + \epsilon\tilde{g}$ belongs to the quadratic module corresponding to

$$\widetilde{G} := \{X_0, X_1,\ X_0X_1^2 - X_2^3, X_0^2 + X_1^2 + X_2^2 - 1, 1 - X_0^2 - X_1^2 - X_2^2\}$$

for every $\epsilon > 0$. Define

$$\widetilde{\mathrm{TH}}_k(\widetilde{G}) := \{(x_1, x_2) \in \mathbb{R}^2 \mid \tilde{l}(1, x_1, x_2) \geq 0 \quad \forall\, \tilde{l} \in \mathcal{Q}_k(\widetilde{G}) \cap \mathbb{P}[X_0, X_1, X_2]_1\},$$

we have $\mathbf{cl}(\mathbf{co}(S)) = \bigcap_{k=1}^{\infty} \widetilde{\mathrm{TH}}_k(\widetilde{G})$.

COROLLARY 3.9. *Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set defined as in* (3.1). *Suppose that Assumption* 3.6 *is satisfied and PP-BDR property holds for $\widetilde{G}$ with order k', then* $\mathbf{cl}(\mathbf{co}(S)) = \widetilde{\mathrm{TH}}_{k'}(\widetilde{G})$.

Proof. By (3.6), we need to verify that $\mathbf{cl}(\mathbf{co}(S)) \supseteq \widetilde{\mathrm{TH}}_{k'}(\widetilde{G})$. Assume that there exists a vector $u \in \widetilde{\mathrm{TH}}_{k'}(\widetilde{G})$ but $u \notin \mathbf{cl}(\mathbf{co}(S))$. According to (3.9), there exists a linear polynomial $\tilde{f} \in \mathbb{R}[\widetilde{X}]$ with $\tilde{f}(0) = 0$ such that $\tilde{f}(1,u) < 0$ and $\tilde{f}(\tilde{x}) > 0$ on $\widetilde{S}$. Since $\widetilde{G}$ satisfies PP-BDR property with order k', we have $\tilde{f} \in \mathcal{Q}_{k'}(\widetilde{G}) \cap \mathbb{P}[\widetilde{X}]_1$. Due to the fact that $\tilde{f}(1,u) < 0$, we derive that $u \notin \widetilde{\mathrm{TH}}_{k'}(\widetilde{G})$. This yields a contradiction. Thus, we have $\mathbf{cl}(\mathbf{co}(S)) = \widetilde{\mathrm{TH}}_{k'}(\widetilde{G})$. ∎

Remark 3.10. Define the *perspective projection* φ as

$$\begin{aligned} \varphi:\ \{(x_0, x) \in \mathbb{R}^{n+1} \mid x_0 > 0\} \to &\quad \mathbb{R}^n, \\ (x_0, x) \to &\quad \frac{x}{x_0}. \end{aligned} \tag{3.11}$$

Let

$$H := \{(0, x) \mid x \in \mathbb{R}^n\}. \tag{3.12}$$

As pointed out by a referee, to approximate $\mathbf{cl}(\mathbf{co}(S))$ of a noncompact S, we can also consider the perspective projections of the classic theta bodies of the compact set $\widetilde{S}$,

$$\begin{aligned}\varphi\left(\mathrm{TH}_k(\widetilde{G})\backslash H\right) &= \{x \in \mathbb{R}^n \mid \exists t > 0 \text{ s.t. } (t, tx) \in \mathrm{TH}_k(\widetilde{G})\} \\ &= \{x \in \mathbb{R}^n \mid \exists t > 0 \text{ s.t. } \tilde{l}(t, tx) \geq 0 \;\forall\, \tilde{l} \in \mathcal{Q}_k(\widetilde{G}) \cap \mathbb{R}[\widetilde{X}]_1\}.\end{aligned} \tag{3.13}$$

It is obvious that

$$\varphi\left(\mathrm{TH}_k(\widetilde{G})\backslash H\right) \subseteq \widetilde{\mathrm{TH}}_k(\widetilde{G}) \quad \text{for each } k \in \mathbb{N}. \tag{3.14}$$

We show in Corollary 3.21 that these two sets are actually the same when $\mathcal{Q}_k(\widetilde{G})$ is closed.

We would like to point out that two conditions in Assumption 3.6 cannot be dropped in Theorem 3.8.

Example 3.5 *continued*. We have shown that S is not closed at ∞. Clearly, we have $\mathbf{cl}(\mathbf{co}(S)) = S$. It is easy to check that $\mathbf{cl}(\widetilde{S}^{\mathrm{o}})$ is convex. Define $\tilde{f}(\widetilde{X}) := X_0 + X_2$, we have $\tilde{f}(\tilde{x}) > 0$ on $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))\backslash\{0\}$ which implies $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is closed and pointed. Clearly, we have $(0, 0, -1), (1, 0, 0) \in \widetilde{S}$, and $(1 - \lambda, 0, -\lambda) \in \mathbf{co}(\widetilde{S}) \subseteq \mathrm{TH}_k(\widetilde{G})$ for every $\lambda \in [0, 1]$. By (3.14), we have

$$\left(0, -\frac{\lambda}{1-\lambda}\right) \in \varphi\left(\mathrm{TH}_k(\widetilde{G})\backslash H\right) \subseteq \widetilde{\mathrm{TH}}_k(\widetilde{G}) \quad \text{for every } \lambda \in [0, 1).$$

It implies that $\{(0, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0\} \subseteq \widetilde{\mathrm{TH}}_k(\widetilde{G})$ for each $k \in \mathbb{N}$. Since $\mathbf{cl}(\mathbf{co}(S)) = S$, we have $\bigcap_{k=1}^{\infty} \widetilde{\mathrm{TH}}_k(\widetilde{G}) \neq \mathbf{cl}(\mathbf{co}(S))$.

Example 3.11. Consider the set $S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2^3 - x_1^2 \geq 0\}$. We have $\mathbf{cl}(\mathbf{co}(S)) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\}$ and

$$\begin{aligned}\widetilde{S}^{\mathrm{o}} &= \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_2^3 - x_0 x_1^2 \geq 0,\ x_0 > 0\}, \\ \widetilde{S}^{\mathrm{c}} &= \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_2^3 - x_0 x_1^2 \geq 0,\ x_0 \geq 0\}.\end{aligned}$$

It can be verified that $\widetilde{S}^{\mathrm{c}}\backslash\widetilde{S}^{\mathrm{o}} = \{(0, x_1, x_2) \in \mathbb{R}^3 \mid x_2 \geq 0\}$. Using arguments similar to Example 2.10 continued, we can show that S is closed at ∞. However, since $\lim_{\epsilon\to 0}(\epsilon, \pm 1, \sqrt[3]{\epsilon}) = (0, \pm 1, 0)$ and $(0, \pm 1, 0) \in \mathbf{cl}(\widetilde{S}^{\mathrm{o}})$, $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is not pointed. Let

$$\widetilde{G} = \{X_0,\ X_2^3 - X_0 X_2^2, X_0^2 + X_1^2 + X_2^2 - 1,\ 1 - X_0^2 - X_1^2 - X_2^2\}.$$

We show that $\widetilde{\mathrm{TH}}_k(\widetilde{G}) = \mathbb{R}^2$ for every $k \in \mathbb{N}$. Assume $c_0 X_0 + c_1 X_1 + c_2 X_2 \in \mathcal{Q}_k(\widetilde{G})$, then

$$c_0 X_0 + c_1 X_1 + c_2 X_2 = \tilde{\sigma}_0 + \tilde{\sigma}_1 X_0 + \tilde{\sigma}_2(X_2^3 - X_0 X_1^2) + \tilde{h}(X_0^2 + X_1^2 + X_2^2 - 1), \tag{3.15}$$

where $\tilde{\sigma}_i \in \Sigma^2, i = 0, 1, 2$, and $\tilde{h} \in \mathbb{R}[\widetilde{X}]$. Substituting $(X_0, X_1, X_2) = (0, \pm 1, 0)$ in (3.15), we derive $c_1 = 0$. Substituting $(X_0, X_1, X_2) = (0, \pm 1, X_2)$ in (3.15), we have

$$c_2 X_2 = \tilde{\sigma}_0(0, \pm 1, X_2) + \tilde{\sigma}_2(0, \pm 1, X_2) X_2^3 + \tilde{h}(0, \pm 1, X_2) X_2^2. \tag{3.16}$$

It is clear that $\tilde{\sigma}_0(0, \pm 1, X_2)$ cannot have a nonzero constant term. Hence, the right side of (3.16) is divisible by X_2^2, which is only possible when $c_2 = 0$. By the definition (3.4), we derive $\widetilde{\mathrm{TH}}_k(\widetilde{G}) = \mathbb{R}^2$ for every $k \in \mathbb{N}$. This shows that the assumption of pointedness of $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ cannot be dropped in Theorem 3.8.

3.2. Spectrahedral approximations of $\mathbf{cl}(\mathbf{co}(S))$. In order to fulfill computations of $\mathbf{cl}(\mathbf{co}(S))$ via semidefinite programming, we study an alternative description of $\widetilde{\mathrm{TH}}_k(\widetilde{G})$ in a dual view and establish the connection between them. In the following, we consider moment sequences y of real numbers indexed by $(n+1)$-tuple $\alpha := (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{N}^{n+1}$. Each y defines a Riesz functional $\mathscr{L}_y$ on $\mathbb{R}[\widetilde{X}]$. Recall that $\tilde{s}(k) = \binom{n+k+1}{n+1}$ and $k_j = \lceil \deg g_j/2 \rceil$.

DEFINITION 3.12. *The modified kth Lasserre's relaxation of noncompact S (3.1) is defined as*

$$\widetilde{\Omega}_k(\widetilde{G}) := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists y \in \mathbb{R}^{\tilde{s}(2k)} \text{ s.t. } \mathscr{L}_y(X_0) = 1, \\ \mathscr{L}_y(X_i) = x_i,\ i = 1, \dots, n, \\ M_{k-1}(X_0 y) \succeq 0,\ M_{k-1}((\|\widetilde{X}\|_2^2 - 1)y) = 0, \\ M_k(y) \succeq 0,\ M_{k-k_j}(\tilde{g}_j y) \succeq 0,\ j = 1, \dots, m \end{array} \right\}. \tag{3.17}$$

PROPOSITION 3.13. *For each $k \in \mathbb{N}$, the modified Lasserre's relaxation $\widetilde{\Omega}_k(\widetilde{G})$ (3.17) is identical to the perspective projection of the classic Lasserre's relaxation $\Omega_k(\widetilde{G})$ (2.3) of $\widetilde{S}$, i.e., $\widetilde{\Omega}_k(\widetilde{G}) = \varphi\left(\Omega_k(\widetilde{G}) \backslash H\right)$.*

Proof. By the definition (2.3), we have

$$\Omega_k(\widetilde{G}) = \left\{ \tilde{x} \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} \exists y \in \mathbb{R}^{\tilde{s}(2k)} \text{ s.t. } \mathscr{L}_y(1) = 1, \\ \mathscr{L}_y(X_i) = x_i,\ i = 0, \dots, n, \\ M_{k-1}(X_0 y) \succeq 0,\ M_{k-1}((\|\widetilde{X}\|_2^2 - 1)y) = 0, \\ M_k(y) \succeq 0,\ M_{k-k_j}(\tilde{g}_j y) \succeq 0,\ j = 1, \dots, m \end{array} \right\}. \tag{3.18}$$

For any $w \in \widetilde{\Omega}_k(\widetilde{G})$, there exists $y = (y_0, 1, w, \dots) \in \mathbb{R}^{\tilde{s}(2k)}$ satisfying the conditions in (3.17). Since $M_k(y) \succeq 0$ and $\mathscr{L}_y(X_0) = 1$, we have $y_0 > 0$. Then, y/y_0 satisfies the conditions in (3.18), i.e., $w = (w/y_0)/(1/y_0) \in \varphi(\Omega_k(\widetilde{G}) \backslash H)$. For any $w \in \varphi(\Omega_k(\widetilde{G}) \backslash H)$, there exist $t > 0$ and $y = (1, t, tw, \dots) \in \mathbb{R}^{\tilde{s}(2k)}$ such that y satisfies the conditions in (3.18). Then, y/t satisfies the conditions in (3.17), which means $w \in \widetilde{\Omega}_k(\widetilde{G})$. ∎

THEOREM 3.14. *We have $\mathbf{cl}(\mathbf{co}(S)) \subseteq \mathbf{cl}(\widetilde{\Omega}_k(\widetilde{G})) \subseteq \widetilde{\mathrm{TH}}_k(\widetilde{G})$ for every $k \in \mathbb{N}$.*

Proof. It is clear that for every $x \in S$, there exists $t > 0$ such that $(t, tx) \in \widetilde{S}$. Then, $(t, tx) \in \Omega_k(\widetilde{G})$ for each $k \in \mathbb{N}$. By Proposition 3.13, we have $x \in \varphi(\Omega(\widetilde{G}) \backslash H) = \widetilde{\Omega}_k(\widetilde{G})$. Since $\mathbf{cl}(\widetilde{\Omega}_k(\widetilde{G}))$ is a closed convex set, we obtain $\mathbf{cl}(\mathbf{co}(S)) \subseteq \mathbf{cl}(\widetilde{\Omega}_k(\widetilde{G}))$.

Since $\Omega_k(\widetilde{G}) \subset \mathrm{TH}_k(\widetilde{G})$, by (3.14) and Proposition 3.13, we have

$$\widetilde{\Omega}_k(\widetilde{G}) = \varphi\left(\Omega_k(\widetilde{G}) \backslash H\right) \subseteq \varphi\left(\mathrm{TH}_k(\widetilde{G}) \backslash H\right) \subseteq \widetilde{\mathrm{TH}}_k(\widetilde{G}).$$

The conclusion follows due to the fact that $\widetilde{\mathrm{TH}}_k(\widetilde{G})$ is closed. ∎

Example 2.10 *continued.* Using the software package Bermeja [18], we draw the third spectrahedron $\widetilde{\Omega}_3(\widetilde{G})$. As shown in Figure 3, our modified Lasserre's relaxation $\widetilde{\Omega}_3(\widetilde{G})$ is a very tight approximation of $\mathbf{cl}(\mathbf{co}(S))$.

COROLLARY 3.15. *Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set defined as in* (3.1). *Suppose that Assumption* 3.6 *is satisfied, then*

1. $\mathbf{cl}(\mathbf{co}(S)) = \bigcap_{k=1}^{\infty} \mathbf{cl}(\widetilde{\Omega}_k(\widetilde{G}))$;
2. *if PP-BDR property holds for $\widetilde{G}$ with order k', then* $\mathbf{cl}(\mathbf{co}(S)) = \mathbf{cl}(\widetilde{\Omega}_{k'}(\widetilde{G}))$.

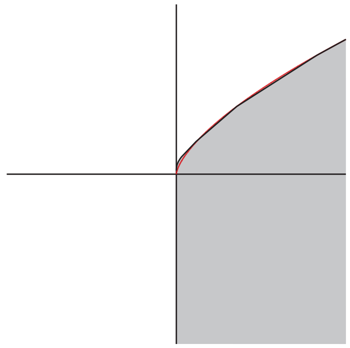

FIG. 3. *The spectrahedral approximation* $\widetilde{\Omega}_3(\widetilde{G})$ *(shown shaded) of* $\mathbf{cl}(\mathbf{co}(S))$ *in Example* 2.10.

Proof. It is straightforward to verify these results via Theorem 3.8, Corollary 3.9 and Theorem 3.14. ∎

Since the PP-BDR property is not generally true, similarly to Theorem 6 in [10, section 2.5], we give an approximate semidefinite representation of $\mathbf{cl}(\mathbf{co}(S))$. For a radius $r \in \mathbb{R}$, let $\mathbf{B}_r := \{x \in \mathbb{R}^n \mid \|x\|_2 \leq r\}$.

THEOREM 3.16. *Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set defined as in* (3.1). *Suppose that Assumption* 3.6 *holds, then for every fixed $\epsilon > 0$ and $r > 0$ with $\mathbf{cl}(\mathbf{co}(S)) \cap \mathbf{B}_r \neq \emptyset$, there exists an integer $k_{r,\epsilon} \in \mathbb{N}$ such that*

$$\mathbf{cl}\,(\mathbf{co}\,(S)) \cap \mathbf{B}_r \subseteq \mathbf{cl}\left(\widetilde{\Omega}_{k_{r,\epsilon}}(\widetilde{G})\right) \cap \mathbf{B}_r \subseteq (\mathbf{cl}\,(\mathbf{co}\,(S)) + \epsilon \mathbf{B}_1) \cap \mathbf{B}_r$$

holds.

Proof. By Proposition 3.13, the conclusion can be proved by similar arguments used in the proof of Theorem 6 in [10, section 2.5]. ∎

Example 3.17. Consider the following semialgebraic set

$$S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^3 - x_2^2 - x_1 + 1 = 0,\ x_2 \geq 0\},$$

which is the red curve shown in Figure 4. We have

$$\widetilde{S}^{\mathrm{o}} = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1^3 - x_0 x_2^2 - x_0^2 x_1 + x_0^3 = 0,\ x_2 \geq 0,\ x_0 > 0\},$$
$$\widetilde{S}^{\mathrm{c}} = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1^3 - x_0 x_2^2 - x_0^2 x_1 + x_0^3 = 0,\ x_2 \geq 0,\ x_0 \geq 0\}.$$

Clearly, $\widetilde{S}^{\mathrm{c}} \backslash \widetilde{S}^{\mathrm{o}} = \{(0, 0, x_2) \in \mathbb{R}^3 \mid x_2 \geq 0\}$. It can be verified that $\widetilde{S}^{\mathrm{c}} = \mathbf{cl}(\widetilde{S}^{\mathrm{o}})$, i.e., S is closed at ∞. Since $X_0 + X_2 > 0$ on $\mathbf{cl}(\widetilde{S}^{\mathrm{o}}) \backslash \{0\}$, we have $X_0 + X_2 > 0$ on $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}})) \backslash \{0\}$ and thus, by Theorem 2.3, $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is closed and pointed. Hence, Assumption 3.6 holds for S. The third projected spectrahedron $\widetilde{\Omega}_3(\widetilde{G})$ is depicted (shaded) in Figure 4.

Remark 3.18. As shown in Theorem 3.16 and Corollary 3.15, the projected spectrahedra $\widetilde{\Omega}_k(\widetilde{G})$ for $k \in \mathbb{N}$ are outer approximations of $\mathbf{cl}(\mathbf{co}(S))$ and converge uniformly to $\mathbf{cl}(\mathbf{co}(S))$ restricted to every fixed ball $\mathbf{B}_r$. If we truncate S first by the ball $\mathbf{B}_r$ and then compute the convex hull of the resulting compact set by Lasserre's relaxations (2.3), in general, we cannot get approximations of the truncation $\mathbf{cl}(\mathbf{co}(S)) \cap \mathbf{B}_r$. Taking Example 3.17 for instance, compared with Figure 4, Lasserre's relaxation of $\mathbf{cl}(\mathbf{co}(S \cap \mathbf{B}_r))$ shown in Figure 5 is not an outer approximation of the truncation $\mathbf{cl}(\mathbf{co}(S)) \cap \mathbf{B}_r$.

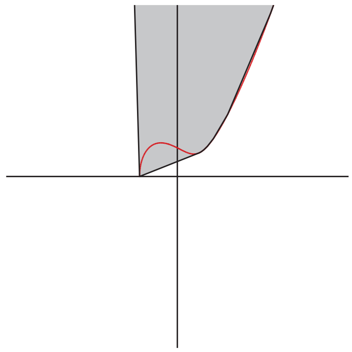

FIG. 4. *The semialgebraic set S (red curve) and projected spectrahedron $\widetilde{\Omega}_3(\widetilde{G})$ (shaded) in Example* 3.17.

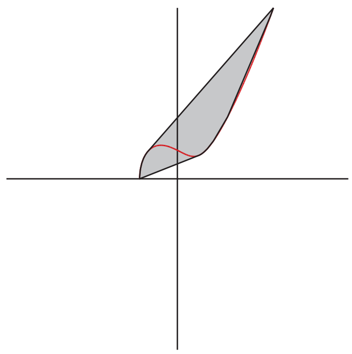

FIG. 5. *Lasserre's relaxation of* $\mathbf{cl}(\mathbf{co}(S)) \cap \mathbf{B}_r$ *in Example* 3.17.

By similar arguments given in [4, 5], we show below that $\widetilde{\mathrm{TH}}_k(\widetilde{G}) = \mathbf{cl}(\widetilde{\Omega}_k(\widetilde{G}))$ for each $k \in \mathbb{N}$ if $\mathcal{Q}_k(\widetilde{G})$ is closed.

Let $M := \{(1, x) \mid x \in \mathbb{R}^n\}$ and $\mathcal{Q}_k^1(\widetilde{G}) = \mathcal{Q}_k(\widetilde{G}) \cap \mathbb{P}[\widetilde{X}]_1$. By the definition of dual cones,

$$\mathcal{Q}_k^1(\widetilde{G})^* \cap M = \{1\} \times \widetilde{\mathrm{TH}}_k(\widetilde{G}). \tag{3.19}$$

Denote $\mathbf{proj}(\mathcal{Q}_k(\widetilde{G})^*)$ as the projection of $\mathcal{Q}_k(\widetilde{G})^*$ onto $(\mathbb{P}[\widetilde{X}]_1)^*$. It is clear that

$$\mathbf{proj}\left(\mathcal{Q}_k(\widetilde{G})^*\right) \cap M = \{1\} \times \widetilde{\Omega}_k(\widetilde{G}). \tag{3.20}$$

If $\mathcal{Q}_k(\widetilde{G})$ is closed, by Proposition 2.1, we have

$$\mathcal{Q}_k^1(\widetilde{G})^* = \mathbf{cl}\left(\mathbf{proj}\left(\mathcal{Q}_k(\widetilde{G})^*\right)\right). \tag{3.21}$$

LEMMA 3.19. *If $\mathcal{Q}_k(\widetilde{G})$ is closed, the hyperplane M intersects* $\mathbf{ri}(\mathbf{proj}(\mathcal{Q}_k(\widetilde{G})^*))$.

Proof. By [17, Theorem 6.3] and (3.21), it is equivalent to prove M intersects $\mathbf{ri}(\mathcal{Q}_k^1(\widetilde{G})^*)$. Fixing a vector $u \in S$, we have $\tilde{l} := (1, u)/\|(1, u)\|_2 \in \widetilde{S}$ and $\tilde{l} \in \mathcal{Q}_k^1(\widetilde{G})^*$. Let

$$D := \{t_0 \in \mathbb{R} \mid \exists\, t \in \mathbb{R}^n \text{ s.t. } (t_0, t) \in \mathcal{Q}_k^1(\widetilde{G})^*\}.$$

Since $X_0 \in \mathcal{Q}_k^1(\widetilde{G})$ and $c \cdot \tilde{l} \in \mathcal{Q}_k^1(\widetilde{G})^*$ for all $c \geq 0$, we get $D = [0, \infty)$ and thus $1 \in \mathbf{ri}(D)$. By [17, Theorem 6.8], we derive that M intersects $\mathbf{ri}(\mathcal{Q}_k^1(\widetilde{G})^*)$. ∎

THEOREM 3.20. *If $\mathcal{Q}_k(\widetilde{G})$ is closed, then $\widetilde{\mathrm{TH}}_k(\widetilde{G}) = \mathbf{cl}(\widetilde{\Omega}_k(\widetilde{G}))$.*

Proof. By (3.19), (3.20), (3.21), [17, Corollary 6.5.1], and Lemma 3.19, we have

$$\begin{aligned}\{1\} \times \mathbf{cl}\left(\widetilde{\Omega}_k(\widetilde{G})\right) &= \mathbf{cl}\left(\mathbf{proj}\left(\mathcal{Q}_k(\widetilde{G})^*\right) \cap M\right)\\ &= \mathbf{cl}\left(\mathbf{proj}\left(\mathcal{Q}_k(\widetilde{G})^*\right)\right) \cap M\\ &= \mathcal{Q}_k^1(\widetilde{G})^* \cap M\\ &= \{1\} \times \widetilde{\mathrm{TH}}_k(\widetilde{G}).\end{aligned}$$

This shows that $\widetilde{\mathrm{TH}}_k(\widetilde{G}) = \mathbf{cl}(\widetilde{\Omega}_k(\widetilde{G}))$. ∎

COROLLARY 3.21. *If $\mathcal{Q}_k(\widetilde{G})$ is closed, then our modified theta body* (3.4) *is identical to the perspective projection of the classic theta body* (2.5) *of $\widetilde{S}$, i.e.,*

$$\widetilde{\mathrm{TH}}_k(\widetilde{G}) = \varphi\left(\mathrm{TH}_k(\widetilde{G}) \backslash H\right).$$

Proof. It is straightforward by (3.14), Proposition 3.13, and Theorem 3.20. ∎

Remark 3.22. Since $\widetilde{S}$ has an empty interior, we cannot check the closedness of $\mathcal{Q}_k(\widetilde{G})$ using the same argument given in Remark 2.9. However, if the original set S has a nonempty interior, then the vanishing ideal of $\widetilde{S}$ is $I = \langle \|\widetilde{X}\|_2^2 - 1\rangle$ and the image of $\mathcal{Q}_k(\widetilde{G})$ in $\mathbb{R}[\widetilde{X}]/I$ is closed in $\mathbb{R}[\widetilde{X}]_{2k}/I$ [12, 15]. Hence, we can show that the equalities in Theorem 3.20 and Corollary 3.21 hold for modified theta bodies and Lasserre's relaxations defined in $\mathbb{R}[\widetilde{X}]/I$. Further research is needed to provide geometric conditions for the quadratic modules $\mathcal{Q}_k(\widetilde{G})$ to be closed.

4. More discussions on Assumption 3.6.

As we have seen, if Assumption 3.6 is satisfied, then we can obtain a hierarchy of nested semidefinite relaxations converging to $\mathbf{cl}(\mathbf{co}(S))$. In this section, we give more discussions on cases where Assumption 3.6 does not hold.

4.1. Closedness at ∞ of S.

We have mentioned that a semialgebraic set is closed at ∞ in general [6]. Unfortunately, as we show below, the closedness condition does not hold on certain kinds of semialgebraic sets.

Let U be a semialgebraic set defined as

$$U := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} h_i(x) = 0,\ i = 1, \ldots, m_1, \\ g_j(x) \geq 0,\ \deg(g_j) \text{ is even},\ j = 1, \ldots, m_2 \end{array} \right\}. \tag{4.1}$$

Denote

$$\begin{aligned}\widetilde{U}^{\mathrm{o}} &= \left\{\tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{h}_i(\tilde{x}) = 0,\ \tilde{g}_j(\tilde{x}) \geq 0,\ x_0 > 0,\quad i = 1, \ldots, m_1,\ j = 1, \ldots, m_2\right\},\\ \widetilde{U}^{\mathrm{c}} &= \left\{\tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{h}_i(\tilde{x}) = 0,\ \tilde{g}_j(\tilde{x}) \geq 0,\ x_0 \geq 0,\quad i = 1, \ldots, m_1,\ j = 1, \ldots, m_2\right\}.\end{aligned}$$

PROPOSITION 4.1. *Suppose U is not compact. If $\mathbf{co}(\mathbf{cl}(\widetilde{U}^{\mathrm{o}}))$ is closed and pointed, then U is not closed at ∞.*

Proof. Since U is not compact, there is a sequence $\{u^{(k)}\}_{k=1}^{\infty} \subseteq U$ satisfying $\lim_{k\to\infty} \|u^{(k)}\|_2 = \infty$. Because $\{(1, u^{(k)})/\|(1, u^{(k)})\|_2\} \subseteq \widetilde{U}^{\mathrm{o}}$ is bounded, there exists a nonzero point $\tilde{u} = (0, u_1, \ldots, u_n) \in \mathbf{cl}(\widetilde{U}^{\mathrm{o}})$.

If $\mathbf{co}(\mathbf{cl}(\widetilde{U}^{\mathrm{o}}))$ is closed and pointed, by Theorem 2.3, we have $-\tilde{u} \notin \mathbf{cl}(\widetilde{U}^{\mathrm{o}})$. However, since $\deg(g_j)$ is even for $j = 1, \ldots, m_2$, it is straightforward to see both $\tilde{u}$ and $-\tilde{u}$ belong to $\widetilde{U}^{\mathrm{c}}$, which implies $\mathbf{cl}(\widetilde{U}^{\mathrm{o}}) \neq \widetilde{U}^{\mathrm{c}}$. Thus, U is not closed at ∞. ∎

Remark 4.2. Consider the semialgebraic set S defined as in (3.1). Let $\hat{g}_i$ be the homogeneous part of the highest degree of g_i for $i = 1, \ldots, m$ and $D_S = \widetilde{S}^{\mathrm{c}} \backslash \mathbf{cl}(\widetilde{S}^{\mathrm{o}})$. If S is not closed at ∞, then

$$\emptyset \neq D_S \subseteq \{(0, x) \in \mathbb{R}^{n+1} \mid \hat{g}_1(x) \geq 0, \ \ldots, \ \hat{g}_m(x) \geq 0\}.$$

Decompose $D_S = D_S^1 \cup D_S^2$, where

$$D_S^1 = \left\{ (0, x) \in \mathbb{R}^{n+1} \;\middle|\; (0, x) \in \widetilde{S}^{\mathrm{c}} \backslash \mathbf{cl}\left(\widetilde{S}^{\mathrm{o}}\right) \text{ but } (0, -x) \in \mathbf{cl}\left(\widetilde{S}^{\mathrm{o}}\right) \right\}$$

and

$$D_S^2 = \left\{ (0, x) \in \mathbb{R}^{n+1} \;\middle|\; (0, x) \in \widetilde{S}^{\mathrm{c}} \backslash \mathbf{cl}\left(\widetilde{S}^{\mathrm{o}}\right) \text{ and } (0, -x) \notin \mathbf{cl}\left(\widetilde{S}^{\mathrm{o}}\right) \right\}.$$

If S is defined by (4.1), then by the proof of Proposition 4.1, $D_S^1 \neq \emptyset$. However, if $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is closed and pointed, there exists a linear function $\tilde{l} \in \mathbb{P}[\widetilde{X}]_1$ such that $\tilde{l}(\tilde{x}) > 0$ on $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}})) \backslash \{0\}$. Adding the inequality $\tilde{l}(\tilde{x}) \geq 0$ to the generators of $\widetilde{S}^{\mathrm{o}}$ or, equivalently, adding $\tilde{l}(1, x) \geq 0$ to the generators of S, it is clear that S and $\widetilde{S}^{\mathrm{o}}$ remain the same but we have $D_S^1 = \emptyset$. As a result, the set S with new generators is more likely to be closed at ∞.

Example 4.3. Consider the quartic bow curve

$$S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^4 - x_1^2 x_2 + x_2^3 = 0\}$$

as shown (red) in Figure 6. We have

$$\widetilde{S}^{\mathrm{o}} = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1^4 - x_0 x_1^2 x_2 + x_0 x_2^3 = 0, \ x_0 > 0\},$$
$$\widetilde{S}^{\mathrm{c}} = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1^4 - x_0 x_1^2 x_2 + x_0 x_2^3 = 0, \ x_0 \geq 0\}.$$

We first show that $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is closed and pointed by proving the polynomial $X_0 - X_2$ is positive on $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}})) \backslash \{0\}$. For every $0 \neq \tilde{u} = (u_0, u_1, u_2) \in \mathbf{cl}(\widetilde{S}^{\mathrm{o}})$, we have $u_1^4 - u_0 u_1^2 u_2 + u_0 u_2^3 = 0$. If $u_2 = 0$, then $u_1 = 0$ and $u_0 - u_2 > 0$. Assume $u_2 \neq 0$, then $u_1^2 u_2 - u_2^3 \neq 0$. Otherwise, we have $u_1^2 = u_2^2$ and $u_2^4 - u_0 u_2^3 + u_0 u_2^3 = 0$, then $u_2 = 0$, a contradiction. Therefore,

$$\begin{aligned} u_0 - u_2 &= \frac{u_1^4}{u_1^2 u_2 - u_2^3} - u_2 \\ &= \frac{(u_1^2 - u_2^2)^2 + u_1^2 u_2^2}{u_1^2 u_2 - u_2^3} \\ &> 0, \end{aligned}$$

which implies $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is closed and pointed. Since S is of form (4.1), by Proposition 4.1, S is not closed at ∞. By Remark 4.2, we add $1 - x_2 \geq 0$ to the generators of S to force it to be closed at ∞. The third spectrahedron $\widetilde{\Omega}_3(\widetilde{G})$ with the new generating set is shown (shaded) in Figure 6.

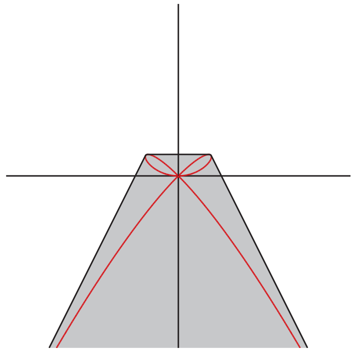

FIG. 6. *The curve S (red curve) and the third spectrahedron $\widetilde{\Omega}_3(\widetilde{G})$ (shaded) in Example* 4.3.

4.2. Pointedness of $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$. When $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is not pointed, we divide S into 2^n parts along each axis. Let

$$\mathcal{E} := \{e = (e_1, \dots, e_n) \in \mathbb{R}^n \mid e_i \in \{0,1\}, \quad i = 1, \dots, n\} \tag{4.2}$$

and for each $e \in \mathcal{E}$,

$$S_e := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, \ (-1)^{e_j} x_j \geq 0, \quad i = 1, \dots, m, \ j = 1, \dots, n\}. \tag{4.3}$$

Then $S = \bigcup_{e \in \mathcal{E}} S_e$ and $|\mathcal{E}| = 2^n$. For each $e \in \mathcal{E}$, define $\widetilde{S}_e^{\mathrm{o}}$, $\widetilde{S}_e^{\mathrm{c}}$, $\widetilde{S}_e$, and $\widetilde{G}_e$ as in (3.2) and (3.3). By Theorem 2.3, both $\mathbf{co}(\mathbf{cl}(\widetilde{S}_e^{\mathrm{o}}))$ and $\mathbf{co}(\widetilde{S}_e^{\mathrm{c}})$ are closed and pointed for each $e \in \mathcal{E}$.

THEOREM 4.4. *Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set defined as in* (3.1)*. Assume that*

1. *S is closed at ∞;*
2. *for each $e \in \mathcal{E}$, PP-BDR property holds for $\widetilde{G}_e$.*

Then $\mathbf{cl}(\mathbf{co}(S))$ is the closure of a projected spectrahedron.

Proof. Fix an integer k' such that PP-BDR property holds for each $\widetilde{G}_e$ with order k'. Note that S_e may not be closed at ∞ for some $e \in \mathcal{E}$. However, we show that

$$\mathbf{cl}\left(\mathbf{co}\left(S_e\right)\right) \subseteq \mathbf{cl}\left(\widetilde{\Omega}_{k'}\left(\widetilde{G}_e\right)\right) \subseteq \mathbf{cl}\left(\mathbf{co}\left(S\right)\right) \quad \forall e \in \mathcal{E}. \tag{4.4}$$

By Theorem 3.14, we get

$$\mathbf{cl}\left(\mathbf{co}\left(S_e\right)\right) \subseteq \mathbf{cl}\left(\widetilde{\Omega}_{k'}\left(\widetilde{G}_e\right)\right) \subseteq \widetilde{\mathrm{TH}}_{k'}(\widetilde{G}_e) \quad \forall e \in \mathcal{E}.$$

Fix a vector $u \notin \mathbf{cl}(\mathbf{co}(S))$. According to (3.7), there exists a polynomial $\tilde{f} \in \mathbb{P}[\widetilde{X}]_1$ such that $\tilde{f}(1,u) < 0$ and $\tilde{f}(\tilde{x}) \geq 0$ on $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$. Since $\mathbf{cl}(\widetilde{S}^{\mathrm{o}}) = \widetilde{S}^{\mathrm{c}}$ and $\widetilde{S}_e^{\mathrm{c}} \subseteq \widetilde{S}^{\mathrm{c}}$ for each $e \in \mathcal{E}$, we have $\tilde{f}(\tilde{x}) \geq 0$ on each $\mathbf{co}(\widetilde{S}_e^{\mathrm{c}})$. Because $\mathbf{co}(\widetilde{S}_e^{\mathrm{c}})$ is closed and pointed, by Theorem 2.3, there exists a polynomial $\tilde{g} \in \mathbb{P}[\widetilde{X}]_1$ such that $\tilde{g}(\tilde{x}) > 0$ on $\mathbf{co}(\widetilde{S}_e^{\mathrm{c}})$. We choose a small $\epsilon > 0$ such that $(\tilde{f} + \epsilon\tilde{g})(1,u) < 0$ and rename $\tilde{f} + \epsilon\tilde{g}$ as $\tilde{f}$, then $\tilde{f}(\tilde{x}) > 0$ on $\widetilde{S}_e^{\mathrm{c}}$. In particular, $\tilde{f}(\tilde{x}) > 0$ on $\widetilde{S}_e$. Since $\widetilde{G}_e$ satisfies PP-BDR property with order k', we have $\tilde{f} \in \mathcal{Q}_{k'}(\widetilde{G}_e)$ and $u \notin \widetilde{\mathrm{TH}}_{k'}(\widetilde{G}_e)$ due to the fact that

$\tilde{f}(1,u) < 0$. It implies $\widetilde{\mathrm{TH}}_{k'}(\widetilde{G}_e) \subseteq \mathbf{cl}(\mathbf{co}(S))$ and (4.4). Therefore, we have

$$\begin{aligned}\mathbf{cl}\,(\mathbf{co}\,(S)) &= \mathbf{cl}\left(\mathbf{co}\left(\bigcup_{e\in\mathcal{E}} S_e\right)\right)\\ &= \mathbf{cl}\left(\mathbf{co}\left(\bigcup_{e\in\mathcal{E}} \mathbf{cl}\,(\mathbf{co}\,(S_e))\right)\right)\\ &\subseteq \mathbf{cl}\left(\mathbf{co}\left(\bigcup_{e\in\mathcal{E}} \mathbf{cl}\left(\widetilde{\Omega}_{k'}\left(\widetilde{G}_e\right)\right)\right)\right)\\ &\subseteq \mathbf{cl}\,(\mathbf{co}\,(\mathbf{cl}\,(\mathbf{co}\,(S))))\\ &= \mathbf{cl}\,(\mathbf{co}\,(S)),\end{aligned}$$

which implies

$$\mathbf{cl}\,(\mathbf{co}\,(S)) = \mathbf{cl}\left(\mathbf{co}\left(\bigcup_{e\in\mathcal{E}} \mathbf{cl}\left(\widetilde{\Omega}_{k'}\left(\widetilde{G}_e\right)\right)\right)\right) = \mathbf{cl}\left(\mathbf{co}\left(\bigcup_{e\in\mathcal{E}} \widetilde{\Omega}_{k'}\left(\widetilde{G}_e\right)\right)\right).$$

Since each $\widetilde{\Omega}_{k'}(\widetilde{G}_e)$ is a projected spectrahedron, by [7, Theorem 2.2], we have

$$\begin{aligned}&\mathbf{cl}\left(\mathbf{co}\left(\bigcup_{e\in\mathcal{E}} \widetilde{\Omega}_{k'}\left(\widetilde{G}_e\right)\right)\right)\\ &\qquad = \mathbf{cl}\left(\left\{\sum_{e\in\mathcal{E}} \lambda_e x^{(e)} \;\middle|\; \sum_{e\in\mathcal{E}} \lambda_e = 1,\ \lambda_e \geq 0,\ x^{(e)} \in \widetilde{\Omega}_{k'}\left(\widetilde{G}_e\right)\right\}\right)\end{aligned}$$

which is the closure of a projected spectrahedron. ∎

Remark 4.5. If $\mathcal{E}'$ is a subset of $\mathcal{E}$ such that $S = \bigcup_{e\in\mathcal{E}'} S_e$, then by the above proof, the conclusion of Theorem 4.4 still holds if we replace $\mathcal{E}$ by $\mathcal{E}'$.

Example 3.11 *continued*. For $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2^3 - x_1^2 \geq 0\}$, we have shown that $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is not pointed, and the modified theta bodies (3.4) do not converge to $\mathbf{cl}(\mathbf{co}(S))$. Due to Remark 4.5, divide S into two parts

$$\begin{aligned}S_{(0,0)} &:= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2^3 - x_1^2 \geq 0,\ x_1 \geq 0,\ x_2 \geq 0\},\\ S_{(1,0)} &:= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2^3 - x_1^2 \geq 0,\ -x_1 \geq 0,\ x_2 \geq 0\}.\end{aligned}$$

It is easy to check that PP-BDR property holds for $\widetilde{G}_{(0,0)}$ and $\widetilde{G}_{(1,0)}$ with order one. Thus for any $k' \geq 1$, we have

$$\begin{aligned}\widetilde{\Omega}_{k'}\left(\widetilde{G}_{(0,0)}\right) &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0\},\\ \widetilde{\Omega}_{k'}\left(\widetilde{G}_{(1,0)}\right) &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0,\ x_2 \geq 0\}.\end{aligned}$$

Then, $\mathbf{cl}(\mathbf{co}(\widetilde{\Omega}_{k'}(\widetilde{G}_{(0,0)}) \cup \widetilde{\Omega}_{k'}(\widetilde{G}_{(1,0)}))) = \mathbf{cl}(\mathbf{co}(S))$ for any $k' \geq 1$.

However, if PP-BDR property does not hold with order k' for some $\widetilde{G}_e$, according to the proof of Theorem 4.4, $\mathbf{cl}(\widetilde{\Omega}_{k'}(\widetilde{G}_e))$ may not be a subset of $\mathbf{cl}(\mathbf{co}(S))$ for some $e \in \mathcal{E}$. In this case, $\mathbf{cl}(\mathbf{co}(\bigcup_{e\in\mathcal{E}} \widetilde{\Omega}_{k'}(\widetilde{G}_e)))$ may contain $\mathbf{cl}(\mathbf{co}(S))$ strictly.

Example 4.6. Rotating the semialgebraic set S in Example 3.11 about the origin 45° counterclockwise, we get

$$S' := \{(x_1, x_2) \in \mathbb{R}^2 \mid -\sqrt{2}(x_1 - x_2)^3 - 2(x_1 + x_2)^2 \geq 0\},$$

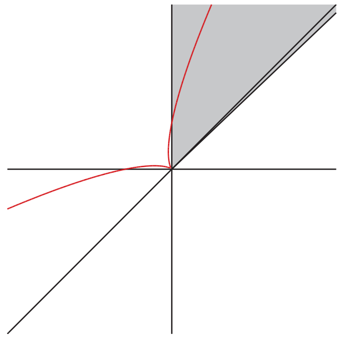

FIG. 7. *The third spectrahedral approximation* $\widetilde{\Omega}_3(\widetilde{G}'_{(0,0)})$ *(shaded) of* $S'_{(0,0)}$ *in Example* 4.6.

which is the left part of $\mathbb{R}^2$ divided by the red curve in Figure 7. Then, $\mathbf{cl}(\mathbf{co}(S))$ is the closed half-plane of $\mathbb{R}^2$ partitioned by the line $X_2 = X_1$.

By Remark 4.5, set $\mathcal{E}' = \{(0,0),(1,0),(1,1)\}$ and divide $S' = \bigcup_{e\in\mathcal{E}'} S'_e$ defined as in (4.3). Clearly, for any integer $k' \geq 1$, we have

$$\widetilde{\Omega}_{k'}\left(\widetilde{G}'_{(1,0)}\right) = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \leq 0,\ x_2 \geq 0\}.$$

The third spectrahedral approximation $\widetilde{\Omega}_3(\widetilde{G}'_{(0,0)})$ of $S'_{(0,0)}$ is shown shaded in Figure 7. As we can see, the support line $X_2 = X_1$ is approximated by $X_2 = aX_1$ with $a < 1$. The same thing happens in the third quadrant. Numerically, we deduce $\mathbf{cl}(\mathbf{co}(\bigcup_{e\in\mathcal{E}'} \widetilde{\Omega}_3(\widetilde{G}'_e))) = \mathbb{R}^2$ which contains $\mathbf{cl}(\mathbf{co}(S))$ strictly.

Therefore, when $\mathbf{co}(\mathbf{cl}(\widetilde{S}^{\mathrm{o}}))$ is not pointed, it becomes much more complicated to approximate $\mathbf{cl}(\mathbf{co}(S))$ properly if PP-BDR property does not hold on $\widetilde{G}_e$ for some $e \in \mathcal{E}$. We leave this case for future investigation.

Acknowledgment. We would like to acknowledge many helpful comments and suggestions from the referees.

REFERENCES

[1] A. Ben-Tal and A. S. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, MOS-SIAM Ser. Optim. 2, SIAM, Philadelphia, 2001.

[2] W. Fenchel, *Convex cones, sets, and functions*, Logistics Research Project, Princeton University, Dept. of Mathematics, Princeton, NJ, 1953.

[3] L. Fialkow and J. Nie, *The truncated moment problem via homogenization and flat extensions*, J. Funct. Anal., 263 (2012), pp. 1682–1700.

[4] J. Gouveia, P. A. Parrilo, and R. R. Thomas, *Theta bodies for polynomial ideals*, SIAM J. Optim., 20 (2010), pp. 2097–2118.

[5] J. Gouveia and R. R. Thomas, *Spectrahedral approximations of convex hulls of algebraic sets*, in Semidefinite Optimization and Convex Algebraic Geometry, G. Blekherman, P. A. Parrilo, and R. R. Thomas, eds., MOS-SIAM Ser. Optim. 13, SIAM, Philadelphia, 2012, pp. 293–340.

[6] F. Guo, L. Wang, and G. Zhou, *Minimizing rational functions by exact Jacobian SDP relaxation applicable to finite singularities*, J. Global Optim., 58 (2014), pp. 261–284.

[7] J. W. Helton and J. Nie, *Sufficient and necessary conditions for semidefinite representability of convex hulls and sets*, SIAM J. Optim., 20 (2009), pp. 759–791.

[8] J. W. Helton and J. Nie, *Semidefinite representation of convex sets*, Math. Program., 122 (2010), pp. 21–64.

[9] J. W. Helton and V. Vinnikov, *Linear matrix inequality representation of sets*, Comm. Pure Appl. Math., 60 (2007), pp. 654–674.

[10] J. B. Lasserre, *Convex sets with semidefinite representation*, Math. Program. Ser. A, 120 (2009), pp. 457–477.

[11] S. Lin, *Unbounded Convex Semialgebraic Sets as Spectrahedral Shadows*, preprint, 2010.

[12] M. Marshall, *Optimization of polynomial functions*, Canad. Math. Bull., (2003), pp. 575–587.

[13] J. Nie, *An exact Jacobian SDP relaxation for polynomial optimization*, Math. Program. Ser. A, 137 (2013), pp. 225–255.

[14] P. A. Parrilo, *Exact semidefinite representations for genus zero curves*, Workshop on Positive Polynomials and Optimization, Banff, Canada, 2006.

[15] V. Powers and C. Scheiderer, *The moment problem for non-compact semialgebraic sets*, Adv. Geom., 1 (2001), pp. 71–88.

[16] M. Putinar, *Positive polynomials on compact semi-algebraic sets*, Indiana Univ. Math. J., 42 (1993), pp. 969–984.

[17] R. T. Rockafellar, *Convex Analysis*, Princeton Landmarks Math. Phys., Princeton University Press, Princeton, NJ, 1996.

[18] P. Rostalski, *Bermeja - software for convex algebraic geometry*, http://www.math.uni-frankfurt.de/˜rostalsk/pmwiki/pmwiki.php?n=Main.HomePage.

[19] P. Rostalski and B. Sturmfels, *Dualities in convex algebraic geometry*, Rend. Mat. (7), 30 (2010), pp. 285–327.

[20] M. Schweighofer, *Optimization of polynomials on compact semialgebraic sets*, SIAM J. Optim., 15 (2005), pp. 805–825.