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The integral closure of a primary ideal is not always primary

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ABSTRACT

In 1936, Krull asked if the integral closure of a primary ideal is still primary. Fifty years later, Huneke partially answered this question by giving a primary polynomial ideal whose integral closure is not primary in a regular local ring of characteristic $p = 2$. We provide counterexamples to Krull's question regarding polynomial rings over any fields. We also find that the Jacobian ideal J of the polynomial $f = x^6 + y^6 + x^4zt + z^3$ given by Briançon and Speder (1975) is a counterexample to Krull's question.

1. Introduction

Krull (1936, p. 577) asked: *Ist etwa bei einem Primärideal q immer auch q_b Primärideal?* For monomial ideals, the answer to Krull's question is yes. The integral closure of a primary monomial ideal is always primary (Jarrah, 2002, p. 5474). However, for non-monomial ideals, Huneke partially answered this question by giving a counterexample in the regular local ring $k[[x, y, z]]$ with $\mathrm{char}(k) = 2$ (Huneke, 1986, Example 3.7). According to Jarrah (2002, p. 5473) there are no known counterexamples for

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rings of characteristic zero. The integral closure of ideals is related to Whitney equisingularity. For instance, Tráng and Teissier (1988, Proposition 1.3.8) (see also Flores and Teissier, 2018, Remark 4.11) gave an algebraic description for Whitney's condition (b) using the integral closure of the sheaf of ideals, which started the modern equisingularity theory. Gaffney (1992, 1996), Gaffney and Kleiman (1999) generalized the theory of integral closure of ideals to modules, and made many applications in studying Whitney equisingularity.

Our main contributions are summarized below:

- We answer Krull's question negatively by giving a sequence of primary ideals

  $$I = \langle x^3, y^3, x^2y, x^2z^n - xy^2\rangle, n \in \mathbb{Z}_+$$

  whose integral closures

  $$\bar{I} = \langle x^3, y^3, x^2y, x^2z^n, xy^2\rangle$$

  are not primary over a field of characteristic zero or positive characteristics. Hence, taking integral closure of a primary polynomial ideal may create embedded primes. On the other hand, we also show that there are examples where the given polynomial ideal is not primary but its integral closure is primary. It implies that taking integral closure may also remove embedded primes. Therefore, the relation between a primary ideal and its integral closure is not clear yet.
- We discover that the Jacobian ideal J of the polynomial

  $$f(x, y, z, t) = x^6 + y^6 + x^4zt + z^3 = 0, \tag{1}$$

  is also a counterexample to Krull's question. This polynomial was first given by Briançon and Speder (1975) and it has been investigated by many researchers ever since (see Parusiński and Păunescu, 2017, Example 7.9; Wall, 2010, p. 354; Zariski, 1977, pp. 14–16). Let $\bar{J}$ be the integral closure of J, V_1 be the hypersurface defined by $f = 0$ and V_2 be the singular locus of V_1. We find that $\bar{J}$ has an embedded prime $P = \langle x, y, z, 4t^3 + 27\rangle$, and the variety of P contains exactly the points where $(V_1 \setminus V_2, V_2)$ fails Whitney's conditions.

The paper is organized as follows. Section 2 is for basic definitions and properties of integral closures of ideals. In Section 3, we present a sequence of primary ideals whose integral closures are not primary over a field of characteristic zero or positive characteristics. Moreover, we present an example to show that taking integral closure may remove embedded primes. Finally, we compute the integral closure $\bar{J}$ of the Jacobian ideal of f defined by (1) and verify that the pair $(V_1 \setminus V_2, V_2)$ fails Whitney's conditions at the variety of the embedded prime of $\bar{J}$.

2. Basic properties

Let us first recall some basic definitions from Huneke and Swanson (2006).

Definition 2.1. Let I be an ideal in a ring R. An element $r \in R$ is said to be integral over I if there exists an integer n and elements $a_i \in I^i$, $i = 1, \ldots, n$, such that

$$r^n + a_1r^{n-1} + a_2r^{n-2} + \cdots + a_{n-1}r + a_n = 0.$$

The set of all integral elements over I is called the *integral closure* of I and is denoted by $\bar{I}$. If $I = \bar{I}$, then I is called integrally closed. Let J be an ideal satisfying $I \subset J$, we say that J is integral over I if $J \subset \bar{I}$.

Definition 2.2. Let k be a field and R be the polynomial ring $k[x_1, \ldots, x_d]$. For any monomial $m = x_1^{n_1}x_2^{n_2}\cdots x_d^{n_d}$, its exponent vector is $(n_1, \ldots, n_d) \in \mathbb{N}^d$. For any monomial ideal I, the set of all exponent vectors of all the monomials in I is called the *exponent set* of I.

The integral closure of a monomial ideal in a polynomial ring is still a monomial ideal (Huneke and Swanson, 2006, Proposition 1.4.2). The following proposition is useful for computing the integral closure of a monomial ideal.

Proposition 2.3 *(Huneke and Swanson, 2006, Proposition 1.4.6). The exponent set of the integral closure of a monomial ideal I equals all the integer lattice points in the convex hull of the exponent set of I.*

3. Counterexamples to Krull's question

In this section, we answer Krull's question negatively by giving a sequence of primary ideals whose integral closures are not primary. We also show that the Jacobian ideal of the ideal defined by $f(x, y, z, t) = x^6 + y^6 + x^4zt + z^3$ is another counterexample to Krull's question.

3.1. A set of counterexamples to Krull's question

Let $I = \langle x^3, y^3, x^2y, x^2z - xy^2\rangle$ and $\bar{I}$ be the integral closure of I. Integral closure of ideals can be computed by the algorithm in de Jong (1998), and primary decomposition of ideals can be computed by classical algorithms in Gianni et al. (1988), Eisenbud et al. (1992), Shimoyama and Yokoyama (1996); all these algorithms have been implemented in Macaulay2 (Grayson and Stillman, 2022). From the computation result of Macaulay2, we get

$$\bar{I} = \langle x^3, y^3, x^2y, x^2z, xy^2\rangle.$$

The ideal $\bar{I}$ has an irredundant primary decomposition:

$$\bar{I} = \langle x^2, y^3, xy^2\rangle \cap \langle x^3, y, z\rangle,$$

which implies that $\bar{I}$ is not primary. In fact, we found that this ideal is a special case of a sequence of counterexamples in the following theorem.

Theorem 3.1. *Let $k[x, y, z]$ be a polynomial ring over a field k. Let n be a positive integer, and*

$$I = \langle x^3, y^3, x^2y, x^2z^n - xy^2\rangle$$

be a polynomial ideal in $k[x, y, z]$. Then the ideal I is primary and its integral closure

$$\bar{I} = \langle x^3, y^3, x^2y, x^2z^n, xy^2\rangle$$

is not primary.

Theorem 3.1 follows from three claims below.

- *Claim 1.* The ideal $I = \langle x^3, y^3, x^2y, x^2z^n - xy^2\rangle \subset k[x, y, z]$ is primary, where $n \in \mathbb{Z}_+$.
  Proof of Claim 1.[1]
  First, we prove that $I : z^\infty = I$. Let T be a new variable. It follows from Cox et al. (2015, Chapter 4, § 4, Theorem 14) that

  $$I : z^\infty = (I + \langle zT - 1\rangle) \cap k[x, y, z].$$

  One can verify that $G = \{x^3, x^2y, xy^2 - x^2z^n, y^3, zT - 1\}$ is a Gröbner basis of the ideal $I + \langle zT - 1\rangle$ with respect to the lexicographic order $T > y > x > z$. Then

[1] This proof is suggested by an anonymous referee. We thank Sizhuo Yan for pointing out the lexicographic order $y > x > z$ and Hao Liang for sharing his knowledge about graded rings and quasi-homogeneous ideals.

$$G \cap k[x, y, z] = \{x^3, x^2y, xy^2 - x^2z^n, y^3\}$$

is a Gröbner basis of the ideal $I : z^\infty$, which implies that $I : z^\infty = I$.
Suppose that the ideal I has an embedded prime $P_1 \supsetneq \sqrt{I} = \langle x, y\rangle$. Then P_1 is quasi-homogeneous since I is quasi-homogeneous with $\deg x = \deg y = n$ and $\deg z = 1$ (see e.g. Bruns and Herzog, 1993, Lemma 1.5.6). Therefore, we have $P_1 = \langle x, y, z\rangle$ because $\langle x, y, z\rangle$ is the only quasi-homogeneous prime ideal that properly contains $\langle x, y\rangle$. By the first uniqueness theorem of primary decomposition (see e.g. Atiyah and Macdonald, 1969, Theorem 4.5), we have $P_1 = \sqrt{I : f}$ for some $f \in k[x, y, z]$ and $f \notin I$. Then there is a positive integer m such that $z^m \in (I : f)$, which implies that $z^m f \in I$ and $f \in I : z^m \subset I : z^\infty = I$. This is a contradiction. Claim 1 is proved. □

- *Claim 2.* For $n \in \mathbb{Z}_+$, the integral closure of the ideal $I = \langle x^3, y^3, x^2y, x^2z^n - xy^2\rangle \subset k[x, y, z]$ is

$$\bar{I} = \langle x^3, y^3, x^2y, x^2z^n, xy^2\rangle.$$

Proof of Claim 2. Let $I_1 = \langle x^3, y^3\rangle \subset I$. By Proposition 2.3, the monomial xy^2 is in the integral closure of I_1, which implies that xy^2 is in the integral closure of I. Therefore $I + \langle xy^2\rangle \subset \bar{I}$. On the other hand, $I + \langle xy^2\rangle = \langle x^3, y^3, x^2y, x^2z^n, xy^2\rangle$, and according to Proposition 2.3, it is integrally closed, which leads to $I + \langle xy^2\rangle = \bar{I}$. Claim 2 is proved. □

- *Claim 3.* The ideal $\bar{I} = \langle x^3, y^3, x^2y, x^2z^n, xy^2\rangle \subset k[x, y, z]$ is not primary, where $n \in \mathbb{Z}_+$.
Proof of Claim 3. Because $x^2z^n \in \bar{I}$ while $x^2 \notin \bar{I}$ and $(z^n)^m \notin \bar{I}$ for any $m \in \mathbb{Z}_+$, the ideal $\bar{I}$ is not primary. Claim 3 is proved. □

In contrast, there exist non-primary ideals whose integral closures are primary. For instance,

$$I = \langle x^2, y^2, xyz\rangle = \langle x^2, xy, y^2\rangle \cap \langle x^2, y^2, z\rangle$$

is not primary. By Proposition 2.3, its integral closure is

$$\bar{I} = \langle x^2, y^2, xy\rangle,$$

which is primary.

3.2. Another counterexample related to Whitney equisingularity

The following example was given in Briançon and Speder (1975) by Briançon and Speder to show that Whitney equisingularity does not imply Zariski equisingularity. We show that the Jacobian ideal J of f is primary and its integral closure $\bar{J}$ is not, which gives another counterexample to Krull's question. Moreover, the embedded prime of $\bar{J}$ happens to be the vanishing ideal of the points where Whitney equisingularity fails.

Example 3.2. Let $f = x^6 + y^6 + x^4zt + z^3 \in \mathbb{Q}[x, y, z, t]$, and its Jacobian ideal

$$J = \langle x^4t + 3z^2,\ x^4z,\ y^5,\ 3x^5 + 2x^3zt\rangle \subset \mathbb{Q}[x, y, z, t].$$

We verified by Macaulay2 that J is a primary ideal of $\mathbb{Q}[x, y, z, t]$, while its integral closure $\bar{J}$ in $\mathbb{Q}[x, y, z, t]$ is not primary, where

$$\begin{aligned}\bar{J} = \langle &3x^2yz + 2yz^2t,\ 3x^3z + 2xz^2t,\ x^4t + 3z^2,\ y^4z,\ x^4z,\ y^5,\ 3x^2y^3 + 2y^3zt,\\ &3x^3y^2 + 2xy^2zt,\ 9x^4y - 4yz^2t^2,\ 3x^5 + 2x^3zt,\ x^3yzt, x^4y^2,\\ &4y^3zt^3 + 27y^3z,\ xy^3zt^2,\ 2x^3y^2t^2 - 9xy^2z,\ 4xy^4t^3 + 27xy^4\rangle.\end{aligned}$$

The associated primes of $\bar{J}$ are

$$\langle z,\ y,\ x\rangle \text{ and } \langle z,\ y,\ x,\ 4t^3 + 27\rangle.$$

As suggested by one of the anonymous referees, one can treat Example 3.2 without using Macaulay2. First, one proves that the Jacobian ideal J of f is primary by the following argument: the associated primes of J in $\mathbb{C}[x, y, z, t]$ are binomial according to Eisenbud and Sturmfels (1996, Theorem 6.1) (also Swanson and Sáenz-de Cabezón, 2017, Theorem 18), and $(t - c)$ is not a zero-divisor of $\mathbb{C}[x, y, z, t]/J$ for any $c \neq 0$. Then one shows that $4t^3 + 27$ is contained in one of the associated primes of $\bar{J}$ by verifying that $(4t^3 + 27)xz^2 \in \bar{J}$ and $xz^2 \notin \bar{J}$.

Since the polynomial $f = x^6 + y^6 + x^4zt + z^3$ is a classical example in the literature (Briançon and Speder, 1975; Parusiński and Păunescu, 2017; Wall, 2010; Zariski, 1977), let us investigate its properties by the integral closure of the Jacobian ideal of f.

Theorem 3.3. *Let $f = x^6 + y^6 + x^4zt + z^3$, V_1 be the hypersurface defined by $f = 0$, V_2 be the singular locus of V_1, and V_3 be the variety of $\langle x, y, z, 4t^3 + 27\rangle$. The pair $(V_1 \setminus V_2,\ V_2)$ does not satisfy Whitney's condition (a) and (b) at the points in V_3.*

Proof. Since Whitney's condition (b) implies Whitney's condition (a), we only need to prove that the pair $(V_1 \setminus V_2, V_2)$ does not satisfy Whitney's condition (a).

Let $\boldsymbol{p} = (0, 0, 0, \xi)$ be a point in V_3, where $\xi = (-\sqrt[3]{27/4})\omega$ and ω is one of the cube roots of unity, i.e. $\omega^3 = 1$. Consider the sequence of points

$$\boldsymbol{p}_\epsilon = (\epsilon, 0, c\epsilon^2, \xi) \text{ where } \epsilon \neq 0 \text{ and } c = (\sqrt[3]{1/2})\omega^2.$$

Note that $\xi = -3c^2$ and $c^3 = 1/2$. For any $\epsilon \neq 0$, we have $\boldsymbol{p}_\epsilon \in V_1 \setminus V_2$ and $\boldsymbol{p}_\epsilon \to \boldsymbol{p}$, as $\epsilon \to 0$. One can verify that $V_2 = \{(0, 0, 0, t) \in \mathbb{C}^4\}$ and $T_{\boldsymbol{p}_\epsilon} V_1 = \{(x, y, z, t) \in \mathbb{C}^4 \mid t = 0\}$, and therefore $T_{\boldsymbol{p}} V_2 = V_2$ is not contained in the limit of $T_{\boldsymbol{p}_\epsilon} V_1$ as $\epsilon \to 0$. □

One can verify by the criterion in Đinh and Jelonek (2021, Lemma 2.8) and/or the algorithm in Helmer and Nanda (2022) that V_3 contains all the points where $(V_1, V_1 \setminus V_2)$ fails Whitney's conditions.

We also noticed that this example implies that the stratification defined by isosingular sets (Hauenstein and Wampler, 2013) is different from Whitney stratification: one can stratify V_1 as $(V_1 \setminus V_2) \cup (V_2 \setminus \{\mathbf{0}\}) \cup \{\mathbf{0}\}$ by isosingular sets, which does not exclude V_3 from V_2.

CRediT authorship contribution statement

Nan Li: Writing – review & editing, Writing – original draft, Methodology, Investigation. **Zijia Li:** Writing – review & editing, Writing – original draft, Methodology, Investigation. **Zhi-Hong Yang:** Writing – review & editing, Writing – original draft, Methodology, Investigation. **Lihong Zhi:** Writing – review & editing, Writing – original draft, Methodology, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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